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ON SPACE-LIKE SURFACES WITH PARALLEL MEAN CURVATURE VECTOR OF AN INDEFINITE SPACE FORM

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

By

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1. Introduction

Let $N_p^{2+p}(c)$ be a $(2+p)$-dimensional connected indefinite Riemannian manifold of index p and of constant curvature c, which is called an indefinite space form of index p. According to $c>0$, $c=0$ or $c<0$, it is a pseudo-Riemannian sphere $S_p^{2+p}(c)$, a pseudo-Euclidean space R_p^{2+p} or a pseudo-Hyperbolic space $H_p^{2+p}(c)$, respectively. A surface M of an indefinite space form $N_p^{2+p}(c)$ is said to be space-like if the induced metric on M from that of $N_p^{2+p}(c)$ is positive definite.

Suppose M is a space-like surface with parallel mean curvature vector $\vec{H}$ in an indefinite space form $N_p^{2+p}(c)$. Let K be the Gauss curvature of M and $H=|\vec{H}|$. We remark that the square norm S of the second fundamental form of M satisfies the following Gauss equation (see (2.6))

$$S=2(2H^2-c+K). \tag{1.1}$$

We first consider the pinching problem for the square norm S of the second fundamental form of a complete space-like surface M with parallel mean curvature vector $\vec{H}$ in $N_p^{2+p}(c)$. From the results by J. Ramanathan [15], U.H. Ki, H.J. Kim and H. Nakagawa [8], we know that for a complete space-like surface M with constant mean curvature H in a 3-dimensional indefinite space form $N_1^3(c)$,

$$2H^2\leqq S\leqq S_+\equiv\max\{2H^2, 4H^2-2c\}, \tag{1.2}$$

where $S\equiv 2H^2$ iff M is totally umbilic, and this estimated value S_+ is the best possible. For the higher co-dimensional case, we have already known that the following result by Q.M. Cheng [4] and Q.M. Cheng and S.M. Choi [5],

(1) $2H^2\leqq S$, $S\equiv 2H^2$ iff M is totally umbilic,

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(2) if $c>0$ and $H^2 \leqq c$, then $S \equiv 2H^2$,

(3) in the other case of (2), $S \leqq -2pc+2(p+1)H^2+2(p-1)H\sqrt{H^2-c}$.

In section 3 of this paper, we show that the estimate (1.2) still holds for the higher codimensional case, that is, we prove the following

THEOREM 3.1. *Let M be a complete space-like surface with parallel mean curvature vector $\vec{H}$ in an indefinite space form $N_p^{2+p}(c)$ and $H=|\vec{H}|$. Then the square norm S of the second fundamental form of M satisfies*

$$2H^2 \leqq S \leqq \max\{2H^2, 4H^2-2c\},$$

where $S \equiv 2H^2$ if and only if M is totally umbilic.

Next, we consider estimating the Gauss curvature of a conformal metric on a space-like surface M with parallel mean curvature vector $\vec{H}$ in $N_p^{2+p}(c)$, while the similar problem for minimal surface M in Riemannian space form $N^{2+p}(c)$ has been studied by Barbosa-Do Carmo [2], Lawson [9] and the first author [10]. Here Riemannian space form $N^{2+p}(c)$ means one of Euclidean sphere $S^{2+p}(c)$, Euclidean space R^{2+p} or Hyperbolic space $H^{2+p}(c)$, according to $c>0$, $c=0$ or $c<0$, respectively. In section 4, we prove the following theorem for space-like surfaces in an indefinite space form $N_p^{2+p}(c)$

THEOREM 4.1. *Let M be a space-like surface with parallel mean curvature vector $\vec{H}$ in $N_p^{2+p}(c)$ and K be the Gauss curvature of M with the induced metric ds_M^2. At non-umbilic points in M, $H^2-c+K>0$. So we can define the conformal metric*

$$\overline{ds}^2=(H^2-c+K)^b ds_M^2 \tag{1.3}$$

for any real number b. Then the Gauss curvature $\bar{K}$ of this metric $\overline{ds}^2$ satisfies

$$\bar{K} \leqq -\frac{(2b-1)K}{(H^2-c+K)^b}, \tag{1.4}$$

and the equality holds in (1.4) if there exists a complete 3-dimensional totally geodesic submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M \subset N_1^3(c)$.

