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BALANCED MODULES AND RINGS OF LEFT COLOCAL TYPE

By

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Introduction

Let M be a left module over a ring R. Then M can be regarded as a right C-module, where $C=\mathrm{End}(_RM)$ is the endomorphism ring of M. Denoting by $\mathrm{BiEnd}(_RM)$ the endomorphism ring $\mathrm{End}(M_C)$ of the right C-module M, the mapping

$$R\ni r \overset{\rho}{\longmapsto} r_m\in \mathrm{BiEnd}(_RM)$$

is a ring homomorphism, where r_m is the left multiplication of r, i.e., $r_m(x)=rx$ for all $x\in M$. If ρ is surjective, or equivalently every element of $\mathrm{BiEnd}(_RM)$ is a left multiplication of an element of R, then M is said to be *balanced*. If every finitely generated faithful left R-module is balanced, then R is called a *left* QF-1 *ring*. Further, R is said to be a *left balanced ring* if R/I is left QF-1 for every two sided ideal I of R; this condition is equivalent to the condition that every finitely generated left module is balanced. The concept of QF-1 rings was introduced by Thrall [18] as a generalization of quasi-Frobenius rings, and in the same paper he proposed to give an internal characterization of QF-1 rings. At the present time, however, this problem is not solved completely, though partial answers are given in several cases (Camillo and Fuller [1], Dlab and Ringel [2], Fuller [7], Makino [8, 9, 10], Morita [11], Ringel [12] and Tachikawa [17]). On the other hand, the structure of balanced rings was completely determined by Dlab and Ringel [2, 3, 4, 5, 6]. Indeed they proved that an indecomposable ring is left balanced if and only if it is a full matrix ring over either a local uniserial ring or an exceptional ring. We should note that exceptional rings are characterized as local rings of either left or right colocal type with the square zero radical (see Section 1 for the precise definition of rings of left colocal type).

Our aim of this paper is to show that in the case of left serial rings, left QF-1ness, which is much more weaker condition than left balancedness, implies rings of left colocal type (Theorem). For the case of a finite dimensional

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algebra over a field, Tachikawa [17] has already obtained the same result, but our proof seems to be different from his proof. Owing to Theorem, we obtain a solution of Thrall's problem for the case of left serial rings (Characterization of left serial QF-1 rings), cf. [8] as to the case of left serial algebras. It should be recognized that there exists an essential difference, caused by problems concerning division rings, between rings of left colocal type and algebras of left colocal type.

Throughout this paper R is a left artinian ring with identity, N the Jacobson radical of R and all R-modules are finitely generated unitary left modules. It is well-known that the balancedness of modules is Morita invariant. So we assume that R is basic. A module is *uniserial* when it has a unique composition series. A ring R is said to be *left serial* if ${}_RR$ is a direct sum of uniserial modules. Let M be a left R-module. We denote by $|M|$ the composition length of M and write $S^i(M)$ for the i-th socle of M, i.e., $S^i(M)=\{x\in M \mid N^i x =0\}$. Especially $S(M)=S^1(M)$ is the socle of M. Homomorphisms always act from the opposite side to scalars; in particular the left R-module M defines a right C-module, where C is the endomorphism ring of ${}_RM$.

1. In this section we explain the definition of rings of left colocal type and obtain some preliminary results. For a uniserial left R-module L with $|L|=n$, let $D(L)=\mathrm{End}({}_RS(L))$ and

$$D_i(L)=\{\theta\in D(L) \mid \theta \text{ is extendable to an endomorphism of } S^i(L)\}$$

for each $i=1, \cdots, n$. Then we have a descending chain of division rings

$$D(L)=D_1(L)\supset D_2(L)\supset \cdots \supset D_n(L).$$

The following was proved by Tachikawa [15].

(Tachikawa) *If every indecomposable left module over a ring R has simple socle, then the following four conditions are satisfied:*

(A) *R is left serial.*

(B) *For uniserial left R-modules K and L with $|K|=|L|\geqq 2$, if $S^2(K)\cong S^2(L)$, then $K\cong L$.*

(C) *$D_2(L)=D_n(L)$ for any uniserial left R-module L with $|L|=n\geqq 2$.*

(D) *$|eN/eN^2|\leqq 2$ for any primitive idempotent e of R.*

Conversely, every indecomposable left R-module has simple scole if the above four conditions plus the following condition are satisfied:

(E) $\dim_{D_2(L)}D_1(L)=\dim D_1(L)_{D_2(L)}$ *for any uniserial left R-module L with $|L|\geqq 2$, where* $\dim_{D_2(L)}D_1(L)$ *and* $\dim D_1(L)_{D_2(L)}$ *are the dimensions of the left* *vector space* $D_1(L)$ *and right vector space* $D_1(L)$ *over* $D_2(L)$, *respectively.*

