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ANR-RESOLUTIONS OF TRIADS

By

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1. Introduction.

By a triad of topological spaces (X, A, A') we mean a topological space X and two subsets $A, A'\subseteqq X$ such that $A\cup A'=X$. By an ANR-triad we mean a triad (X, A, A') such that A and A' are closed subsets of X and X, A, A' and $A\cap A'$ are ANR's (for metric spaces). A map of triads $f:(X, A, A')\rightarrow(Y, B, B')$ is a map $f: X\rightarrow Y$ such that $f(A)\subseteqq B$, $f(A')\subseteqq B'$.

An inverse system of triads $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')=((X, A, A')_{\lambda}, p_{\lambda\lambda'}, \Lambda)$ consists of a directed index set Λ, of a collection of triads $(X, A, A')_{\lambda}=(X_{\lambda}, A_{\lambda}, A'_{\lambda})$, $\lambda\in\Lambda$, and of maps triads $p_{\lambda\lambda'}:(X, A, A')_{\lambda'}\rightarrow(X, A, A')_{\lambda}$, $\lambda\leqq\lambda'$, such that $p_{\lambda\lambda}=1_{X_{\lambda}}$, $\lambda\in\Lambda$ and $p_{\lambda\lambda'}p_{\lambda'\lambda''}=p_{\lambda\lambda''}$, $\lambda\leqq\lambda'\leqq\lambda''$.

By a morphism $\boldsymbol{p}=(p_{\lambda}):(X, A, A')\rightarrow(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ of a triad into an inverse system of triads we mean a collection of maps of triads $p_{\lambda}:(X, A, A')\rightarrow(X, A, A')_{\lambda}$, $\lambda\in\Lambda$, such that $p_{\lambda\lambda'}p_{\lambda'}=p_{\lambda}$, $\lambda\leqq\lambda'$.

A resolution of a triad (X, A, A') is a morphism $\boldsymbol{p}=(p_{\lambda}):(X, A, A')\rightarrow(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ which satisfies the following two conditions:

(R1) Let (P, Q, Q') be an ANR-triad, let $\mathcal{V}$ be an open covering of P and $f:(X, A, A')\rightarrow(P, Q, Q')$ a map of triads. Then there exist a $\lambda\in\Lambda$ and a map of triads $g:(X, A, A')_{\lambda}\rightarrow(P, Q, Q')$ such that the maps gp_{λ} and f are $\mathcal{V}$-near maps.

(R2) Let (P, Q, Q') be an ANR-triad and let $\mathcal{V}$ be an open covering of P. Then there exists an open covering $\mathcal{V}'$ of P such that whenever $\lambda\in\Lambda$ and $g, g':(X, A, A')_{\lambda}\rightarrow(P, Q, Q')$ are maps such that the maps gp_{λ} and $g'p_{\lambda}$ are $\mathcal{V}'$-near, then there exists a $\lambda'\geqq\lambda$ such that the maps $gp_{\lambda\lambda'}$ and $g'p_{\lambda\lambda'}$ are $\mathcal{V}$-near.

If all $(X, A, A')_{\lambda}$, $\lambda\in\Lambda$, are ANR-triads, $\boldsymbol{p}:(X, A, A')\rightarrow(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ is called an ANR-resolution of the triad (X, A, A').

Note that the definition of a resolution of triads given in the present paper differs from the definition given in [3].

In an analogous way one defines resolutions and ANR-resolutions of pairs of spaces $(X, A)\rightarrow(\boldsymbol{X}, \boldsymbol{A})=((X, A)_{\lambda}, p_{\lambda\lambda'}, \Lambda)$ and of single spaces $X\rightarrow\boldsymbol{X}=(X_{\lambda}, p_{\lambda\lambda'}, \Lambda)$

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(see [7], [8], [5], [6]). Note that an *ANR*-pair (X, A) consists of *ANR*'s (for metric spaces) X, A such that A is a closed subset of X.

Resolutions for single spaces were introduced in [5] and [6] (also see [8]) and can be viewed as special inverse limits. K. Morita has recently shown [10] that they coincide with the proper morphisms $X \to \boldsymbol{X}$ introduced in his paper [9]. In [10] Morita also gave an internal characterization of resolutions. Another internal characterization is due to T. Watanabe [11]. Resolutions for pairs were introduced in [8] and studied and characterized in [7].

ANR-resolutions are essentially used in [1] in constructing the Steenrod-Sitnikov homology for arbitrary spaces. In order to prove the excision axiom for this homology theory, we need several facts concerning *ANR*-resolutions of triads. To establish these facts is the main purpose of the present paper. The obtained results, together with results in [4], show that our homology indeed satisfies the excision axiom.

The main result of the paper is Theorem 3, which asserts that every triad of topological spaces admits an *ANR*-resolution. Moreover, the *ANR*-resolution, which we shall construct, will have some additional properties (see (4.1)), needed in establishing the excision axiom.

2. A factorization theorem for maps of triads.

The least cardinal of subsets dense in a space X is called the density of X and will be denoted by $s(X)$. Note that for any map $f: X \to Y$ one has $s(f(X)) \leqq s(X)$. If (X, A, A') is a triad, then

$$s(X) \leqq s(A) + s(A') \leqq \max(s(A),\ s(A'),\ \aleph_0)\,.$$

Moreover, for any metric pair (X, A) one has $s(A) = s(\bar{A}) \leqq s(X)$.

