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WEAKLY DIVISIBLE AND DIVISIBLE MODULES

(Dedicated to Prof. K. Murata for his sixtieth birthday)

By

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1. Introduction.

Recently in his paper [5], M. Sato introduced the concept of weakly divisible modules and called a preradical t pseudo-cohereditary if every weakly divisible module is divisible. On the other hand, J. Jirásko [2] called a preradical t pseudo-cohereditary if every homomorphic image of $M/(M\cap t(E(M)))$ is torsionfree for each module M, where $E(M)$ denotes the injective hull of M. Now we call an exact squence $0\rightarrow A\xrightarrow{j}B$ of modules coindependent (resp. weakly coindependent) if $B=j(A)+t(B)$ (resp. $B/(j(A)+t(B))$ is torsion). By means of these notions, we give, in the first half of this paper, some characterizations of weakly divisible and divisible modules. In the latter half, we inquire into relations between two pseudo-cohereditarities in the sense of Sato and Jirásko.

2. Weakly divisible modules and coindependent sequences.

Throughout this note R means a ring with identity and modules mean unitary left R-modules. We denote the category of left R-modules by R-mod. A subfunctor t of the identity functor of R-mod is called a *preradical* of R-mod. It is called *idempotent* if $t(t(M))=t(M)$ and a *radical* if $t(M/t(M))=0$ for all modules M. Also, it is called *left exact* if $t(N)=t(M)\cap N$ and *cohereditary* if $t(M/N)=(t(M)+N)/N$ for all modules M and submodules N. To each preradical t of R-mod, we put

$$T(t)=\{M\in R\text{-mod}\,|\,t(M)=M\}\quad\text{and}\quad F(t)=\{M\in R\text{-mod}\,|\,t(M)=0\}.$$

In general, $T(t)$ is closed under homomorphic images and direct sums, while $F(t)$ is closed under submodules and direct products.

DEFINITION 2.1. For a preradical t, a module M is called *weakly divisible* (resp. *divisible*) with respect to t if the functor $\mathrm{Hom}_R(-, M)$ preserves the ex-

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actness of all exact sequences of modules $0\to A\to B\to C\to 0$ with $B\in T(t)$ (resp. $C\in T(t)$). When there is no confusion, we simply say that M is *weakly divisible* (resp. *divisible*).

Clearly every injective module is divisible and every divisible module is weakly divisible, but in general, the converses are not true.

From this definition we have

LEMMA 2.2. *Let t be a preradical and let M be a module and N its submodule. Then*

(1) *If M is weakly divisible and $t(M)\subseteqq N$, then N is also weakly divisible.*

(2) *If t is idempotent, N is essential in M and is weakly divisible, then $t(M)\subseteqq N$.*

PROOF. (1) is easy. (2). Since $t(N)\subseteqq N\cap t(M)\subseteqq N$ and N is weakly divisible, $N\cap t(M)$ is also weakly divisible by (1). Hence the exact sequence $0\to N\cap t(M)\to t(M)$ splits. There exists a submodule L of $t(M)$ such that $t(M)=(N\cap t(M))\oplus L$. However $0=N\cap t(M)\cap L=N\cap L$ and so L must be zero by assumption. Thus we have $t(M)=N\cap t(M)$ and $t(M)\subseteqq N$.

From this lemma it follows that, in case t is idempotent, if M is weakly divisible, then $t(M)$ is also weakly divisible and M is weakly divisible if and only if $t(M)=t(E(M))$ (cf. [5, Theorem 2.1] and [6, Lemma 1.3]).

DEFINITION 2.3. Let t be a preradical. An exact sequence $0\to A\xrightarrow{j} B$ of modules is called *coindependent* (resp. *weakly coindependent*) with respect to t if $B=j(A)+t(B)$ (resp. $B/(j(A)+t(B))\in T(t)$).

We note that for a preradical t any exact sequence $0\to A\xrightarrow{j} B$ with *Coker* $(j)\in T(t)$ is weakly coindependent, however the converse is not true in general.