If a ring R satisfies the above five conditions (A), (B), (C), (D) and (E), then we shall call R a ring of *left colocal type.*

For left R-modules M and P, let ΣM^P (resp. $\cap M^P$) denote the sum (resp. the intersection) of the images (resp. the kernels) of all homomorphisms from P into M (resp. from M into P) and all endomorphisms of M that are not automorphisms, that is,

$$\Sigma M^P=\Sigma\{\operatorname{Im} f \mid f\in \operatorname{Hom}_R(P, M) \text{ or } f\in \operatorname{End}(_RM)\setminus \operatorname{Aut}(_RM)\}$$

and

$$\cap M^P=\cap\{\operatorname{Ker} g \mid g\in \operatorname{Hom}_R(M, P) \text{ or } g\in \operatorname{End}(_RM)\setminus \operatorname{Aut}(_RM)\}.$$

Now we state the important lemma, which is a slight extension of [10, Lemma 1.4], and the idea of the proof is due to Morita [11].

LEMMA 1.1. *Let M be an indecomposable left R-module and P a left R-module. Let U be a C-submodule of M, where C is the endomorphism ring of $_RM$. Suppose that $M\oplus P$ is balanced and that $\Sigma M^P+U\neq M$. Then, for given elements x in $M\setminus\Sigma M^P+U$ and y in $\cap M^P$, we find an element a of R such that $aP=0$, $a(\Sigma M^P+U)=0$ and $ax=y$.*

PROOF. Let $G=M\oplus P$ and $\mathcal{D}=\operatorname{End}(_RG)$, and let $e: G\to M$ be the projection (regarded as an element of $\mathcal{D}$). Then $(\Sigma M^P+U)\oplus P$ is a $\mathcal{D}$-submodule of G and $X=G/(\Sigma M^P+U)\oplus P$ is a semi-simple $\mathcal{D}$-module with $X=Xe$. Also, letting $i: M\to G$ be the injection, $Y=i(\cap M^P)$ is a semi-simple $\mathcal{D}$-module with $Y=Ye$. Since $\overline{(x, 0)}=(x, 0)+(\Sigma M^P+U)\oplus P$ is a non-zero element of X by hypothesis, it follows that there exists a $\mathcal{D}$-homomorphism $\alpha: X\to Y$ such that $\alpha\overline{(x, 0)}=(y, 0)$. By $a: G\to G$, we denote the composed $\mathcal{D}$-homomorphism

$$G\xrightarrow{\varepsilon} X\xrightarrow{\alpha} Y\xrightarrow{\iota} G,$$

where ε is the canonical epimorphism and ι the inclusion. Since G is balanced, we may consider a an element of R. Then a is a required element, so the proof is completed.

For a subset X of an R-module M, the annihilator $\{a\in R \mid aX=0\}$ of X will be denoted by $\operatorname{Ann}(X)$.

COROLLARY 1.2. *Let M be an indecomposable left R-module and P a left R-module. Suppose that $M\oplus P$ is balanced. Then*

(a) *For any R-submodule H of M, either $H\subset\Sigma M^P$ or $H\supset\cap M^P$ holds.*

(b) *Let X be a subset of M such that* $\mathrm{Ann}(X)\cap\mathrm{Ann}(P)=0$. *Suppose that M is not cogenerated by P. Then*

$$XC+\sum M^{P}=M,$$

where $C=\mathrm{End}(_{R}M)$.

PROOF. (a) Suppose that neither $H\subset\sum M^{P}$ nor $H\supset\cap M^{P}$. Then we can take elements $x\in H$ such that $x\in M\setminus\sum M^{P}$ and $y\in\cap M^{P}$ such that $y\notin H$. From Lemma 1.1, it follows that there exists an element a of R such that $ax=y$, but this contradicts that H is an R-submodule of M.

(b) From the hypothesis $H=\cap\{\mathrm{Ker}\,g\mid g\in\mathrm{Hom}_{R}(M, P)\}$ is a non-zero C-submodule of M. Thus we have $\cap M^{P}=S(M_{C})\cap H\neq 0$. Hence, assuming $XC+\sum M^{P}\neq M$, by Lemma 1.1 we find a non-zero element a of R such that a annihilates both $XC+\sum M^{P}$ and P. But this contradicts that $\mathrm{Ann}(X)\cap\mathrm{Ann}(P)=0$, so (b) holds.

REMARK. Corollary 1.2 (a) has been already shown in [8]. Letting $P=0$, Corollary 1.2 (b) yields [5, Lemma II. 2.1], and from Lemma 1.1 we can immediately obtain a part of Morita's criterion [11, Theorem 1.1].

