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3-DIMENSIONAL ISOTROPIC SUBMANIFOLDS OF SPHERES

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1. Introduction

In this paper, we study 3-dimensional isotropic submanifolds in spheres. The notion of an isotropic submanifold of an arbitrary Riemannian manifold was first introduced by B. O'Neill in [O]$_1$. The basic equations for isotropic submanifolds are recalled in Section 2.

Isotropic immersions of submanifolds into spheres have been studied by, amongst others, T. Itoh, H. Nakagawa, K. Ogiue and K. Sakamoto in [I], [N-I], [I-O] and [S]. Here, we will prove the two following theorems.

THEOREM 3.1. *Let $x: M\to S^n$ be a constant isotropic immersion such that* $\dim(\mathrm{im}(h))\leqq 3$. *Then, one of the following holds:*

(a) *M is totally geodesic in S^n,*
(b) *There exists a totally geodesic S^4 in S^n, such that the image of M is an open part of a small hypersphere of S^4,*
(c) *There exists a totally geodesic S^7 in S^n, such that the image of M is congruent with an open part of $j\left(\boldsymbol{R}\times S^2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\right)$ in S^7, where j is defined in Section* 3.

THEOREM 3.2. *Let M be a* 3*-dimensional, minimal, isotropic submanifold in S^n. Then, M has constant sectional curvature.*

2. Preliminaries

In this section M will always denote a 3-dimensional totally real submanifold of $S^n(1)$. We will denote the curvature tensor of M by R. The formulas of Gauss and Weingarten are given by

$$(2.1)\qquad D_XY=\nabla_XY+h(X, Y)\quad\text{and}\quad D_X\zeta=-A_\zeta X+\nabla_X^\perp\zeta,$$

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where X and Y are tangent vector fields and ζ is a normal vector field on M. The space spanned by the image of h, will be called the first normal space. The equations of Gauss, Codazzi and Ricci for a submanifold of $S^n(1)$ are given by

(2.2) $$R(X, Y)Z=\langle Y, Z\rangle X-\langle X, Z\rangle Y+A_{h(Y,Z)}X-A_{h(X,Z)}Y,$$

(2.3) $$(\nabla h)(X, Y, Z)=(\nabla h)(Y, X, Z),$$

(2.4) $$\langle R^{\perp}(X, Y)\zeta, \eta\rangle=\langle[A_{\zeta}, A_{\eta}]X, Y\rangle,$$

for tangent (resp. normal) vector fields X, Y and Z (resp. ζ and η) and $R^{\perp}$ (resp. $\tilde{R}$) denotes the curvature tensor of $\nabla^{\perp}$ (resp. D).

From now on, we will also assume that M is an isotropic submanifold, i. e. in each point p of M, $\|h(v, v)\|$ is independent of the unit vector v. Hence, we obtain a function λ on M by

(2.5) $$\lambda(p)=\|h(v, v)\|,$$

where $v\in UM_p$. If the function λ is also independent of the point p, we say that M is constant isotropic. In that case, we obtain from [O]$_1$ the following conditions for orthonormal tangent vectors X, Y, Z and W:

(2.6) $$\langle h(X, Y), h(X, X)\rangle=0,$$

(2.7) $$\lambda^2-\langle h(X, X), h(Y, Y)\rangle-2\langle h(X, Y), h(X, Y)\rangle=0,$$

(2.8) $$\langle h(Y, Z), h(X, X)\rangle+2\langle h(X, Y), h(X, Z)\rangle=0,$$

(2.9) $$\langle h(X, Y), h(Z, W)\rangle+\langle h(X, Z), h(W, Y)\rangle+\langle h(X, W), h(Y, Z)\rangle=0.$$

3. Proof of the theorems

Let M be a 3-dimensional, isotropic submanifold of $S^n(1)$ and let $p\in M$. Then, we choose an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM in the following way. Let $S=\{(u, v)\mid u, v\in T_pM$ with $\langle u, v\rangle=0$ and $\|u\|=\|v\|=1\}$. We define a function f on S by

$$f((u, v))=\|h(u, v)\|^2.$$

Since S is compact, we can choose (e_1, e_2) as a point in which the function f attains a maximum. To conclude the choice of our basis, we choose e_3 such that e_3 is orthogonal to both e_1 and e_2. Since (e_1, e_2) is an absolute maximum we obtain that

$$\langle h(e_1, e_2), h(e_1, e_3)\rangle=0,$$

$$\langle h(e_1, e_2), h(e_2, e_3)\rangle=0.$$

LEMMA 3.1. *Let M be a 3-dimensional isotropic submanifold with* $\dim(\mathrm{im}(h))$ $\leqq 3$ *and let $p \in M$. Then there exists an orthonormal basis $\{e_2, e_2, e_3\}$ of T_pM such that one of the following holds.*