Generalizing Lemma 3 of [7], we will now establish a factorization theorem needed in § 3.

THEOREM 1. *Let $f: (X, A, A') \to (Y, B, B')$ be a map of triads, where (Y, B, B') is an ANR-triad. Then there exists an ANR-triad (Z, C, C') and there exist maps of triads $g: (X, A, A') \to (Z, C, C')$, $h: (Z, C, C') \to (Y, B, B')$ such that $f = hg$ and the following inequalities hold*:

(1) $$s(Z) \leqq s(X),$$

(2) $$s(C) \leqq s(A),$$

(3) $$s(C') \leqq s(A').$$

The proof repeatedly uses the following simple lemma.

LEMMA 1. *Let M be a metric space, P an ANR and $f: M \to P$ a map. Then there exist an ANR N and a map $g: N \to P$ such that M is a closed subset of N, $g|M=f$ and $s(N)=s(M)$. Moreover, if M is finite, then $N=M$.*

PROOF. If M is finite, we put $N=M$ and $g=f$. Now assume that M is infinite. By the Kuratowsky-Wojdisławski embedding theorem (see [8], I, § 3.1, Theorem 2), one can assume that M is embedded in a normed vector space X and is closed in the convex hull L of M. Note that $s(L)=s(M)$ because M is infinite. The map $f: M \to P$ extends to a map $g: N \to P$, where N is an open neighborhood of M in L. Since L is an *AR*, N is an *ANR*. M is closed in N. Moreover, $s(N)=s(M)$, because $M \subseteqq N \subseteqq L$ implies $s(M) \leqq s(N) \leqq s(L)$.

PROOF OF THEOREM 1. Let $\overline{f(A)}$, $\overline{f(A')}$ denote the closures in Y of the sets $f(A)$ and $f(A')$ respectively. Since B and B' are closed sets, we have $\overline{f(A)} \subseteqq B$, $\overline{f(A')} \subseteqq B'$. By Lemma 1, there is an *ANR* D and there is a map $h_0: D \to B \cap B'$ such that $\overline{f(A)} \cap \overline{f(A')}$ is closed in D, $h_0|\overline{f(A)} \cap \overline{f(A')}$ is the inclu sion map and

$$(4) \qquad s(D)=s(\overline{f(A)} \cap \overline{f(A')}) \leqq \min(s(f(A)),\ s(f(A'))) \,.$$

Let E be the metric space obtained from the topological sum $D \sqcup \overline{f(A)}$ by identifying the two copies of $\overline{f(A)} \cap \overline{f(A')}$. Note that D and $\overline{f(A)}$ are closed subsets of E and

$$(5) \qquad s(E) \leqq s(D)+s(f(A)) \,.$$

Since $s(D) \leqq s(f(A))$, we see that $s(D)+s(f(A))=s(f(A))$, whenever $f(A)$ is infinite, and thus

$$(6) \qquad s(E) \leqq s(f(A)) \leqq s(A)$$

(6) also holds if $f(A)$ is finite because then also $\overline{f(A)} \cap \overline{f(A')}$ is finite, $D=\overline{f(A)} \cap \overline{f(A')}$ and $E=f(A)$. Let $h_1: E \to B$ be the only map such that $h_1|D=h_0$ and $h_1|\overline{f(A)}$ is the inclusion map.

By Lemma 1, there is an *ANR* C and there is a map $h_2: C \to B$ such that E is a closed subset of C, h_2 extends h_1 and

$$(7) \qquad s(C)=s(E) \,.$$

Note that $\overline{f(A)}$ and D are closed subsets of C, $h_2|\overline{f(A)}$ is the inclusion map and $h_2|D=h_0$. If $f(A)$ is finite, then $C=E=f(A)$ and $h_2=h_1$.

In the same way we define an *ANR* C' and a map $h_2': C' \to B'$ such that

$\overline{f(A')}$ and D are closed subsets of C', $h_2'|\overline{f(A')}$ is the inclusion map, $h_2'|D=h_0$ and

(8) $$s(C')\leqq s(f(A'))\leqq s(A')\,.$$

Moreover, if $f(A')$ is finite, then $C'=f(A')$.

We now form a new space Z. It is obtained from the topological sum $C\sqcup C'$ by identifying the two copies of D. Note that C and C' are closed in Z, $C\cap C'=D$ and $C\cup C'=Z$. By the sum theorem for ANR's we see that Z is an ANR and therefore (Z, C, C') is an ANR-triad.

We take for $h:Z\to Y$ the unique map such that $h|C=h_2$, $h|C'=h_2'$. Clearly, h is a map of triads $h:(Z, C, C')\to(Y, B, B')$. We define the map $g:X\to Z$ by requiring that

$$g|A=f|A:A\to\overline{f(A)}\subseteqq C\subseteqq Z\,,$$

$$g|A'=f|A':A'\to\overline{f(A')}\subseteqq C'\subseteqq Z\,.$$

Clearly, g is a map of triads $g:(X, A, A')\to(Z, C, C')$ and $hg=f$.