On the other hand, in [9], H.B. Lawson has studied the case which an isometric immersion $f: M \to N^3(c)$ has constant mean curvature H and Gauss curvature K with induced metric ds_M^2, in this case we get $K \leqq H^2+c$, and for the conformal metric

$$\overline{ds}^2=\sqrt{H^2+c-K}\,ds_M^2,$$

the Gauss curvature $\bar{K}$ is zero. Conversely, for a simply connected Riemannian surface (M, ds_M^2) and any non-negative real number $(\bar{H})^2$ satisfying $K<(\bar{H})^2$ and flatness of the metric $\overline{ds}^2=\sqrt{(\bar{H})^2-K}ds_M^2$, there exists a family of isometric immersion $M\rightarrow N^3(c)$ $(c\leqq(\bar{H})^2)$ with constant mean ɔurvature $H=\sqrt{(\bar{H})^2-c}$. In section 4, for space-like surfaces in an indefinite space form $N_p^{2+p}(c)$ we establish the counterpart of above Lowson's result (see Theorem 4.2).

Finally, for a maximal space-like surface M in an indefinite space form $N_p^{2+p}(c)$, T. Ishihara introduced its Gauss map $g: M\rightarrow G$ which is defined by carrying each point $x\in M$ to the totally geodesic space-like surface tangent to M at x in M, where G is the Grassmannian manifold (definition of G see section 4). Let Gauss curvature of M be K with induced metric ds_M^2, then the induced metric of g is $ds_g^2=g*(ds_G^2)=(K-c)ds_M^2$. We prove the following result

THEOREM 4.3. *Let M be a maximal space-like surface in $N_p^{2+}(c)$ and K be the Gauss curvature of M. At non-umbilic points in M, the Gauss curvature K_g of the Gauss image $g(M)$ satisfies*

$$K_g\leqq -1-\frac{c}{K-c}, \tag{1.5}$$

and the equality holds in (1.5) *if and only if there exists a complete* 3*-dimensional submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M\subset N_1^3(c)$.*

2. Preliminaries

Let M be a 2-dimensional space-like surface of $N_p^{2+p}(c)$. We choose a local field of pseudo-Riemannian orthonormal frames $e_1, \cdots, e_{2+p}$ in $N_p^{2+p}(c)$ in such that, at each point of M, e_1, e_2 spans the tangent space of M and forms an orthonormal frame there. We make use of the following convention on the ranges of indices:

$$1\leqq A, B, C\leqq 2+p; \quad 1\leqq i, j, k\leqq 2; \quad 3\leqq\alpha, \beta, \gamma\leqq 2+p.$$

We shall agree the repeated indices are summed over the respective ranges. Let $\omega_1, \cdots, \omega_{2+p}$ be its dual frame field so that the pseudo-Riemannian metric of $N_p^{2+p}(c)$ is given by $ds_N^2=\sum_i\omega_i^2-\sum_\alpha\omega_\alpha^2=\varepsilon_A\sum_A\omega_A^2$, where $\varepsilon_i=1$ for $1\leqq i\leqq 2$ and $\varepsilon_\alpha=-1$ for $3\leqq\alpha\leqq 2+p$. Then the structure equations of $N_p^{2+p}(c)$ are given by

$$\begin{cases} d\omega_A=\sum_B\varepsilon_B\omega_{AB}\wedge\omega_B, \quad \omega_{AB}+\omega_{BA}=0, \\ d\omega_{AB}=\sum\varepsilon_C\omega_{AC}\wedge\omega_{CB}-\frac{1}{2}\sum_{C,D}\varepsilon_C\varepsilon_D K_{ABCD}\omega_C\wedge\omega_D, \\ K_{ABCD}=\varepsilon_A\varepsilon_B(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC})c. \end{cases} \tag{2.1}$$

We restrict these forms to M, then

(2.2) $$\omega_\alpha=0, \quad 3\leqq\alpha\leqq 2+p,$$

and the Riemannian metric of M is written as $ds_M^2=\omega_1^2+\omega_2^2$. We may put

(2.3) $$\omega_{i\alpha}=\sum_j h_{ij}^\alpha\omega_j, \quad h_{ij}^\alpha=h_{ij}^\alpha.$$

Then h_{ij}^α are components of the second fundamental form of M. From (2.1), we obtain the structure equations of M