2. Throughout this section we assume that R is a left serial ring. Let K and L be uniserial left R-modules with $|K|, |L|\geqq 2$. Let $\theta: S^{l}(K)\to S^{l}(L)$, $1\leqq l<\min\{|K|, |L|\}$, be an isomorphism satisfying that the isomorphism $\bar{\theta}$: $S^{l}(K)/S^{l-1}(K)\to S^{l}(L)/S^{l-1}(L)$ induced by θ can not be extended to an isomorphism from $S^{l+1}(K)/S^{l-1}(K)$ into $S^{l+1}(L)/S^{l-1}(L)$. In this case, we shall call $\theta: S^{l}(K)\to S^{l}(L)$ a *tightly lacing isomorphism*, and say that l is a *lacing length* of K and L. It may occur that two different integers are lacing lengths of K and L. We should note the following; For the interlacing module by the tightly lacing isomorphism $\theta: S^{l}(K)\to S^{l}(L)$

$$I=\mathrm{Int}_{\theta}(K, L)=K\oplus L/\{(x, x\theta)\mid x\in S^{l}(K)\},$$

the l-th socle $S^{l}(I)$ is uniserial by [15, Lemma 1.3], because $\theta_{i}: S^{l}(K)/S^{i}(K)\to S^{l}(L)/S^{i}(L)$ induced by θ is not extendable to a homomorphism from $S^{l+1}(K)/S^{i}(K)$ into $L/S^{i}(L)$ for each $i=0, 1, \cdots, l-1$.

By Π let denote the totality of non-isomorphic primitive idempotents and by Λ the subset of Π such that $\oplus_{e_{\lambda}\in\Lambda}Re_{\lambda}$ is minimal faithful, that is, $\oplus_{e_{\lambda}\in\Lambda}Re_{\lambda}$ is faithful and the deletion of any direct summand makes it unfaithful. The meaning of Π and Λ will be retained throughout. One can easily see that every primitive left ideal Re, $e\in\Pi$, is embedded into Re_{λ} for some $e_{\lambda}\in\Lambda$.

The following proposition is an extension of [17, Theorem 3.1] to the case of artinian rings, and our proof is shorter and seems to be different from that in [17].

PROPOSITION 2.1. *Let R be a left serial ring. Suppose that R is left QF-1. Then the largest lacing length l of uniserial left R-modules is at most one.*

PROOF. Let $\theta: S^l(K)\to S^l(L)$ be a tightly lacing isomorphism of uniserial left R-modules K and L. We may assume that $K=Re_\kappa$ for some $e_\kappa\in\Lambda$ by the maximality of l. Construct the interlacing module by θ

$$M=\mathrm{Int}_\theta(Re_\kappa,\ L)=Re_\kappa\oplus L/H,$$

where $H=\{(x,\ x\theta)\,|\,x\in S^l(Re_\kappa)\}$, and let

$$P=(\oplus_{e_\lambda\in\Lambda,\,e_\lambda\neq e_\kappa}Re_\lambda)\oplus Re_\kappa/S(Re_\kappa)\quad\text{and}\quad G=M\oplus P.$$

Then G is faithful since Re_λ is embedded into G for any $e_\lambda\in\Lambda$. It follows that G is balanced by the hypothesis. Letting $e_\kappa u$ be a generator of Re_κ, it is easily seen that $\sum M^P\not\supset R\overline{(e_\kappa u,\ 0)}$. (Here we write $\overline{(x,\ y)}=(x,\ y)+H(x\in Re_\kappa,\ y\in L)$ for an element of M.) Also, considering the fact that $S^l(M)$ is uniserial, we see that $\cap M^P\supset S^l(M)$ $(=S^l(R\overline{(e_\kappa u,\ 0)}))$.

Now suppose that $l\geqq 2$. Then we can take an element $y\in\cap M^P$ such that $y\in R\overline{(e_\kappa u,\ 0)}\setminus S(R\overline{(e_\kappa u,\ 0)})$. Applying Lemma 1.1, for an element $x\in R\overline{(e_\kappa u,\ 0)}$ with $x\notin\sum M^P$ we find an element $a\in R$ such that $ax=y$ and $aP=0$. Then $a(R\overline{(e_\kappa u,\ 0)}/S(R\overline{(e_\kappa u,\ 0)}))\neq 0$, but the element a annihilates the direct summand $Re_\kappa/S(Re_\kappa)$ of P. This contradicts $Re_\kappa/S(Re_\kappa)\cong R\overline{(e_\kappa u,\ 0)}/S(R\overline{(e_\kappa u,\ 0)})$ as R-modules. As a consequence, we have shown that $l<2$, so the proof of the proposition is completed.