By (6), (7) and (8), we have

(9) $$s(Z)\leqq s(C)+s(C')\leqq s(f(A))+s(f(A'))\,.$$

If at least one of the sets $f(A)$, $f(A')$ is infinite, then $s(f(A))+s(f(A'))=\max(s(f(A)),\ s(f(A')))\leqq s(f(X))\leqq s(X)$, and thus (1) holds. If both sets $f(A)$, $f(A')$ are finite, then $C=f(A)$, $C'=f(A')$ and therefore $Z=f(X)$, which again implies (1).

3. An approximate factorization theorem.

The following approximate factorization theorem will be used in § 4. in the proof of the main theorem (existence of ANR-resolutions).

THEOREM 2. *Let $f:(X, A, A')\to(Y, B, B')$ be a map of triads, let (Y, B, B') be an ANR-triad and let $\mathcal{V}$ be an open covering of Y. Then there exists an ANR-triad (Z, C, C') and there exist maps of triads*

$$g:(X, A, A')\to(Z, C, C')\,,\quad h:(Z, C, C')\to(Y, B, B')$$

such that the maps hg and f are $\mathcal{V}$-near and the following relations hold:

(1) $$s(C)\leqq\max(s(A),\ \aleph_0)\,,\quad s(C')\leqq\max(s(A'),\ \aleph_0)\,,$$

(2) $$s(Z)\leqq\max(s(X),\ \aleph_0)\,,$$

(3) $$g(A)\subseteqq \mathrm{Int}_Z(C)\,,\quad g(A')\subseteqq \mathrm{Int}_Z(C')\,.$$

PROOF. In view of Theorem 1 there is no loss of generality in assuming that

(4) $$s(Y)\leqq s(X),$$

(5) $$s(B)\leqq s(A),\quad s(B')\leqq s(A').$$

We define (Z, C, C') by putting

(6) $$C=((B\cap B')\times I)\cup(B\times 1)\subseteqq Y\times I,$$

(7) $$C'=((B\cap B')\times I)\cup(B'\times 0)\subseteqq Y\times I,$$

where $I=[0, 1]$,

(8) $$Z=C\cup C'\subseteqq Y\times I.$$

Clearly, C, C', $C\cap C'$ and Z are ANR's and $C, C'\subseteqq Z$ are closed subsets, so that (Z, C, C') is an ANR-triad. Moreover,

(8) $$s(C)\leqq s(B\times I)=\max(s(B), \aleph_0)\leqq\max(s(A), \aleph_0),$$

(9) $$s(C')\leqq\max(s(B'), \aleph_0)\leqq\max(s(A'), \aleph_0),$$

(10) $$s(Z)\leqq s(C)+s(C')\leqq\max(s(B), s(B'), \aleph_0)\leqq\max(s(Y), \aleph_0)\leqq\max(s(X), \aleph_0).$$

Let $h:Z\to Y$ be the restriction to $Z\subseteqq Y\times I$ of the first projection $Y\times I\to Y$. Note that h is a map of triads $h:(Z, C, C')\to(Y, B, B')$.

We will also define a map $\phi:(Y, B, B')\to(Z, C, C')$ such that $h\phi$ and the identity 1_Y are $\mathcal{V}$-near maps and

(11) $$\phi(B)\subseteqq \mathrm{Int}_Z(C),\quad \phi(B')\subseteqq \mathrm{Int}_Z(C').$$

To complete the proof, it then suffices to put $g=\phi f:(X, A, A')\to(Z, C, C')$, because $hg=h\phi f$ and f are $\mathcal{V}$-near maps and (3) is a consequence of (11) and

(12) $$g(A)=\phi f(A)\subseteqq\phi(B),\quad g(A')\subseteqq\phi(B').$$

In order to define ϕ we use the following lemma.

LEMMA 2. *Let (B, D) be an ANR-pair and let $\mathcal{U}$ be an open covering of B. Then there exists a map $\varphi:B\to(D\times I)\cup(B\times 0)\subseteqq B\times I$ such that $p\varphi$ and 1_B are $\mathcal{U}$-near maps, where p denotes the first projection $p:B\times I\to B$. Moreover, $\varphi(x)=(x, 1)$ for $x\in D$.*

The map $\phi:(Y, B, B')\to(Z, C, C')$ is constructed as follows. We apply Lemma 2 to the ANR-pair (B, D), where $D=B\cap B'$, and to the open covering $\mathcal{U}=\mathcal{V}|B$. We obtain a map

$$\varphi: B \to \left((B\cap B')\times\left[\frac{1}{2}, 1\right]\right)\cup(B\times 1)\subseteqq C$$

such that

(13) $$\varphi(x)=\left(x, \frac{1}{2}\right), \qquad x\in B\cap B'$$

and the maps $h\varphi$ and 1_B are $\mathcal{V}$-near.

The same lemma, applied to $(B', B\cap B')$ yields a map

$$\varphi': B' \to \left((B\cap B')\times\left[0, \frac{1}{2}\right]\right)\cup(B'\times 0)\subseteqq C'$$

such that

(14) $$\varphi'(x)=\left(x, \frac{1}{2}\right), \qquad x\in B\cap B'.$$

and the maps $h\varphi'$ and $1_{B'}$ are $\mathcal{V}$-near.

Because of (13) and (14), the two maps φ, φ' extend to a unique map $\phi: Y\to Z$, which is a map of triads $\phi:(Y, B, B')\to(Z, C, C')$. Clearly, $h\phi$ and 1_Y are $\mathcal{V}$-near maps. Moreover, $\phi(B)=\varphi(B)\subseteqq \mathrm{Int}_Z(C)$, because

(15) $$\left((B\cap B')\times\left[\frac{1}{2}, 1\right]\right)\cup(B\times 1)\subseteqq \mathrm{Int}_Z(C).$$

Similarly, $\phi(B')\subseteqq \mathrm{Int}_Z(C')$.