(2.4) $$\begin{cases} d\omega_i=\sum_j \omega_{ij}\wedge\omega_j, \\ d\omega_{ij}=\sum_k \omega_{ik}\wedge\omega_{kj}-\dfrac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l, \end{cases}$$

and the Gauss equation

(2.5) $$R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})-\sum(h_{ik}^\alpha h_{jl}^\alpha-h_{il}^\alpha h_{jk}^\alpha).$$

Let S be the square norm of the second fundamental form of M, $\vec{H}$ denote the mean curvature vector of M and H the mean curvature of M:

$$S=\sum_{\alpha,i,j}(h_{ij}^\alpha)^2, \quad \vec{H}=\frac{1}{2}\sum_\alpha(\sum_i h_{ii}^\alpha)e_\alpha, \quad H=|\vec{H}|.$$

From the Gauss equation (2.5), we know that the Gauss curvature K of M satisfies

(2.6) $$2K=2c-4H^2+S.$$

We also have

(2.7) $$\begin{cases} d\omega_{\alpha\beta}=-\sum_\gamma \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}-\dfrac{1}{2}\sum_{i,j} R_{\alpha\beta ij}\omega_i\wedge\omega_j, \\ R_{\alpha\beta ij}=\sum_l (h_{il}^\alpha h_{lj}^\beta-h_{jl}^\alpha h_{li}^\beta). \end{cases}$$

We call M a surface with parallel mean curvature vector if $D\vec{H}\equiv 0$ on M, where D is the normal connection of M in $N_p^{2+p}(c)$. M is said to be *maximal* if $H\equiv 0$ on M. Let h_{ijk}^α (resp. h_{ijkl}^α) denote the coveriant derivative of h_{ij}^α (resp. h_{ijk}^α), we have (see [7]) the following Codazzi equation and Ricci formula

(2.8) $$h_{ijk}^\alpha=h_{ikj}^\alpha,$$

(2.9) $$h_{ijkl}^\alpha-h_{ijlk}^\alpha=\sum_m h_{im}^\alpha R_{mjkl}+\sum_m h_{mj}^\alpha R_{mikl}+\sum_\beta h_{ij}^\beta R_{\alpha\beta kl}.$$

We need the following generalized maximum principle (see Omori [14] or Yau [16]) in order to prove our results.

LEMMA 2.1 (Omori-Yau). *Let M be a complete Riemannian manifold with*

Ricci curvature bounded from below. Let f be a C^2 function which bounded from below on M. Then there is a sequence of points $\{p_k\}$ in M such that

$$\lim_{k\to\infty} f(p_k)=\inf(f), \quad \lim_{k\to\infty}|\nabla f(p_k)|=0, \quad \lim_{k\to\infty}\inf \Delta f(p_k)\geqq 0.$$

3. The square norm of the second fundamental form

In this section, we prove

THEOREM 3.1. *Let M be a complete space-like surface with parallel mean curvature vector $\vec{H}$ in an indefinite space form $N_p^{2+p}(c)$ and $H=|\vec{H}|$. Then the square norm S of the second fundamental form of M satisfies*

$$2H^2 \leqq S \leqq \max\{2H^2,\ 4H^2-2c\},$$

where $S\equiv 2H^2$ if and only if M is totally umbilic.

COROLLARY 3.1 ([1] or [8]). *Let M be a complete space-like surface with constant mean curvature H in $N_1^3(c)$. Then $2H^2\leqq S\leqq \max\{2H^2, 4H^2-2c\}$.*

COROLLARY 3.2. *Let M be a complete maximal space-like surface in $H_p^{2+p}(c)$ $(c<0)$. Then $0\leqq S\leqq -2c$. $S\equiv 0$ if and only if M is totally geodesic, and $S\equiv -2c$ if and only if there exists a complete 3-dimensional totally geodesic submanifold $N_1^3(c)$ such that M is the hyperbolic cylinder $H^1(2c)\times H^1(2c)$ in $N_1^3(c)$.*

REMARK 3.1. Corollary 3.2 improves Theorem 1.2 of [7] in case $n=2$.