The largest lacing length has relation to the conditions for rings of left colocal type as follows (cf. [13, Lemma 3.1]).

LEMMA 2.2. *The largest lacing length l of uniserial left R-modules is at most one if and only if R satisfies both the conditions* (B) *and* (C) *for rings of left colocal type.*

PROOF. Let L be a uniserial left R-module with $|L|=n\geqq 2$. Let $\theta_i: S(L)\to S(L)$, $i=1,\ 2$, be isomorphisms and $\tau_i: S^k(L)\to S^k(L)$ $(k\geqq 2)$ extensions of θ_i respectively. Then we remark that $\theta_1=\theta_2$ if and only if $\bar{\tau}_1=\bar{\tau}_2$, where $\bar{\tau}_i$: $S^k(L)/S^{k-1}(L)\to S^k(L)/S^{k-1}(L)$ is the homomorphism induced by τ_i.

Now assume $l<2$. Then (B) holds trivially. Let $\theta: S(L)\to S(L)$ be any isomorphism belonging to $D_k(L)$ and $\tau: S^k(L)\to S^k(L)$ an extension of θ, where

$k \geqq 2$ (so that $k>l$). Then, since τ is not a tightly lacing isomorphism, there exists an isomorphism $\bar{\rho}: S^{k+1}(L)/S^{k-1}(L) \to S^{k+1}(L)/S^{k-1}(L)$ which is an extension of the isomorphism $\bar{\tau}: S^k(L)/S^{k-1}(L) \to S^k(L)/S^{k-1}(L)$ induced by τ. We can lift $\bar{\rho}$ to $\rho: S^{k+1}(L) \to S^{k+1}(L)$. Taking account of the fact remarked above, we see that the restriction $\rho | S(L): S(L) \to S(L)$ coincides with θ. This implies that $\theta \in D_{k+1}(L)$, thus $D_k(L) = D_{k+1}(L)$. It follows that $D_2(L) = D_n(L)$, so (C) holds.

Conversely, assume that (B) and (C) hold. Let K and L be uniserial modules, and let $\theta: S^k(K) \to S^k(L)$ be an isomorphism, where $2 \leqq k < \min\{|K|, |L|\}$. We show that θ is not a tightly lacing isomorphism. Since $k \geqq 2$, we may assume that $K = L$ by (B). By (C) we have $D_k(L) = D_{k+1}(L)$, and hence there exists an isomorphism $\tau: S^{k+1}(K) \to S^{k+1}(L)$ such that $\theta | S(L) = \tau | S(L)$ for restrictions. Then $\bar{\tau}: S^{k+1}(L)/S^{k-1}(L) \to S^{k+1}(L)/S^{k-1}(L)$ induced by τ is an extension of $\bar{\theta}: S^k(L)/S^{k-1}(L) \to S^k(L)/S^{k-1}(L)$ induced by θ. This shows that θ is not tightly lacing isomorphism, as required.

Combining Proposition 2.1 and Lemma 2.2 we have

PROPOSITION 2.3. *Let R be a left serial ring. If R is left QF-1, then R satisfies the conditions* (B) *and* (C) *for rings of left colocal type.*

3. Our purpose of this section is to prove the following proposition.

PROPOSITION 3.1. *Let R be a left serial ring. If R is left QF-1, then R satisfies the condition* (D) *for rings of left colocal type.*

We need the following lemma, which was shown in [15].

LEMMA 3.2. *Let R be a left serial ring with $N^2=0$ and e, f primitive idempotents such that $eNf \neq 0$. Put $L = Rf$. Then*

$$\dim D_1(L)_{D_2(L)} = \dim eNf_{fRf/fNf}.$$

PROOF. See [15] and the proof of [13, Lemma 3.2].

From now on throughout this section we shall assume that R is a left serial ring satisfying the conditions (B) and (C) for rings of left colocal type, and moreover assume that

$$e_1R/e_1N \oplus e_2R/e_2N \oplus e_3R/e_3N$$

is embedded into eN/eN^2, where e, e_1, e_2, $e_3 \in \Pi$ (so that $|eN/eN^2| \geqq 3$). In order to construct a non-balanced R-module which is faithful we distinguish three cases; and according as the cases, we choose uniserial left R-modules

L_1, L_2, L_3 and isomorphisms $\theta: S(L_1)\to S(L_2)$, $\tau: S(L_1)\to S(L_3)$ as follows.