(3.1) (a)
$$h(e_1, e_1)=h(e_2, e_2)=h(e_3, e_3)=0 .$$
$$h(e_1, e_3)=h(e_1, e_3)=h(e_2, e_3)=0 ,$$

or

(3.2) (b)
$$h(e_1, e_1)=h(e_2, e_2)=h(e_3, e_3)\neq 0 ,$$
$$h(e_1, e_2)=h(e_1, e_3)=h(e_2, e_3)=0 ,$$

or

(3.3) (c)
$$h(e_1, e_1)=-h(e_2, e_2)=h(e_3, e_3)=\lambda g_1 ,$$
$$h(e_1, e_2)=\lambda g_2 ,$$
$$h(e_1, e_2)=0 ,$$
$$h(e_2, e_3)=\lambda g_3 ,$$

where g_1, g_2, g_3 are unit normal vectors at the point p and $\lambda\neq 0$.

PROOF. First, we assume that $\dim(\mathrm{im}(h_p))=0$. This means that p is a totally geodesic point. Therefore, we obtain (a).

Next, we assume that $\dim(\mathrm{im}(h_p))=1$. Since h is symmetric, this implies that $\lambda(p)\neq 0$. We choose an orthonormal basis of T_pM as shown above. Then, it follows from the first isotropy condition (2.6) that $h(e_1, e_2)$ is orthogonal to $h(e_1, e_1)$. Since $\dim(\mathrm{im}(h_p))=1$ and $\lambda(p)\neq 0$ this implies that $h(e_1, e_2)=0$. Similarly, we also obtain that $h(e_1, e_3)=h(e_2, e_3)=0$.

From the second isotropy condition it then follows that

$$0=\lambda^2-\langle h(e_1, e_1), h(e_2, e_2)\rangle .$$

Hence, since M is isotropic and $\lambda\neq 0$, we obtain by applying the Cauchy Schwartz inequality that $h(e_1, e_1)=h(e_2, e_2)$. Similarly, we also obtain that $h(e_3, e_3)=h(e_1, e_1)$. This proves (b).

Next, we assume that $\dim(\mathrm{im}(h_p))=2$. First, we assume that the function f defined above is identically zero. In this case, we obtain similar as in the previous case that (b) holds. This is in contradiction with the assumption that $\dim(\mathrm{im}(h_p))=2$. Therefore f is not identically zero. Thus, if we choose an orthonormal basis indicated above, we obtain that $\|h(e_1, e_2)\|=\mu\neq 0$. Therefore $h(e_1, e_1)$ and $h(e_1, e_2)$ span the first normal space at the point p. By our choice of orthonormal basis and by the isotropy conditions, we know that $h(e_1, e_3)$ and $h(e_2, e_3)$ are orthogonal to $h(e_1, e_1)$ and $h(e_1, e_2)$. Thus $h(e_1, e_3)=h(e_2, e_3)=0$.

From the isotropy conditions and the Cauchy-Schwartz ineqality it then follows that $h(e_1, e_1)=h(e_3, e_3)=h(e_2, e_2)$. By applying then again (2.7), we obtain that $\mu=0$, which is again a contradiction.

Finally, we assume that $\dim(\mathrm{im}(h))=3$. By a similar argument as in the previous case, we obtain that the function f is not identically zero. Therefore, if we choose an orthonormal basis in the same way as in the previous case, we have that $\|h(e_1, e_2)\|=\mu\neq 0$. Then, we obtain from the isotropy conditions that

$$\langle h(e_1, e_3), h(e_2, e_2)\rangle=-2\langle h(e_1, e_2), h(e_2, e_3)\rangle=0.$$

Thus we see that $h(e_1, e_3)$ and $h(e_2, e_3)$ are orthogonal to $h(e_1, e_1)$, $h(e_2, e_2)$ and $h(e_1, e_2)$. If $h(e_1, e_1)$, $h(e_2, e_2)$ and $h(e_1, e_2)$ span the first normal space, we obtain that $h(e_1, e_3)=h(e_2, e_3)=0$. From this, we obtain in the same way as in the previous case that $\mu=0$. Therefore, we may assume that $h(e_1, e_1)$, $h(e_1, e_2)$ and $h(e_2, e_2)$ are linearly dependent. The first isotropy condition then implies that $h(e_2, e_2)$ only has a component in the direction of $h(e_1, e_1)$ and the second isotropy condition then implies that $\mu=\lambda$ and that $h(e_1, e_1)=-h(e_2, e_2)$. These formulas imply that there exist orthonormal normal vectors f_1, f_2 and f_3 such that

$$\begin{aligned}
&h(e_1, e_1)=\lambda f_1,\\
&h(e_2, e_2)=-\lambda f_1,\\
&h(e_1, e_2)=\lambda f_2,\\
&h(e_1, e_3)=\nu_1 f_3,\\
&h(e_2, e_3)=\nu_2 f_3,\\
&h(e_3, e_3)=\alpha f_1+\beta f_2.
\end{aligned}$$

