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NEW CRITERIA FOR MEROMORPHIC p-VALENT STARLIKE FUNCTIONS

By

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Abstract. Let $B_n(\alpha)$ be the class of functions of the form

$$f(z)=\frac{a_{-p}}{z^p}+\sum_{k=0}^{\infty} a_k z^k \quad (a_{-p}\neq 0,\ p\in N=\{1, 2, \cdots\})$$

which are regular in the punctured disc $U^*=\{z: 0<|z|<1\}$ and satisfying

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-(p+1)\right\}<-\alpha \quad (n\in N_0=\{0, 1, \cdots\}, \quad |z|<1,\ 0\leqq\alpha<p),$$

where

$$D^n f(z)=\frac{a_{-p}}{z^p}+\sum_{m=1}^{\infty}(p+m)^n a_{m-1}z^{m-1}.$$

It is proved that $B_{n+1}(\alpha)\subset B_n(\alpha)$. Since $B_0(\alpha)$ is the class of meromorphically p-valent starlike functions of order α, all functions in $B_n(\alpha)$ are p-valent starlike. Further a property preserving integrals is considered.

1. Introduction.

Let Σ_p denote the class of functions of the form

$$f(z)=\frac{a_{-p}}{z^p}+\sum_{k=0}^{\infty} a_k z^k \quad (a_{-p}\neq 0,\ p\in N=\{1, 2, \cdots\}) \tag{1.1}$$

which are regular in the punctured disc $U^*=\{z: 0<|z|<1\}$. Define

$$D^0 f(z)=f(z), \tag{1.2}$$

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$$D^1 f(z)=\frac{a_{-p}}{z^p}+(p+1)a_0+(p+2)a_1 z+(p+3)a_2 z^2+\cdots$$

$$=\frac{(z^{p+1}f(z))'}{z^p}. \tag{1.3}$$

$$D^2 f(z)=D(D^1 f(z)), \tag{1.4}$$

and for $n=1, 2, \cdots$,

$$D^n f(z)=D(D^{n-1}f(z))=\frac{a_{-p}}{z^p}+\sum_{m=1}^{\infty}(p+m)^n a_{m-1}z^{m-1}$$

$$=\frac{(z^{p+1}D^{n-1}f(z))'}{z^p}. \tag{1.5}$$

In this paper, we shall show that a function $f(z)$ in Σ_p, which satisfies one of the conditions

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-(p+1)\right\}<-\alpha, \quad (z\in U=\{z:|z|<1\}), \tag{1.6}$$

for some $\alpha(0\leqq\alpha<p)$ and $n\in N_0=\{0, 1, 2, \cdots\}$, is meromorphically p-valent starlike in U^*. More precisely, it is proved that, for the classes $B_n(\alpha)$ of functions in Σ_p satisfying (1.6).

$$B_{n+1}(\alpha)\subset B_n(\alpha) \tag{1.7}$$

holds. Since $B_0(\alpha)$ equals $\Sigma_p^*(\alpha)$ (the class of meromorphically p-valent starlike functions of order α), the starlikeness of members of $B_n(\alpha)$ is a consequence of (1.7). Further for $c>0$, let

$$F(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt. \tag{1.8}$$

It is shown that $F(z)\in B_n(\alpha)$ whenever $f(z)\in B_n(\alpha)$. Some known results of Bajpai [1], Goel and Sohi [2] and Uralegaddi and Somanatha [5] are extended. In [4] Ruscheweyh obtained the new criteria for univalent functions.

2. Properties of the class $B_n(\alpha)$.

In proving our main results (Theorem 1 and Theorem 2 below), we shall need the following lemma due to Jack [3].

LEMMA. *Let $w(z)$ be non-constant regular in $U=\{z:|z|<1\}$, $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at z_0, we have $z_0 w'(z_0)=kw(z_0)$ where k is a real number, $k\geqq 1$.*

THEOREM 1. *$B_{n+1}(\alpha)\subset B_n(\alpha)$ fer each integer $n\in N_0$.*

PROOF. Let $f(z)\in B_{n+1}(\alpha)$. Then

$$\operatorname{Re}\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-(p+1)\right\}<-\alpha, \quad |z|<1. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)}-(p+1)\right\}<-\alpha. \tag{2.2}$$

Define a regular function $w(z)$ in U by

$$\frac{D^{n+1}f(z)}{D^{n}f(z)}-(p+1)=-\frac{p+(2\alpha-p)w(z)}{1+w(z)}. \tag{2.3}$$

Clearly $w(0)=0$. Equation (2.3) may be written as

$$\frac{D^{n+1}f(z)}{D^{n}f(z)}=\frac{1+(2p+1-2\alpha)w(z)}{1+w(z)}. \tag{2.4}$$

Differentiating (2.4) logarithmically and using the identity (easy to verify)

$$z(D^n f(z))'=D^{n+1}f(z)-(p+1)D^n f(z), \tag{2.5}$$

we obtain

$$\frac{\dfrac{D^{n+2}f(z)}{D^{n+1}f(z)}-(p+1)+\alpha}{p-\alpha}=\frac{2zw'(z)}{(1+w(z))(1+(2p+1-2\alpha)w(z))}-\frac{1-w(z)}{1+w(z)}. \tag{2.6}$$

We claim that $|w(z)|<1$ in U. For otherwise (by Jack's lemma) there exists a point z_0 in U such that

$$z_0 w'(z_0)=k\,w(z_0) \tag{2.7}$$

where $|w(z_0)|=1$ and $k\geqq 1$. From (2.6) and (2.7), we obtain

$$\frac{\dfrac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)}-(p+1)+\alpha}{p-\alpha}=\frac{2k\,w(z_0)}{(1+w(z_0))(1+(2p+1-2\alpha)w(z_0))}-\frac{1-w(z_0)}{1+w(z_0)}. \tag{2.8}$$

Thus

$$\operatorname{Re}\left\{\frac{\dfrac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)}-(p+1)+\alpha}{p-\alpha}\right\}\geqq\frac{1}{2(1+p-\alpha)}>0, \tag{2.9}$$

which contradicts (2.1). Hence $|w(z)|<1$ in U and from (2.3) it follows that $f(z)\in B_n(\alpha)$.