In order to prove Lemma 2, we need the following lemma (see [8], I, 6.5. Lemma 4).

LEMMA 3. *Let (B, D) be an ANR-pair and let $\mathcal{U}$ be an open covering of B. Then there exists an open neighborhood V of D in B and a map $k: B\to B$ such that $k|V$ is a retraction $V\to D$ and k is $\mathcal{U}$-near 1_B.*

PROOF OF LEMMA 2. We choose V and k according to Lemma 3. Let $\chi: B\to I$ be a map such that

(16) $$\chi|D=1, \ \chi|B\setminus V=0.$$

We then define $\varphi: B\to B\times I$ by

(17) $$\varphi(x)=(k(x), \chi(x)), \qquad x\in B.$$

If $x\in D$, then $\varphi(x)=(x, 1)$. If $x\in V$, then $\varphi(x)\in D\times I$ and if $x\in B\setminus V$, then $\varphi(x)=(k(x), 0)\in B\times 0$. Consequently, $\varphi(B)\subseteqq(D\times I)\cup(B\times 0)$. Furthermore, 1_B and $p\varphi=k$ are $\mathcal{U}$-near maps.

4. Existence of *ANR*-resolutions of triads.

THEOREM 3. *Every triad of topological spaces (X, A, A') admits an ANR-resolution $\boldsymbol{p}=(p_\lambda):(X, A, A')\to(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ indexed by a cofinite set and such that for every $\lambda\in\Lambda$ one has*

$$(1)\qquad X_\lambda=\mathrm{Int}_{X_\lambda}A_\lambda\cup\mathrm{Int}_{X_\lambda}A'_\lambda .$$

In [7, Theorem 6], it was shown that every pair of spaces admits an ANR-resolution of pairs. Although the present proof proceeds along same general plan, one must take into account the new additional requirements.

We say that two maps of triads $q_1:(X, A, A')\to(Y_1, B_1, B'_1)$, $q_2:(X, A, A')\to(Y_2, B_2, B'_2)$ are equivalent provided there is a homeomorphism $h:(Y_1, B_1, B'_1)\to(Y_2, B_2, B'_2)$ such that

$$hq_1=q_2 .$$

Consider all maps of triads $q:(X, A, A')\to(Y, B, B')$ such that (Y, B, B') is an ANR-triad and

$$(2)\qquad s(Y)\leqq\max(s(X), \aleph_0),$$

$$(3)\qquad s(B)\leqq\max(s(A), \aleph_0),\quad s(B')\leqq\max(s(A'), \aleph_0),$$

$$(4)\qquad q(A)\subseteq\mathrm{Int}_Y(B),\quad q(A')\subseteq\mathrm{Int}_Y(B'),$$

where Int_Y denotes interior with respect to Y. Note that (2) implies that the weight $w(Y)=s(Y)\leqq\max(s(X), \aleph_0)$ and $\mathrm{card}(Y)\leqq 2^{w(Y)}\leqq\max(2^{s(X)}, 2^{\aleph_0})$. Therefore, the equivalence classes of the maps q form a set Γ. We choose for each $\gamma\in\Gamma$ a unique representative $q_\gamma:(X, A, A')\to(Y, B, B')_\gamma$ of the class γ.

Let Δ be the set of all finite subsets of Γ, ordered by inclusion. If $\delta=\{\gamma_1, \cdots, \gamma_n\}\in\Delta$, we define a triad $(X, B, B')_\delta$ by putting

$$(5)\qquad B_\delta=B_{\gamma_1}\times\cdots\times B_{\gamma_n},\quad B'_\delta=B'_{\gamma_1}\times\cdots\times B'_{\gamma_n},$$

$$(6)\qquad Y_\delta=B_\delta\cup B'_\delta\subseteq Y_{\gamma_1}\times\cdots\times Y_{\gamma_n} .$$

Since B_γ, B'_γ are ANR's, which are closed in Y_γ, it follows that B_δ, B'_δ are ANR's closed in Y_δ. Moreover,

$$(7)\qquad B_\delta\cap B'_\delta=(B_{\gamma_1}\cap B'_{\gamma_1})\times\cdots\times(B_{\gamma_n}\cap B'_{\gamma_n})$$

is an ANR, because $B_{\gamma_i}\cap B'_{\gamma_i}$ are ANR's. Therefore, by the sum theorem for ANR's, Y_δ is also an ANR and $(Y, B, B')_\delta$ is an ANR-triad.