Case (i) $e_i\not\cong e_j\,(i\neq j)$ for the primitive idempotents e_1, e_2 and e_3. We take uniserial modules L_i, $i=1, 2, 3$, with $S^2(L_i)\cong Re_i/N^2e_i$. Note that $S(L_i)\cong Re/Ne$, $i=1, 2, 3$. Take any isomorphisms $\theta: S(L_1)\to S(L_2)$ and $\tau: S(L_1)\to S(L_3)$. By (B), we may assume that L_1 and L_2 are projective, and thus that $L_1=Re_\kappa$ and $L_2=Re_\mu$ for some e_κ, $e_\mu\in\Lambda$.

Case (ii) $e_1\cong e_2\not\cong e_3$. Let L be a uniserial R-module such that $S^2(L)\cong Re_1/N^2e_1$. Then we have $\dim D_1(L)_{D_2(L)}\geqq 2$ by Lemma 3.2. Since the largest lacing length <2 by Lemma 2.2, it follows that L is projective. Thus we may assume $L=Re_\kappa$ for some $e_\kappa\in\Lambda$. Let $L_1=L_2=L$ and L_3 a uniserial module such that $S^2(L_3)\cong Re_3/N^2e_3$. Let $\theta: S(L)\to S(L)$ be an isomorphism such that 1 and θ are linearly independent in the right vector space $D_1(L)_{D_2(L)}$, and $\tau: S(L_1)\to S(L_3)$ any isomorphism.

Case (iii) $e_1\cong e_2\cong e_3$. Similarly as Case (ii), there exists a uniserial R-module L such that $L=Re_\kappa$ for some $e_\kappa\in\Lambda$ and $S^2(L)\cong Re_1/N^2e_1$. Note that $\dim D_1(L)_{D_2(L)}\geqq 3$ by Lemma 3.2. We let $L_1=L_2=L_3=L$, and $\theta: S(L)\to S(L)$ and $\tau: S(L)\to S(L)$ isomorphisms such that 1, θ and τ are linearly independent in $D_1(L)_{D_2(L)}$.

Now, from L_1, L_2, L_3, θ, τ taken above, make the interlacing module

$$I=\mathrm{Int}_{(\theta,\tau)}(L_1, L_2, L_3)=L_1\oplus L_2\oplus L_3/H,$$

where $H=\{(x, x\theta, x\tau)\mid x\in S(L_1)\}$. We shall write

$$[x_1]_1+[x_2]_2+[x_3]_3$$

for the coset of (x_1, x_2, x_3) modulo H, where $x_i\in L_i$, i.e.,

$$[x_1]_1+[x_2]_2+[x_3]_3=(x_1, x_2, x_3)+H$$

as an element of I.

LEMMA 3.3. *Any homomorphism $\phi: I\to I$ can be lifted to a homomorphism $\Phi: L_1\oplus L_2\oplus L_3\to L_1\oplus L_2\oplus L_3$.*

PROOF. Let f_iu_i, $i=1, 2, 3$, be generators of L_i respectively, where $f_i\in\Pi$. Let

$$[f_iu_i]_i\phi=[f_ix_i]_i+[f_ix_k]_k+[f_ix_l]_l,$$

where $\{i, k, l\}=\{1, 2, 3\}$. Let af_i be an element of Rf_i such that $af_iu_i=0$. Then $af_i\in N^{|L_i|}f_i$, so $af_ix_i=0$. Also we have $af_ix_k=0$ and $af_ix_l=0$, since $0=a[f_iu_i]_i=[af_ix_k]_k+[af_ix_l]_l$ and $R[f_ku_k]_k\oplus R[f_lu_l]_l\,(\subset I)$. Thus there exists a homomorphism $\phi_{ij}: L_i\to L_j$ such that $f_iu_i\phi_{ij}=f_ix_j$, $j=1, 2, 3$. If we let $\Phi: L_1\oplus L_2\oplus L_3\to L_1\oplus L_2\oplus L_3$ be the homomorphism given by the matrix (ϕ_{ij}), then the diagram

$$\begin{array}{ccc} L_1\oplus L_2\oplus L_3 & \xrightarrow{\Phi} & L_1\oplus L_2\oplus L_3 \\ \downarrow & & \downarrow \\ I & \xrightarrow{\phi} & I \end{array}$$

is commutative. So the lemma is shown.

Let P be the direct sum of all Re_λ, $e_\lambda\in\Lambda$, which are isomorphic to no L_i, $i=1, 2, 3$, that is,

$$P=\bigoplus_{e_\lambda\in\Lambda,\, Re_\lambda\not\cong L_1, L_2, L_3} Re_\lambda$$

and put $F=I\oplus P$. Proposition 3.1 will be proved if we show the following proposition.

PROPOSITION 3.4. *F is faithful and non-balanced.*

PROOF. Obviously Re_λ is embedded into F for any $e_\lambda\in\Lambda$. Thus F is faithful. It was shown by Tachikawa [15] that the module I is indecomposable in each case, and this fact is essential for proving non-balancedness of F.