EXAMPLE 2.4. Let K be a field, R the ring of all 3×3 upper triangular matrices over K. If we put $I=\begin{pmatrix} 0 & K & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $J=\begin{pmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then both I and J are two-sided ideals in R and $I^2=0$. Let t be the left exact preradical corresponding to the left linear topology having the smallest element I. Then, since $I\subseteqq t(R)$, $R/t(R)\in T(t)$ and $0\to J\to R$ is weakly coindependent, but $R/J\notin T(t)$ because $J\subsetneqq I$.

We also note that, in case t is an idempotent radical, the converse is always true and weakly coindependent sequences are exactly those exact sequences $0\to A\xrightarrow{j} B$ with $Coker(j)\in T(t)$.

THEOREM 2.5. *Let t be an idempotent preradical. Then the following conditions are equivalent for a module M:*

(1) *M is weakly divisible(resp. divisible).*

(2) *For every coindependent (resp. weakly coindependent) sequence $0 \to A \xrightarrow{j} B$ and every homomorphism $f: A \to M$, there exists a homomorphism $g: B \to M$ such that $g \circ j = f$.*

(3) *For every coindependent (resp. weakly coindependent) sequence $0 \to A \xrightarrow{j} B$ with $j(A)$ essential in B and every homomorphism $f: A \to M$, there exists a homomorphism $g: B \to M$ such that $g \circ j = f$.*

(4) *Every coindependent sequence $0 \to M \xrightarrow{j} N$ splits.*

(5) *Every coindependent sequence $0 \to M \xrightarrow{j} N$ with $j(M)$ essential in N splits.*

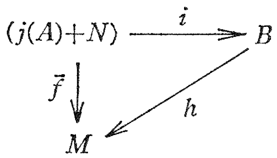
PROOF. (1)$\Rightarrow$(2). Let $0 \to A \xrightarrow{j} B$ be weakly coindependent and $f: A \to M$ a homomorphism. Then the diagram

$$\begin{array}{ccc} j^{-1}(t(B)) & \xrightarrow{j'} & t(B) \\ i \downarrow & & \downarrow k \\ A & \xrightarrow[j]{} & (j(A)+t(B)) \end{array}$$

is a pushout diagram, where i and k are inclusion maps and j' is a restriction map of j. Since M is weakly divisible, we get a commutative diagram

$$\begin{array}{ccc} j^{-1}(t(B)) & \xrightarrow{j'} & t(B) \\ i \downarrow & & \downarrow f' \\ A & \xrightarrow[f]{} & M \end{array}$$

and therefore there exists a (unique) homomorphism $g': (j(A)+t(B)) \to M$ such that $f = g' \circ j$. Further, in case $0 \to A \xrightarrow{j} B$ is coindependent $j(A)+t(B) = B$ by the definition and in case M is divisible g' can be extended to a homomorphism $g: B \to M$, for $B/(j(A)+t(B)) \in T(t)$. (2)$\Rightarrow$(3) is obvious. (3)$\Rightarrow$(1). Let $0 \to A \xrightarrow{j} B$ be an exact sequence with $B \in T(t)$ (resp. $B/j(A) \in T(t)$) and $f: A \to M$ a homomorphism. Then there exists a submodule N of B such that $j(A)\cap N=0$ and $(j(A)+N)$ is essential in B. Since the sequence $0\to(j(A)+N)\xrightarrow{i}B$ is coindependent (resp. weakly coindependent), where i means the inclusion map, and $(j(A)+N)$ is essential in B, there exists a homomorphism $h: B\to M$ making the diagram



commutative, where $\bar{f}(j(a)+x)=f(a)$ for $a\in A$ and $x\in N$. Especially, $\bar{f}(j(a))=f(a)$ for $a\in A$ and hence h is the desired map. (2)⇒(4) and (4)⇒(5) are obvious. (5)⇒(1). Clearly the sequence $0\to M\to(M+t(E(M)))$ is coindependent and M is essential in $(M+t(E(M)))$. So M is a direct summand of $(M+t(E(M)))$ by assumption, namely, $M=M+t(E(M))$. Therefore M is weakly divisible by Lemma 2.2.