If $\delta\leqq\delta'=\{\gamma_1, \cdots, \gamma_n, \cdots, \gamma_m\}$, we define $q_{\delta\delta'}:(Y, B, B')_{\delta'}\to(Y, B, B')_\delta$ as the restriction to $Y_{\delta'}$ of the projection $Y_{\gamma_1}\times\cdots\times Y_{\gamma_n}\times\cdots\times Y_{\gamma_m}\to Y_{\gamma_1}\times\cdots\times Y_{\gamma_n}$. We also define $q_\delta:(X, A, A')\to(Y, B, B')$ as the map

$$q_\delta=q_{\gamma_1}\times\cdots\times q_{\gamma_n}: X\to Y_{\gamma_1}\times\cdots\times Y_{\gamma_n}.$$

Since $q_\delta(A)\subseteq B_{\gamma_1}\times\cdots\times B_{\gamma_n}=B_\delta$ and $q_\delta(A')\subseteq B'_\delta$, we see that $q_\delta(X)\subseteq B_\delta\cup B'_\delta=Y_\delta$. Clearly, $(\boldsymbol{Y}, \boldsymbol{B}, \boldsymbol{B}')=((Y, B, B')_\delta, q_{\delta\delta'}, \Delta)$ is an inverse system of ANR-triads and $\boldsymbol{q}=(q_\delta):(X, A, A')\to(\boldsymbol{Y}, \boldsymbol{B}, \boldsymbol{B}')$ is a morphism.

We will now show that

$$q_\delta(A)\subseteq \mathrm{Int}_{Y_\delta}(B_\delta), \quad q_\delta(A')\subseteq \mathrm{Int}_{Y_\delta}(B'_\delta), \tag{8}$$

so that q_δ also satisfies (4). Indeed, if $\delta=\{\gamma_1, \cdots, \gamma_n\}$, then

$$q_\delta(A)\subseteq q_{\gamma_1}(A)\times\cdots\times q_{\gamma_n}(A)\subseteq \mathrm{Int}_{Y_{\gamma_1}}(B_{\gamma_1})\times\cdots\times \mathrm{Int}_{Y_{\gamma_n}}(B_{\gamma_n}). \tag{9}$$

Clearly, $\mathrm{Int}_{Y_{\gamma_1}}(B_{\gamma_1})\times\cdots\times \mathrm{Int}_{Y_{\gamma_n}}(B_{\gamma_n})$ is an open set of $Y_{\gamma_1}\times\cdots\times Y_{\gamma_n}$, contained in $B_\delta\subseteq Y_\delta$, and therefore it is an open set of Y_δ. Consequently, (9) implies the first of the formulas (8). The second one is established analogously. Note that (8) implies

$$q_\delta(X)\subseteq \mathrm{Int}_{Y_\delta}(B_\delta)\cup \mathrm{Int}_{Y_\delta}(B'_\delta)\subseteq Y_\delta. \tag{10}$$

We now define a new directed set M. Its elements are pairs $\mu=(\delta, U)$, where $\delta\in\Delta$ and U is an open neighborhood of $q_\delta(X)$ in Y_δ contained in $\mathrm{Int}_{Y_\delta}(B_\delta)\cup \mathrm{Int}_{Y_\delta}(B'_\delta)$.

We put $\mu=(\delta, U)\leqq(\delta', U')=\mu'$ provided $\delta\leqq\delta'$ and $q_{\delta\delta'}(U')\subseteq U$. The set M is directed. Indeed, if $\mu_i=(\delta_i, U_i)\in M$, $i=1, 2$, we first choose $\delta\geqq\delta_1, \delta_2$. Note that

$$q_{\delta_i\delta}(q_\delta(X))=q_{\delta_i}(X)\subseteq U_i, \qquad i=1, 2. \tag{11}$$

Therefore, the open set

$$U=(q_{\delta_1\delta})^{-1}(U_1)\cap(q_{\delta_2\delta})^{-1}(U_2)\subseteq Y_\delta \tag{12}$$

satisfies

$$q_\delta(X)\subseteq U, \tag{13}$$

$$q_{\delta_i\delta}(U)\subseteq U_i, \qquad i=1, 2, \tag{14}$$

so that $(\delta_i, U_i)\leqq(\delta, U)$, $i=1, 2$.

For $\mu=(\delta, U)$ we put

$$X_\mu=U, \quad A_\mu=U\cap B_\delta, \quad A'_\mu=U\cap B'_\delta. \tag{15}$$

Note that X_μ, A_μ, A'_μ and $A_\mu\cap A'_\mu$ are ANR's because they are open sets of the ANR's Y_δ, B_δ, B'_δ and $B_\delta\cap B'_\delta$ respectively. Furthermore, A_μ and A'_μ are closed in $X_\mu=U$, because B_μ and B'_μ are closed in Y_μ. Also $A_\mu\cup A'_\mu=X_\mu$, so that $(X, A, A')_\mu$ is an ANR-triad. This triad satisfies (1). Indeed, the set $U\cap \mathrm{Int}_{Y_\delta}B_\delta$

is open in $U=X_\mu$ and is contained in $A_\mu=U\cap B_\delta$. Therefore,

(16) $$U\cap \mathrm{Int}_{Y_\delta}B_\delta\subseteqq \mathrm{Int}_{X_\mu}A_\mu .$$

An analogous formula holds for B'_δ and A'_μ. Consequently,

(17) $$X_\mu=U=U\cap(\mathrm{Int}_{Y_\delta}B_\delta\cup \mathrm{Int}_{Y_\delta}B'_\delta)\subseteqq \mathrm{Int}_{X_\mu}A_\mu\cup \mathrm{Int}_{X_\mu}A'_\mu .$$

We now define maps $r_{\mu\mu'}:(X, A, A')_{\mu'}\to(X, A, A')_\mu$, $\mu\leqq\mu'$, and $r_\mu:(X, A, A')\to(X, A, A')_\mu$ as $q_{\delta\delta'}|U'$ and $q_\delta: X\to q_\delta(X)\subseteqq U=X_\mu$ respectively. Clearly, we obtain an inverse system of ANR-triads $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')=((X, A, A')_\mu, r_{\mu\mu'}, M)$ and a morphism $\boldsymbol{r}=(r_\mu):(X, A, A')\to(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$.