Now let ϕ be any element of the radical $\mathcal{W}$ of the endomorphism ring C of ${}_RI$. According to Lemma 3.3, ϕ can be lifted to $\Phi: L_1\oplus L_2\oplus L_3\to L_1\oplus L_2\oplus L_3$. We write Φ as a matrix (ϕ_{ij}), where $\phi_{ij}: L_i\to L_j$. We show that if all ϕ_{ij}, $1\leqq i, j\leqq 3$, are non-isomorphisms, then F is non-balanced. To see this, assume that ϕ_{ij} are non-isomorphisms for any $\phi\in\mathcal{W}$. Then

$$I\mathcal{W}\subset N[f_1u_1]_1+N[f_2u_2]_2+R[f_3u_3]_3,$$

where f_iu_i are generators of L_i respectively, and

$$S(I_C)\supset R[ew_1]_1\oplus R[ew_2]_2 (=R[ew_2]_2\oplus R[ew_3]_3 =R[ew_3]_3\oplus R[ew_1]_1),$$

where ew_i are generators of $S(L_i)$ respectively. Since L_i, $i=1, 2$, are not generated by P, we have

$$X=\sum\{\mathrm{Im} f\mid f\in\mathrm{Hom}_R(P, I)\}\subset N[f_1u_1]_1+N[f_2u_2]_2+R[f_3u_3]_3.$$

Also we have

$$Y=\cap\{\mathrm{Ker}\, g\mid g\in\mathrm{Hom}_R(I, P)\}\supset R[ew_1]_1\oplus R[ew_2]_2,$$

since L_i, $i=1, 2$, are not embedded into P. Therefore we conclude that

$$\sum I^P = I\mathcal{W} + X \subset N[f_1u_1]_1 + N[f_2u_2]_2 + R[f_3u_3]_3$$

and

$$\cap I^P = S(I_C) \cap Y \supset R[ew_1]_1 \oplus R[ew_2]_2.$$

From this, it follows that $R[f_1u_1]_1$ is a submodule of I satisfying $R[f_1u_1]_1 \not\subset \sum I^P$ and $R[f_1u_1]_1 \not\supset \cap I^P$. By Corollary 1.2 (a), this implies that F is not balanced. So we have shown that if the assumption holds, then F is non-balanced. To complete the proof, we distinguish the above three Cases (i), (ii) and (iii).

Case (i) For $\phi \in \mathcal{W}$, suppose that ϕ_{11} is an isomorphism. Then it can be easily seen that if x_i, $i=1, 2, 3$, are elements of L_i such that $|Rx_1| > |Rx_2|$, $|Rx_3|$, then the image of $[x_1]_1 + [x_2]_2 + [x_3]_3$ by ϕ is written as

$$([x_1]_1 + [x_2]_2 + [x_3]_3)\phi = [y_1]_1 + [y_2]_2 + [y_3]_3$$

with $|Rx_1| = |Ry_1| > |Ry_2|$, $|Ry_3|$. It follows that $[f_1u_1]_1\phi^n \neq 0$ for all $n > 0$, a contradiction of the nilpotency of ϕ. Thus ϕ_{11} is a non-isomorphism, and similarly so are ϕ_{22} and ϕ_{33}. Since L_i, $i=1, 2, 3$, are mutually non-isomorphic, all ϕ_{ij}, $1 \leqq i, j \leqq 3$, are non-isomorphisms. Therefore, by what we have shown above, F is non-balanced.

Case (ii) In this case there are monomorphisms from L_3 into neither L_1 nor L_2. Thus, for $\phi \in \mathcal{W}$, similarly as Case (i) we see that ϕ_{33} is a non-isomorphism. Hence ϕ_{ij} is a non-isomorphism if $i=3$ or $j=3$. Taking this into account, we have

$$0 = ([x]_1 + [x\theta]_2 + [x\tau]_3)\phi = [x\phi_{11} + x\theta\phi_{21}]_1 + [x\phi_{12} + x\theta\phi_{22}]_2$$

for all $x \in S(L)$ $(L = L_1 = L_2)$ since $[x]_1 + [x\theta]_2 + [x\tau]_3 = 0$. Thus

$$x\phi_{11} + x\theta\phi_{21} = x\phi_{12} + x\theta\phi_{22} = 0$$

for all $x \in S(L)$, and hence

$$\phi_{11}^* + \theta\phi_{21}^* = \phi_{12}^* + \theta\phi_{22}^* = 0$$

as elements of $D_1(L)$, where $(\cdot)^*$ denotes the element of $D_1(L)$ obtained by the restriction of $(\cdot)$ to $S(L)$. Since $\phi_{ij}^* \in D_2(L)$, $i, j=1, 2$, and $1, \theta$ are linearly independent in $D_1(L)_{D_2(L)}$, it follows that $\phi_{ij}^* = 0$ so that ϕ_{ij}, $i, j=1, 2$, are non-isomorphisms. As a consequence, all ϕ_{ij}, $1 \leqq i, j \leqq 3$, are non-isomorphisms. From this, we conclude that F is non-balanced.