3. Pseudo-cohereditary preradicals.

DEFINITION 3.1. Let t be a preradical. We call t *pseudo-cohereditary in the sense of Sato* if every weakly divisible module is divisible.

DEFINITION 3.2. Let t be a preradical. t is called *pseudo-cohereditary in the sense of Jirásko* if every homomorphic image of $M/(t(E(M))\cap M)$ is in $F(t)$ for every module M.

First we investigate relations between these two pseudo-cohereditarities.

THEOREM 3.3. *For an idempotent preradical t, the following conditions are equivalent:*

(1) *t is pseudo-cohereditary in the sense of Sato.*

(2) *For any weakly divisible module M, every homomorphic image of $M/t(M)$ is in $F(t)$.*

(3) *t is pseudo-cohereditary in the sense of Jirásko.*

PROOF. (1)⇒(2). Assume (1) and let M be a weakly divisible module and N any submodule of M containing $t(M)$. Put $t(M/N)=M'/N$, where M' is a submodule of M containing N. Since N and M' are weakly divisible by Lemma 2.2, these are divisible by assumption. Hence N is a direct summand of M', i.e.

there exists a submodule K in $T(t)$ such that $M'=N\oplus K$. Since $t(M')=t(N)+K$ and $t(M')=t(N)$, K must be zero and hence $M'=N$. Thus M/N is in $F(t)$.

(2)$\Rightarrow$(3). Assume (2) and let N be any module. Then $M=N+t(E(N))$ is weakly divisible and $N/(t(E(N))\cap N)\cong(N+t(E(N)))/t(E(N))=M/t(M)$. So by assumption t is pseudo-cohereditary in the sense of Jirásko. (3)$\Rightarrow$(1). Assume (3) and M be a weakly divisible module. Then $t(M)=t(E(M))$ and so $E(M)/M$ is a homomorphic image of $E(M)/t(M)=E(M)/(t(E(M))\cap E(M))$. Hence by assumption $E(M)/M$ is in $F(t)$, which means that M is divisible.

For other characterizations of pseudo-cohereditary sub-torsion theories see [5, Theorem 4.2] and [6, Theorem 1.7 and 1.8].

For a preradical t, we now consider the following condition:

(*) *Every weakly coindependent sequence is coindependent.*

An idempotent cohereditary preradical is called a *cotorsion radical*. Beachy [1] has given some characterizations of this radical.

THEOREM 3.4. *Let t be an idempotent preradical. Then t satisfies the condition (*) if and only if t is a cotorsion radical.*

PROOF. Suppose that t satisfies the condition (*) and let M be a module and N any submodule of M. We put $t(M/N)=L/N$ for some submodule L of M. Then $(t(M)+N)\subseteq L$ and $0\to(t(M)+N)\to L$ is a weakly coindependent sequence. Hence $L=t(M)+N$ by assumption and thus t is cohereditary. Conversely suppose that t is a cotorsion radical. Let $0\to A\xrightarrow{j}B$ be a weakly coindependent sequence. Then $B/t(B)$ is in $F(t)$ and $B/(j(A)+t(B))$ is a homomorphic image of $B/t(B)$. Hence $B/(j(A)+t(B))$ is also in $F(t)$ and so $B=j(A)+t(B)$. Thus t satisfies the condition (*).

Combining this theorem with Theorem 3.3, we see that an idempotent preradical satisfying the condition (*) is pseudo-cohereditary in the sense of both Sato and Jirásko. However we have

COROLLARY 3.5. *For a left exact preradical t, the following conditions are equivalent:*

(1) *t is an exact radical.*

(2) *t is pseudo-cohereditary in the sense of Sato.*

(3) *t is pseudo-cohereditary in the sense of Jirásko.*

(4) *t satisfies the condition (*).*

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