We will now show that $\boldsymbol{r}$ is a resolution. We first establish property (R2). Let (P, Q, Q') be an ANR-triad and let $\mathcal{V}$ be an open covering of P. Let $\mu=(\delta, U)\in M$ and let $g, g':(X, A, A')_\mu\to(P, Q, Q')$ be maps of triads such that gr_μ and $g'r_\mu$ are $\mathcal{V}$-near maps. Since $r_\mu=q_\delta$ and $q_\delta(X)\subseteqq U=X_\mu$, we see that $g|q_\delta(X)$ and $g'|q_\delta(X)$ are $\mathcal{V}$-near maps. Therefore, every point $z\in q_\delta(X)$ admits a $V(z)\in\mathcal{V}$ such that $g(z), g'(z)\in V(z)$. By continuity, there exists an open neighborhood $U(z)$ of z in U such that for any $z'\in U(z)$ the points $g(z'), g'(z')\in V(z)$. Let U' be the union of all $U(z)$, when z ranges over $q_\delta(X)$. Then U' is an open neighborhood of $q_\delta(X)$ in U. Moreover, the maps $g|U'$, $g'|U'$ are $\mathcal{V}$-near. Note that $U'\subseteqq \mathrm{Int}_{Y_\delta}(B_\delta)\cup \mathrm{Int}_{Y_\delta}(B'_\delta)$ because $U'\subseteqq U$. Consequently, $\mu'=(\delta, U')$ belongs to M, $\mu\leqq\mu'$ and the maps $gr_{\mu\mu'}=g|U'$, $g'r_{\mu\mu'}=g'|U'$ are $\mathcal{V}$-near.

We will now establish property (R1). Let $f:(X, A, A')\to(P, Q, Q')$ be a map of triads, let (P, Q, Q') be an ANR-triad and let $\mathcal{V}$ be an open covering of P. It suffices to find an ANR-triad (Y, B, B'), which satisfies (2), (3) and (4), and to find maps of triads $q:(X, A, A')\to(Y, B, B')$, $h:(Y, B, B')\to(P, Q, Q')$ such that q satisfies (4) and the maps hq and f are $\mathcal{V}$-near. In that case q is equivalent to q_γ for some $\gamma\in\Gamma$ and we can assume that $q=q_\gamma$. If we now take any $\mu=(\delta, U)\in M$ such that $\delta=\{\gamma\}$, then $h'=h|U: X_\mu\to P$ is a map such that $h'r_\mu=hq$ is $\mathcal{V}$-near the map f.

That such an ANR-triad (Y, B, B') and such a map q exist follows from Theorem 2.

In order to complete the proof of Theorem 3, we will now replace $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ by a new inverse system $(\boldsymbol{Z}, \boldsymbol{C}, \boldsymbol{C}')$, which is indexed by the set Λ of all finite subsets of M and is therefore cofinite. We choose an increasing function $\varphi:\Lambda\to M$ such that $\varphi(\{\mu\})=\mu$. We then put $(Z, C, C')_\lambda=(X, A, A')_{\varphi(\lambda)}$, $\lambda\in\Lambda$, $s_{\lambda\lambda'}=r_{\varphi(\lambda)\varphi(\lambda')}$, $\lambda\leqq\lambda'$, $s_\lambda=r_{\varphi(\lambda)}$, $\lambda\in\Lambda$. It is easy to see that $\boldsymbol{s}=(s_\lambda):(X, A, A')\to(\boldsymbol{Z}, \boldsymbol{C}, \boldsymbol{C}')$ is a resolution of triads with all the desired properties. This well-known argument is described in more details in the case of pairs in [7].

5. Induced resolutions of pairs.

Let $\boldsymbol{p}=(p_\lambda):(X, A, A')\to(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ be a morphism of a triad into an inverse system of triads. This morphism induces several morphisms of pairs into systems of pairs. In particular, we have the morphisms

$$\boldsymbol{p}_{(X,A)}:(X, A)\to(\boldsymbol{X}, \boldsymbol{A}), \quad \boldsymbol{p}_{(X,A')}:(X, A')\to(\boldsymbol{X}, \boldsymbol{A}')$$

and

$$\boldsymbol{p}_{(X,B)}:(X, B)\to(\boldsymbol{X}, \boldsymbol{B}),$$

where $B=A\cap A'$, $B_\lambda=A_\lambda\cap A'_\lambda$, $(X, B)_\lambda=(X_\lambda, B_\lambda)$ and $(\boldsymbol{X}, \boldsymbol{B})=((X, B)_\lambda, p_{\lambda\lambda'}, \Lambda)$. We also have morphisms $\boldsymbol{p}_{(A,B)}:(A, B)\to(\boldsymbol{A}, \boldsymbol{B})$ and $\boldsymbol{p}_{(A',B)}:(A', B)\to(\boldsymbol{A}', \boldsymbol{B})$, where $(\boldsymbol{A}, \boldsymbol{B})=((A, B)_\lambda, p_{\lambda\lambda'}, \Lambda)$, $(A, B)_\lambda=(A_\lambda, B_\lambda)$.