Case (iii) For the proof in this case we refer to that of [5, Lemma II. 2.2]. Suppose that F is balanced. Then, by the fact shown above, there exists an element $\phi \in \mathcal{W}$ such that $\phi_{ij}: L \to L$ is an isomorphism for some i, j, $1 \leqq i, j \leqq 3$. Then it is obvious that $[f_iu_i]_i\phi$, $[f_ku_k]_k$ and $[f_lu_l]_l$ generate I, where k, l are integers such that $\{j, k, l\} = \{1, 2, 3\}$ (in fact $f_i = f_k = f_l = e_\kappa$). Put $m=$

$[f_i u_i]_i$. If $Rm\phi \not\cong Re_\kappa$, then by a length argument we have that $Rm\phi$ is a direct summand of I, a contradiction. Thus $Rm\phi \cong Re_\kappa$. From this together with the choice of P, we have $\mathrm{Ann}(Rm\phi) \cap \mathrm{Ann}(P)=0$. It is easily seen that I is not cogenerated by P. Considering the assumption that F is balanced, we have

$$Rm\phi\mathcal{C}+\sum I^P=I$$

by Corollary 1.2 (b). Since $\phi \in \mathcal{W}$, it follows $\sum I^P=I$. But this does not occur by the choice of P. As a consequence, we get a contradiction. The proof of the proposition is completed.

4. To fulfill our aim, we have only to show the following proposition.

PROPOSITION 4.1. *Let R be a left serial ring. Suppose that R is left QF-1. Then R satisfies the condition* (E) *for rings of left colocal type.*

PROOF. We refer the proof of [5, Proposition II. 2.4]. By Propositions 2.3 and 3.1, R satisfies the conditions (B), (C) and (D). Let L be any uniserial left R-module with $|L|=n \geqq 2$. Put $D=D_1(L)$ and $D^*=D_2(L)$. Then $\dim D_{D^*} \leqq 2$ by (D). We want to show that $\dim D_{D^*}=\dim_{D^*}D$. This is trivial in case $\dim D_{D^*}=1$. So assume that $\dim D_{D^*}=2$. Then, by Proposition 2.1, we may assume that $L=Re_\kappa$ for some $e_\kappa \in \Lambda$.

Now take an element $\theta \in D \setminus D^*$, and make the interlacing module by θ: $S(L) \to S(L)$

$$M=\mathrm{Int}_\theta(L,\ L)=L \oplus L/H,$$

where $H=\{(x,\ x\theta) \mid x \in S(L)\}$. Let

$$P=\oplus_{e_\lambda \in \Pi,\, e_\lambda \neq e_\kappa} Re_\lambda$$

and put $G=M \oplus P$. Then G is faithful, and thus G is balanced by the hypothesis that R is left QF-1. This fact is needed below.

Let $e_\kappa u$ be a generator of L, and consider the element $\overline{(0,\ e_\kappa u)}$ of M, where we write $\overline{(x,\ y)}=(x,\ y)+H$ $(x,\ y \in L)$ for an element of M. Then $\mathrm{Ann}(\overline{(0,\ e_\kappa u)}) \cap \mathrm{Ann}(P)=0$; and, as easily seen, M is not cogenerated by P. Thus, applying Corollary 1.2 (b) to the module $G=M \oplus P$, we obtain

$$(*) \qquad \overline{(0,\ e_\kappa u)}\mathcal{C}+\sum M^P=M,$$

where $\mathcal{C}=\mathrm{End}({}_R M)$. It should be noted that $N^{n-1}M\mathcal{W}=0$ where $\mathcal{W}$ is the radical of $\mathcal{C}$, and thus we have $N^{n-1}\sum M^P=0$ by the choice of P.