THEOREM 2. *Let $f(z)\in\Sigma_p$ satisfy the condition*

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)}-(p+1)\right\}<-\alpha+\frac{p-\alpha}{2(p-\alpha+c)} \quad (z\in U) \tag{2.10}$$

for a given $n\in N_0$ and $c>0$. Then

$$F(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt$$

belongs to $B_n(\alpha)$.

PROOF. From the definition of $F(z)$, we have

$$z(D^nF(z))'=cD^nf(z)-(c+p)D^nF(z) \tag{2.11}$$

and also

$$z(D^nF(z))'=D^{n+1}F(z)-(p+1)D^nF(z). \tag{2.12}$$

Using (2.11) and (2.12) the condition (2.10) may be written as

$$\operatorname{Re}\left\{\frac{\dfrac{D^{n+2}F(z)}{D^{n+1}F(z)}+(c-1)}{1+(c-1)\dfrac{D^nF(z)}{D^{n+1}F(z)}}-(p+1)\right\}<-\alpha+\frac{p-\alpha}{2(p-\alpha+c)}. \tag{2.13}$$

We have to prove that (2.13) implies the inequality

$$\operatorname{Re}\left\{\frac{D^{n+1}F(z)}{D^nF(z)}-(p+1)\right\}<-\alpha. \tag{2.14}$$

Define $w(z)$ in U by

$$\frac{D^{n+1}F(z)}{D^nF(z)}-(p+1)=-\frac{p+(2\alpha-p)w(z)}{1+w(z)}. \tag{2.15}$$

Clearly $w(z)$ is regular and $w(0)=0$. The equation (2.15) may be written as

$$\frac{D^{n+1}F(z)}{D^nF(z)}=\frac{1+(2p+1-2\alpha)w(z)}{1+w(z)}. \tag{2.16}$$

Differentiating (2.16) logarithmically and using (2.5), we obtain

$$\frac{D^{n+2}F(z)}{D^{n+1}F(z)}-\frac{D^{n+1}F(z)}{D^nF(z)}=\frac{2(p-\alpha)zw'(z)}{(1+w(z))(1+(2p+1-2\alpha)w(z))}. \tag{2.17}$$

The above equation may be written as

$$\frac{\dfrac{D^{n+2}F(z)}{D^{n+1}F(z)}+(c-1)}{1+(c-1)\dfrac{D^nF(z)}{D^{n+1}F(z)}}-(p+1)=\frac{D^{n+1}F(z)}{D^nF(z)}-(p+1).$$

$$+\left[\frac{2(p-\alpha)zw'(z)}{(1+w(z))(1+(2p+1-2\alpha)w(z))}\right]\cdot\left[\frac{1}{1+(c-1)\dfrac{D^nF(z)}{D^{n+1}F(z)}}\right].$$

which, by using (2.15) and (2.16), reduces to

$$\frac{\dfrac{D^{n+2}F(z)}{D^{n+1}F(z)}+(c-1)}{1+(c-1)\dfrac{D^nF(z)}{D^{n+1}F(z)}}-(p+1)=-\left[\alpha+(p-\alpha)\frac{1-w(z)}{1+w(z)}\right]$$

$$+\frac{2(p-\alpha)zw'(z)}{(1+w(z))[c+(c+2(p-\alpha))w(z)]}.$$

The remaining part of the proof is similar to that of Theorem 1.

REMARKS. (i) Putting $p=1$, $a_{-1}=1$, $n=0$ and $\alpha=0$ in Theorem 2, we get the result of Goel and Sohi [2, Corollary 1].

(ii) For $p=1$, $a_{-1}=1$, $n=0$, $\alpha=0$ and $c=1$ the above theorem extends a result of Bajpai [1, Theorem 1].

THEOREM 3. *$f(z)\in B_n(\alpha)$ if and only if*

$$F(z)=\frac{1}{z^{1+p}}\int_0^z t^p f(t)dt\in B_{n+1}(\alpha).$$

PROOF. From the definition of $F(z)$ we have

$$D^n(zF'(z))+(1+p)D^nF(z)=D^nf(z).$$

That is,

$$z(D^nF(z))'+(1+p)D^nF(z)=D^nf(z). \tag{2.18}$$

By using the identity (2.5), (2.18) reduces to $D^nf(z)=D^{n+1}F(z)$. Hence $D^{n+1}f(z)=D^{n+2}F(z)$.

Therefore

$$\frac{D^{n+1}f(z)}{D^nf(z)}=\frac{D^{n+2}F(z)}{D^{n+1}F(z)}$$

and the result follows.

REMARK. Putting $p=1$ in the above theorems, we get the results obtained by Uralegaddi and Somanatha [5].

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