REMARK 3.2. Let M be a complete space-like surface in $N_p^{2+p}(c)$ with parallel mean curvature vector $\vec{H}$. We know that M must be one of the following surfaces: (1) maximal space-like surfaces in $N_p^{2+p}(c)$, (2) maximal space-like surfaces of a totally umbilical hypersurfaces $N_{p-1}^{2+p-1}(c')$ in $N_p^{2+p}(c)$, or (3) space-like surfaces with constant mean curvature of a totally umbilic 3-dimensional submanifold $N_1^3(c')$ in $N_p^{2+p}(c)$, (This can be proved by use of the same method as in [3]). Therefore it is easy to see that Theorem 3.1 is equivalent to the combination of Corollary 3.1 and Corollary 3.2.

PROOF OF THEOREM 3.1. Making use of the parallelism of $\vec{H}$ and the equations (2.7), (2.8) and (2.9), we can compute the Laplacian ΔS of S as follows:

$$\frac{1}{2}\Delta S=|\nabla h|^2+\sum_{\alpha,i,j,m,k} h_{ij}^{\alpha}(h_{km}^{\alpha}R_{mijk}+h_{mi}^{\alpha}R_{mkjk})+\frac{1}{2}\sum_{\alpha,\beta,j,k}(R_{\beta\alpha jk})^2. \tag{3.1}$$

Since M is 2-dimensional, $R_{mijk}=K(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})$. Therefore we get

$$\frac{1}{2}\Delta S=|\nabla h|^2+2K(S-2H^2)+\frac{1}{2}\sum_{\alpha,\beta,j,k}(R_{\beta\alpha jk})^2 \tag{3.2}$$
$$\geqq(2c-4H^2+S)(S-2H^2).$$

From (2.6), we know that the Gauss curvature K of M is bounded from below. For any given positive constant δ, define the positive smooth function u on M by $u=(S-2H^2+\delta)^{-1/2}$. Through a simple calculation by use of (3.2), the Laplacian Δu of u satisfies

$$u\Delta u\leqq 3|\nabla u|^2-\frac{(2c-4H^2+S)(S-2H^2)}{(S-2H^2+\delta)^2}. \tag{3.3}$$

Now we know that Omori and Yau's generalized maximum principal (Lemma 2.1) can be apllied to the function u on M. Then there is a sequence of points $\{p_k\}$ in M such that

$$\lim_{k\to\infty} u(p_k)=\inf(u), \quad \lim_{k\to\infty}|\nabla u(p_k)|=0, \quad \lim_{k\to\infty}\inf\Delta u(p_k)\geqq 0.$$

Suppose $\inf(u)\neq 0$ and then $\lim_{k\to\infty} S(p_k)=\sup S<\infty$. Approching limit of the both sides of inequality (3.3), we get

$$0\leqq\lim_{k\to\infty}\inf u(p_k)\Delta u(p_k) \tag{3.4}$$
$$\leqq-\frac{(2c-4H^2+\sup S)(\sup S-2H^2)}{(\sup S-2H^2+\delta)^2}.$$

Since $S\geqq 2H^2$, this inequality implies that $\sup S=2H^2$ or $\sup S\leqq 4H^2-2c$. If $\inf(u)=0$, from (3.3), we conclude a contradiction $0\leqq -1$.
We have completed the proof of Theorem 3.1.

PROOF OF COROLLARY 3.2. We only need to prove the last statement of Corollary 3.2. It is enough to show maximal surface M in $N_p^{2+p}(c)$ satisfying $S\equiv -2c$ is contained in a 3-dimensional totally geodesic submanifold $N_1^3(c)$ of $N_p^{2+p}(c)$, because it is already proved by T. Ishihara [7] that the hyperbolic cylinder $H^1(2c)\times H^1(2c)$ is the only complete maximal surface in $N_1^3(c)$ satisfying $S\equiv -2c$.

Since $H\equiv 0$ and $S\equiv -2c$, it follows from (3.2) that the second fundamental form h is parallel and the normal connection is flat. In this case, we can choose a local field of orthonormal frames $\{e_1, e_2\}$ on M such that

$$(h_{ij}^\alpha)=\begin{pmatrix}\lambda^\alpha & 0\\ 0 & -\lambda^\alpha\end{pmatrix}, \qquad \text{for } \alpha=3, \cdots, 2+p$$

for some functions λ^α ($\alpha=3, \cdots, 2+p$) on M. This means that the first normal space $N_1:=\sup\{h(X, Y)|X, Y$ are any tangent vectors on $M\}$ at any non-umbilic point of M is 1-dimensional, because $N_1=\mathrm{span}\{\sum_\alpha \lambda^\alpha e_\alpha\}$. Since h is parallel, the first normal space at non-umbilic points of M are parallel. We also note that the umbilic points of M are isolated even if they exist. Thus, applying the well-known theorem of reducing codimension (see, for example [3] or [12]), there exists a complete 3-dimensional totally geodesic submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M\subset N_1^3(c)$.