Now let ξ be any element of D. Since $\{1,\ \theta\}$ is a basis of D_{D^*}, we have

$\xi=\rho^*+\theta\sigma^*$ for some $\rho^*, \sigma^*\in D^*$. We extend σ^* to a homomorphism $\sigma: L\to L$. For the element $\overline{(0, e_\kappa u\sigma)}$ of M, by (*) there exist $\phi\in C$ and $w\in\sum M^P$ such that

$$\overline{(0, e_\kappa u\sigma)}=\overline{(0, e_\kappa u)}\phi+w\,.$$

Let x be any element of $S(L)$, and take an element $b\in N^{n-1}$ such that $be_\kappa u=x$. Then we have

$$\begin{aligned}\overline{(0, x\sigma)}&=\overline{(0, be_\kappa u\sigma)}=b\overline{(0, e_\kappa u\sigma)}\\&=b\overline{(0, e_\kappa u)}\phi+bw=\overline{(0, x)}\phi+bw\,.\end{aligned}$$

Since $bw=0$, we have

(⁑) $$\overline{(0, x\sigma)}=\overline{(0, x)}\phi \qquad \text{for any} \quad x\in S(L).$$

Now we lift ϕ to a homomorphism $\Phi: L\oplus L\to L\oplus L$, and write Φ as a matrix (ϕ_{ij}), where $\phi_{ij}: L\to L$ is a homomorphism. Then, noting (⁑) we get

$$\overline{(0, x\theta\sigma)}=\overline{(0, x\theta)}\phi=\overline{(-x, 0)}\phi=\overline{(-x\phi_{11}, -x\phi_{12})}$$

for $x\in S(L)$, and thus

$$\overline{(x\phi_{11}, x\theta\sigma+x\phi_{12})}=0\,.$$

It follows that

$$x\phi_{11}\theta=x\theta\sigma+x\phi_{12} \qquad \text{for all} \quad x\in S(L),$$

and this implies that

$$\phi_{11}{}^*\theta=\theta\sigma^*+\phi_{12}{}^*$$

as elements of D, where $(\cdot)^*$ is the restriction of $(\cdot)$ to $S(L)$. Hence ξ is expressed as

$$\xi=(\rho^*-\phi_{12}{}^*)+\phi_{11}{}^*\theta\,,$$

and this implies that $\{1, \theta\}$ is also a basis of the left vector space ${}_{D^*}D$. So $\dim_{D^*}D=\dim D_{D^*}$, as required. The proof of Proposition 4.1 is completed.

As a consequence, we have shown

THEOREM. *Let R be a left serial ring. If R is left QF-1, then R is a ring of left colocal type.*

It is known that as to the structure of indecomposable left modules, rings of left colocal type does not differ from algebras of left colocal type (cf. Tachikawa [15] and Sumioka [13]). For example, an indecomposable left module over a ring of left colocal type is either uniserial or a module written as $K\triangledown L$. Here, for submodules K and L of a module M, we denote the sum $K+L$ by $K\triangledown L$ in case (1) K and L are uniserial modules with $|K|, |L|\geqq 2$, and (2) $K\cap$

$L=S(K+L)$. It is not difficult to see that the argument in [8] is valid for artinian rings. So, by Theorem, we know that the characterization of left serial QF-1 algebras stated in the introduction of [8] is true for artinian rings, that is, we have

(CHARACTERIZATION OF LEFT SERIAL QF-1 RINGS) *Let R be a left serial ring. Then, in order that R may be left QF-1, it is necessary and sufficient that the following conditions are satisfied:*

(1) *R is a ring of left colocal type (defined in Section 1).*

(2) *If Ne is projective, then every composition factor of Re is isomorphic to the socle of Re_λ for some $e_\lambda \in \Lambda$, where $e \in \Pi$.*

(3) *For an indecomposable left R-module $K \nabla L$, every composition factor of $K \nabla L$ is isomorphic to Re_λ / Ne_λ for some $e_\lambda \in \Lambda$.*

(4) *For an indecomposable left R-module $K \nabla L$ such that K is projective, $L/S(L)$ is embedded into Re_λ for some $e_\lambda \in \Lambda$.*

REMARKS. (1) The module F constructed in Section 3 is minimal faithful. So, assuming that the largest lacing length <2, in Proposition 3.1 "QF-1" can be replaced by "weakly QF-1". Here, weakly QF-1 rings are those each of whose minimal faithful modules is balanced (cf. [8]).

(2) By the condition (C), a ring of left colocal type is necessarily right artinian. Thus, from Theorem we know that for a left serial ring R if R is left QF-1, then R is right artinian. Sumioka [13, 14] showed that for a ring R of left colocal type, every indecomposable right R-module is constructed by the dual method to the case of left modules. Thus, using the above characterization of left serial QF-1 rings and arguing dually to [8], we can show without difficulty that if a ring of left colocal type is left QF-1, then it is also right QF-1. So our Theorem yields the result that left QF-1ness implies right QF-1ness for left serial rings. In general, the following questions arise.

QUESTIONS: (1) *If a left artinian ring is QF-1, then is it also right artinian?*
(2) *For artinian rings, is QF-1ness left-right symmetry?*

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