REMARK 1. If $\boldsymbol{p}$ is a resolution, then so are $\boldsymbol{p}_{(X,A)}$ and $\boldsymbol{p}_{(X,A')}$. To verify properties (R1) and (R2) it suffices to associate with every *ANR*-pair (P, Q) the *ANR*-triad (P, Q, Q'), where $Q'=P$.

By imposing rather mild restrictions on (X, A, A') we can show that the analogous assertion holds also in the case of the induced morphism $\boldsymbol{p}_{(X,B)}$. The argument uses some ideas from a proof presented in [3].

THEOREM 4. *Let $\boldsymbol{p}:(X, A, A')\to(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ be a resolution of triads. If the spaces X, X_λ, $\lambda\in\Lambda$, are normal and the sets A, $A'\subseteq X$ are closed, then the induced morphism $\boldsymbol{p}_{(X,B)}:(X, B)\to(\boldsymbol{X}, \boldsymbol{B})$ is a resolution of pairs.*

COROLLARY 1. *If $\boldsymbol{p}:(X, A, A')\to(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ is an ANR-resolution of triads, X is a normal space and A, $A'\subseteq X$ are closed sets, then $\boldsymbol{p}_{(X,B)}:(X, B)\to(\boldsymbol{X}, \boldsymbol{B})$ is an ANR-resolution of pairs.*

PROOF. First note that the induced morphism $\boldsymbol{p}_X:X\to\boldsymbol{X}$ is a resolution [7]. Therefore, the assertion of Theorem 4 will be proved if we show that $p_{(X,B)}$ satisfies the following condition (B1)** (see [7], Theorem 2):

For every $\lambda\in\Lambda$ and every normal covering $\mathcal{V}$ of X_λ there exists a $\lambda''\geqq\lambda$ such that

$$p_{\lambda\lambda''}(B_{\lambda''})\subseteq \mathrm{St}(p_\lambda(B), \mathcal{V}). \tag{1}$$

In order to verify this condition note that $\overline{p_\lambda(B)}$ is contained in $G=\mathrm{St}(p_\lambda(B), \mathcal{V})$. Therefore, there is an open neighborhood G_0 of $\overline{p_\lambda(B)}$ such that

$$\overline{p_\lambda(B)}\subseteq G_0\subseteq \overline{G}_0\subseteq G. \tag{2}$$

Note that $\mathcal{G}=\{p_\lambda^{-1}(G_0), X\backslash A, X\backslash A'\}$ is an open covering of X, because $B=$

$A\cap A'\subseteq p_\lambda^{-1}(G_0)$. This covering is normal because it is finite and X is a normal space. By a well-known property of resolutions (see [8], I, § 6.2, Theorem 4), there exists a $\lambda'\geq\lambda$ and a normal covering $\mathcal{V}$ of X'_λ such that $(p_{\lambda'})^{-1}(\mathcal{V})$ refines $\mathcal{G}$.

We now put

$$(3)\qquad H=(p_{\lambda\lambda'})^{-1}(G)\,,$$

$$(4)\qquad H_0=(p_{\lambda\lambda'})^{-1}(G_0)\,.$$

Note that

$$(5)\qquad \overline{H}_0\subseteq H\,.$$

Moreover, since

$$(6)\qquad p_{\lambda\lambda'}(\overline{p_{\lambda'}(B)})\subseteq\overline{p_\lambda(B)}\subseteq G\,,$$

we see that

$$(7)\qquad \overline{p_{\lambda'}(B)}\subseteq H\,.$$

Clearly, the sets $\overline{p_{\lambda'}(A)}\backslash H$ and $\overline{p_{\lambda'}(A')}\backslash H$, are closed subsets of $X_{\lambda'}$. We claim that they are disjoint. Assume to the contrary that there exists a point

$$(8)\qquad y\in(\overline{p_{\lambda'}(A)}\backslash H)\cap(\overline{p_{\lambda'}(A')}\backslash H)\,.$$

Let V be a member of $\mathcal{V}$, which contains y. For any open neighborhood W of y, which is contained in V, there exist points $a\in A$, $a'\in A'$ such that

$$(9)\qquad \{p_{\lambda'}(a),\ p_{\lambda'}(a')\}\subseteq W\,.$$

The set

$$(p_{\lambda'})^{-1}(W)\subseteq(p_{\lambda'})^{-1}(V)$$

must be contained in one of the sets $X\backslash A$, $X\backslash A'$ or $p_\lambda^{-1}(G_0)$. It cannot be contained in $X\backslash A$ because $a\in(p_{\lambda'})^{-1}(W)$. Similarly, the point $a'\in(p_{\lambda'})^{-1}(W)$ rules out the set $X\backslash A'$. Hence, we must have

$$(10)\qquad (p_{\lambda'})^{-1}(W)\subseteq p_\lambda^{-1}(G_0)=(p_{\lambda'})^{-1}(H_0)\,.$$

However, (9) and (10) imply

$$(11)\qquad \{p_{\lambda'}(a),\ p_{\lambda'}(a')\}\subseteq H_0\cap W\,.$$

This shows that every sufficiently small open neighborhood W of y intersects H_0 and therefore $y\in\overline{H}_0\subseteq H$, which, however, contradicts (8).