We have completed the proof of Corollary 3.2.

4. The Gauss curvature of a conformal metric

Let M be a space-like surface with parallel mean curvature vector in $N_p^{2+p}(c)$ and ds_M^2 be the induced metric from $N_p^{2+p}(c)$. From Gauss equation (2.6), we can write the following nonnegative function

$$\sigma=\frac{1}{2}S-H^2=H^2-c+K, \tag{4.1}$$

where K is the Gauss curvature of M. Obviously, $\sigma(p)=0$ if and only if p is umbilic.

We consider non-umbilic points in M, $\sigma=(1/2)S-H^2>0$. In this section, we estimate the Gauss curvature $\bar{K}$ of the conformal metric $\overline{ds}^2=\sigma^b ds_M^2$, where b is an arbitrary real number. In fact, we prove

THEOREM 4.1. *Let M be a space-like surface with parallel mean curvature vector $\vec{H}$ in $N_p^{2+p}(c)$ and K be the Gauss curvature of M with induced metric ds_M^2. For non-umbilic points in M, the Gauss curvature $\bar{K}$ of the conformal metric $\overline{ds}^2=(H^2-c+K)^b ds_M^2$ satisfies*

$$\bar{K}\leqq-\frac{(2b-1)K}{(H^2-c+K)^b}, \tag{4.2}$$

and the equality holds in (4.2) *if there exists a complete* 3*-dimensional totally geodesic submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M\subset N_1^3(c)$.*

COROLLARY 4.1. *Let M be a space-like surface with parallel mean curvature vector $\vec{H}$ in $N_p^{2+p}(c)$. For non-umbilic points in M, the Gauss curvature $\bar{K}$ of the conformal metric $\overline{ds}^2=(H^2-c+K)ds_M^2$ satisfies*

$$\bar{K}\leqq-1+\frac{H^2-c}{H^2-c+K}, \tag{4.3}$$

and the equality holds in (4.3) *if there exists a complete* 3*-dimensional totally geodesic submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M\subset N_1^3(c)$.*

COROLLARY 4.2. *Let M be a space-like surface with parallel mean curvature vector $\vec{H}$ in $N_p^{2+p}(c)$. For non-umbilic points in M, the Gauss curvature $\bar{K}$ of the conformal metric $d\bar{s}^2=\sqrt{H^2-c+K}ds_M^2$ satisfies $\bar{K}\leqq 0$, and the equality holds if there exists a complete 3-dimensional totally geodesic submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M\subset N_1^3(c)$.*

REMARK 4.1. The counterparts of Corollary 4.1 and 4.2 for surfaces with constant mean curvature in $N^3(c)$ can be found in Li [11].

We need the following Lemma 4.1 in order to prove Theorem 4.1

LEMMA 4.1. *Let M be a space-like surface with parallel mean curvature vector $\vec{H}$ in $N_p^{2+p}(c)$. Then*

$$|\nabla S|^2\leqq 2(S-2H^2)|\nabla h|^2, \tag{4.4}$$

and equality holds in (4.4) *if there exists a complete 3-dimensional totally geodesic submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M\subset N_1^3(c)$.*

PROOF. Since M is 2-dimensional, by use of the symmetry of the second fundamental form and the parallelism of $\vec{H}$, each term in (4.4) can be computed as follows:

$$2(S-2H^2)|\nabla h|^2=4\sum_\alpha[(h_{11}^\alpha-h_{22}^\alpha)^2+4(h_{12}^\alpha)^2][\sum_\alpha[(h_{111}^\alpha)^2+(h_{112}^\alpha)^2], \tag{4.5}$$

$$|\nabla S|^2=4\{\sum_\alpha[(h_{11}^\alpha-h_{22}^\alpha)h_{111}^\alpha+2h_{12}^\alpha h_{112}^\alpha]\}^2+4\{\sum_\alpha[(h_{11}^\alpha-h_{22}^\alpha)h_{112}^\alpha-2h_{12}^\alpha h_{111}^\alpha]\}^2. \tag{4.6}$$

We can define the functions p_α, q_α, ϕ_α, ψ_α and θ_α on M by

$$\begin{cases}\frac{1}{2}(h_{11}^\alpha-h_{22}^\alpha)=p_\alpha\cos\phi_\alpha\\ h_{12}^\alpha=p_\alpha\sin\phi_\alpha\\ p_\alpha\geqq 0\end{cases},\qquad \begin{cases}h_{111}^\alpha=q_\alpha\cos\psi_\alpha\\ h_{112}^\alpha=q_\alpha\sin\psi_\alpha,\quad \theta_\alpha=\phi_\alpha-\psi_\alpha.\\ q_\alpha\geqq 0\end{cases}$$

Combining with these notations, we follows from (4.5) and (4.6) that

$$2(S-2H^2)|\nabla h|^2=16\sum_{\alpha,\beta}p_\alpha^2q_\beta^2, \tag{4.7}$$

$$|\nabla S|^2=16\sum_{\alpha,\beta}p_\alpha p_\beta q_\alpha q_\beta\cos(\theta_\alpha-\theta_\beta). \tag{4.8}$$

Thus we get

$$\begin{aligned}&|\nabla S|^2-2(S-2H^2)|\nabla h|^2\\ &=-8\sum_{\alpha,\beta}(p_\alpha q_\beta-p_\beta q_\alpha)^2-16\sum_{\alpha,\beta}p_\alpha p_\beta q_\alpha q_\beta[1-\cos(\theta_\alpha-\theta_\beta)]\\ &\leqq 0.\end{aligned}$$

We have completed the proof of Lemma 4.1.

PROOF OF THEOREM 4.1. It is well known that the Gauss curvatures $\bar{K}$ and K associated with the metric $\overline{ds}^2$ and ds_M^2 are related by the equation

$$\bar{K}=\sigma^{-b}(K-\Delta\log\sigma^{b/2})=\sigma^{-b}\Big[K-\frac{b}{2}\Big(\frac{\Delta\sigma}{\sigma}-\frac{|\sigma|^2}{\sigma^2}\Big)\Big]. \tag{4.9}$$

Since $\Delta\sigma=(1/2)\Delta S$, it follows from the equation (3.2) and the estimate (4.4)

$$\begin{aligned}\Delta\sigma&=|\nabla h|^2+(2c-4H^2+S)(S-2H^2)+\frac{1}{2}\sum_{\alpha,\beta,j,k}(R_{\alpha\beta jk})^2\\&\geqq\frac{|\nabla\sigma|^2}{\sigma}+4\sigma(\sigma+c-H^2),\end{aligned} \tag{4.10}$$

where the equality holds if there exists a complete 3-dimensional totally geodesic submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M\subset N_1^3(c)$.

Combining (4.9) with (4.10) completes the proof of Theorem 4.1.

Let M be a space-like surface with constant mean curvature H in $N_1^3(c)$, by Corollary 4.2, we know that the Gauss curvature $\bar{K}\equiv 0$ of the conformal metric $\overline{ds}^2=\sqrt{H^2-c+K}ds_M^2$. Now we prove that the converse of this result still holds, i.e. we have

THEOREM 4.2. *Let M be a space-like surface with constant mean curvature H in $N_1^3(c)$ and K be the Gauss curvature of M. For the conformal metric $\overline{ds}^2=\sqrt{H^2-c+K}ds_M^2$ (well-defined at non-umbilic points on M), the Gauss curvature $\bar{K}$ is zero.*

Conversely, let ds_M^2 be a C^3-Riemannian metric defined over a simply-connected surface M with the Gauss curvature K and let $\bar{H}^2$ be any non-negative real number. Suppose $\bar{H}^2+K>0$ and that the metric $\overline{ds}^2=(\bar{H}^2+K)^{1/2}ds_M^2$ is flat, then for each constant $c\geqq-\bar{H}^2$ there exists a differentiable, 2π-periodic family of isometric space-like immersions

$$\Psi_{c,\theta}: M\longmapsto N_1^3(c)$$

of constant mean curvature $H=(\bar{H}^2+c)^{1/2}$.