We now choose disjoint open set K, $L\subseteq X_{\lambda'}$, such that

$$(12)\qquad \overline{p_{\lambda'}(A)}\backslash H\subseteq K\,,\quad \overline{p_{\lambda'}(A')}\backslash H\subseteq L\,.$$

We then put

$$K^* = K \cup H, \quad L^* = L \cup H. \tag{13}$$

These are open sets in $X_{\lambda'}$ such that

$$\overline{p_{\lambda'}(A)} \subseteq K^*, \quad \overline{p_{\lambda'}(A')} \subseteq L^*, \tag{14}$$

$$K^* \cap L^* = H. \tag{15}$$

Therefore,

$$\overline{p_{\lambda'}(B)} \subseteq \overline{p_{\lambda'}(A)} \cap \overline{p_{\lambda'}(A')} \subseteq K^* \cap L^* = H. \tag{16}$$

Now consider an open set $K_1^* \subseteq X_{\lambda'}$ such that

$$\overline{p_{\lambda'}(A)} \subseteq K_1^* \subseteq \bar{K}_1^* \subseteq K^*. \tag{17}$$

$\mathcal{W} = \{K^*, X \backslash \bar{K}_1^*\}$ is a normal covering of $X_{\lambda'}$. Therefore, by property (B1)** applied to $\boldsymbol{p}_{(X,A)}$ ([7], Theorem 2), there is a $\lambda'' \geqq \lambda'$ such that

$$p_{\lambda'\lambda''}(A_{\lambda''}) \subseteq \mathrm{St}(p_{\lambda'}(A), \mathcal{W}). \tag{18}$$

However, $\mathrm{St}(p_{\lambda'}(A), \mathcal{W}) = K^*$ so that (18) becomes

$$p_{\lambda'\lambda''}(A_{\lambda''}) \subseteq K^*. \tag{19}$$

Similarly, we argue with A' and L^*. Therefore, we can assume that λ'' also satisfies

$$p_{\lambda'\lambda''}(A'_{\lambda''}) \subseteq L^*. \tag{20}$$

It now follows, by (16), that

$$p_{\lambda\lambda''}(B_{\lambda''}) \subseteq p_{\lambda\lambda'}(K^* \cap L^*) = p_{\lambda\lambda'}(H). \tag{21}$$

Consequently, (3) yields the desired result

$$p_{\lambda\lambda''}(B_{\lambda''}) \subseteq G. \tag{22}$$

In the next theorem we consider the induced morphism $\boldsymbol{p}_{(A,B)}$.

THEOREM 5. *Let $\boldsymbol{p}: (X, A, A') \to (\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ be a resolution of triads. Let the spaces X, X_λ, $\lambda \in \Lambda$, be normal, let the sets $A, A' \subseteq X$ be closed and let the sets $A \subseteq X$ and $A_\lambda \subseteq X_\lambda$, $\lambda \in \Lambda$, be normally embedded. Then the induced morphism $\boldsymbol{p}_{(A,B)}: (A, B) \to (\boldsymbol{A}, \boldsymbol{B})$ is a resolution of pairs.*

COROLLARY 2. *If $\boldsymbol{p}: (X, A, A') \to (\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}')$ is an ANR-resolution of triads, X is a normal space, $A, A' \subseteq X$ are closed sets and A is normally embedded in X, then the induced morphism $\boldsymbol{p}_{(A,B)}: (A, B) \to (\boldsymbol{A}, \boldsymbol{B})$ is an ANR-resolution of pairs.*

We say that $A \subseteq X$ is normally embedded in X (or $\mathcal{P}$-embedded) provided every normal covering $\mathcal{V}$ of A admits a normal covering $\mathcal{U}$ of X such that $\mathcal{U}|A$ refines $\mathcal{V}$.

PROOF OF THEOREM 5. By [7, Theorem 2], it suffices to prove that the induced morphism $\boldsymbol{p}_A: A \to \boldsymbol{A}$ is a resolution and $\boldsymbol{p}_{(A,B)}$ has property (B1)**. Since $\boldsymbol{p}_{(X,A)}$ is a resolution and also $A \subseteq X$ and $A_\lambda \subseteq X_\lambda$, $\lambda \in \Lambda$, are normally embedded, [7, Theorem 3] implies that $\boldsymbol{p}_A$ is a resolution.

In order to establish (B1)** for $\boldsymbol{p}_{(A,B)}$, we apply Theorem 4 and conclude that $\boldsymbol{p}_{(X,B)}$ is a resolution. Therefore, $\boldsymbol{p}_{(X,B)}$ has property (B1)**. Consequently, for any $\lambda \in \Lambda$ and any normal covering $\mathcal{U}$ of X_λ there is a $\lambda'' \geqq \lambda$ such that (1) holds. Now let $\mathcal{V}$ be a normal covering of A_λ. Since A_λ is normally embedded in X_λ, we can choose $\mathcal{U}$ such that $\mathcal{U}|A$ refines $\mathcal{V}$. Then the star $\mathrm{St}_{A_\lambda}(p_\lambda(B), \mathcal{V})$ (star with respect to A_λ) clearly contains $A_\lambda \cap \mathrm{St}(p_\lambda(B), \mathcal{U})$, which, by (1), contains $p_{\lambda\lambda''}(B_{\lambda''})$. This establishes (B1)** for $\boldsymbol{p}_{(A,B)}$.

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