REMARK 4.2. The counterpart of Theorem 4.2 for surfaces with constant mean curvature in $N^3(c)$ can be found in Lawson [9].

REMARK 4.3. It is interesting to note that the Euclidean case and Minkowski's case are dual to each other. That is, if $\sqrt{|K|}ds_M^2$ is flat for a Riemannian surface (M, ds_M^2), then it can be realized as a minimal surface in R^3 or a maximal surface in R_1^3 depending on $K<0$ or $K>0$.

PROOF OF THEOREM 4.2. Because Theorem 4.2 is the counterpart of Theorem 8 of Lawson [9], we can prove it by use of the same method as Lawson's. We omit it here.

Similar to Obata's Gauss map [13] for a minimal surface $M \to N^n(c)$, T. Ishihara [6] introduced the Gauss map for a space-like surface M in $N_p^{2+p}(c)$ as follows: the Gauss map $g: M \to G$ is defined by carrying each point $x \in M$ to the totally geodesic space-like surface tangent to M at x in M, where G is a Grassmannian manifold with the structure of $o(3, p)/o(3) \times o(p)$, $o(2, p)/o(2) \times o(p)$, or $o(2, p+1)/o(2) \times o(p+1)$ according to $c>0$, $c=0$ or $c<0$, respectively, where $o(n, q)$ is the orthogonal group consisted of all linear isometries on indefinite Euclidean space R_n^{n+q} and $o(n)$ is the orthogonal group consisted of all linear isometries on Euclidean space R^n. Then Gauss map $g: M \to G$ describes a maximal immersion with singularities occuring precisely at the points where the Gauss curvature K of M is c, the induced Riemannan metric of g is

$$ds_g^2 = g*(ds_G^2) = (K-c)ds_M^2 .$$

We establish the following result

THEOREM 4.3. *Let M be a maximal space-like surface in $N_p^{2+p}(c)$ and K be the Gauss curvature of M. At non-umbilic points in M, the Gauss curvature K_g of the Gauss image $g(M)$ satisfies*

$$K_g \leqq -1 - \frac{c}{K-c}, \tag{4.11}$$

and the equality holds in (4.11) *if and only if there exists a complete* 3*-dimensional submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M \subset N_1^3(c)$.*

PROOF OF THEOREM 4.3. Let (M, ds_M^2) be a maximal surface in $N_p^{2+p}(c)$ with the Gauss curvature K. The first part of Theorem 4.3 comes from Theorem 4.1 directly. We only need to show the last part of Theorem 4.3, that is, we only need to prove that if the Gauss curvature K_g of the Gauss image $g(M)$ $K_g = -1 - (c/K-c)$, then there exists a complete 3-dimensional totally geodesic submanifold $N_1^3(c)$ in $N_p^{2+p}(c)$ such that $M \subset N_1^3(c)$.

In this case, since the equality in (4.10) holds, the normal connection of M is flat. Then, by the same argument in the proof of Corollary 3.2, the first normal space N_1 of M is 1-dimensional. Thus we can choose a local field of frames $e_1, \cdots, e_{2+p}$ on M such that

$$(h^3_{ij})=\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}, \qquad (h^4_{ij})=\cdots=(h^{2+p}_{ij})=0 \tag{4.12}$$

for some function $\lambda\neq 0$ (at non-umbilic points) on M. Using the same notations in the proof of Lemma 4.1, we get $p_3=\lambda$ and $p_4=\cdots=p_{2+p}=0$. Since the equality in (4.4) holds, $p_\alpha q_\beta=q_\alpha p_\beta$ for any α and β. This implies that $q_4=\cdots=q_{2+p}=0$ and then $h^\mu_{ijk}=0$ ($\mu=4, \cdots, 2+p$). But by

$$h^\mu_{ijk}\omega_k=dh^\mu_{ij}+h^\mu_{kj}\omega_{ki}+h^\mu_{ik}\omega_{kj}+h^3_{ij}\omega_{3\mu},$$

we have

$$\omega_{3\mu}=0, \qquad \mu=4, \cdots, 2+p, \tag{4.13}$$

Combining (4.12) with (4.13), we know that the first normal spaces are parallel and M is contained in a 3-dimensional totally geodesic submanifold $N^3_1(c)$ of $N^{2+p}_p(c)$. *We have completed the proof of Theorem* 4.3.

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