Then, the isotropy conditions are equivalent with

$$\begin{aligned}
&2\nu_1^2=\lambda^2-\alpha\lambda,\\
&2\nu_2^2=\lambda^2+\alpha\lambda,\\
&\beta\lambda+2\nu_1\nu_2=0.
\end{aligned}$$

From the first two equations, we see that we can put $\nu_1=\sin(\theta)\lambda$ and $\nu_2=\cos(\theta)\lambda$. But then it is clear from the last two equations that $\alpha=(2\cos^2(\theta)-1)\lambda$ and $\beta=-2\sin(\theta)\cos(\theta)\lambda$. But then if we put $u_3=e_3$, $g_3=f_3$, $g_1=\cos(2\theta)f_1-\sin(2\theta)f_2$, $g_2=\cos(2\theta)f_2+\sin(2\theta)f_1$, $u_1=\cos(\theta)e_1-\sin(\theta)e_2$ and $u_2=\sin(\theta)e_1+\cos(\theta)e_2$ we obtain (c). This completes the proof of the lemma. ∎

LEMMA 3.2. *Let M be as in Lemma* 3.1.

(a) *If* (3.1) *holds at a point of M, then $K_p \equiv 1$,*

(b) *If* (3.2) *holds at a point p of M, then $K_p \equiv 1+\lambda^2$,*

(c) *If* (3.3) *holds at a point p of M, then the sectional curvatures K of M at the point p satisfy $1-2\lambda^2 \leqq K \leqq 1+\lambda^2$. Furthermore, $K_p = 1-2\lambda^2$ for every plane through e_2 and $K_p = 1+\lambda^2$ only for the plane through e_1 and e_3.*

PROOF. (a) and (b) immediately follow from (3.1) and (3.2). To prove (c), we take an arbitrary tangent plane σ at p. Then, we can find an orthonormal basis $\{X, Y\}$ of σ such that $X = \cos\theta e_1 + \sin\theta e_3$ and $Y = -\cos\phi \sin\theta e_1 + \sin\phi e_2 + \cos\phi \cos\theta e_3$, where $\theta, \phi \in \boldsymbol{R}$. Then

$$\begin{aligned}\langle R(X, Y)Y, X\rangle &= \cos^2\theta \langle R(e_1, Y)Y, e_1\rangle + 2\cos\theta \sin\theta \langle R(e_1, Y)Y, e_3\rangle \\ &\quad + \sin^2\theta \langle R(e_3, Y)Y, e_3\rangle \\ &= (1+\lambda^2)\cos^2\phi - 3\lambda^2 \sin^2\phi .\end{aligned}$$

From this formula, (c) follows immediately. ∎

Let us now assume that M is *constant* isotropic, i.e. λ is a constant on M. Then it follows from Lemma 3.1, Lemma 3.2 and the connectedness of M that either

(a) (3.1) holds everywhere on M, i.e. M is totally geodesic,

or

(b) (3.2) holds everywhere on M, i.e. M is totally umbilical,

or

(c) (3.3) holds everywhere on M.

Totally geodesic and totally umbilical submanifolds of spheres are well known ([C], [O]$_2$). So the only case we still have to consider is the case that (3.3) holds everywhere on M with $\lambda \neq 0$. Let $p \in M$. Since in that case the sectional curvature equals $1+\lambda^2$ only for the plane through e_1 and e_3, we see that at each point p the vector e_2 is uniquely determined, namely e_2 is the vector orthogonal to the unique plane with sectional curvature $1+\lambda^2$. From this it follows that we can choose differentiable vector fields E_1, E_2, E_3, defined on a neighbourhood U of p, such that $\{E_1(q), E_2(q), E_3(q)\}$ satisfies (3.3) for every $q \in U$. Therefore, we also obtain orthonormal normal vector fields g_1, g_2 and g_3 such that

$$h(E_1, E_1) = -h(E_2, E_2) = h(E_3, E_3) = \lambda g_1 ,$$

$$h(E_1, E_2) = \lambda g_2 ,$$

$$h(E_1, E_3)=0,$$

$$h(E_2, E_3)=\lambda g_3.$$

Then, we have the following lemma.

LEMMA 3.3. *Let us assume that* (3.3) *holds on* M, *where* λ *is a constant different from zero. Then* $\lambda=\frac{1}{\sqrt{2}}$, M *is locally isometric with* $\boldsymbol{R}\times S^2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$ *and after identification* E_2 *is tangent to* $\boldsymbol{R}$. *Further,* $\nabla_X E_2=0$, *for every tangent vector field* X *and we can still choose locally* E_1 *and* E_3 *in such a way that they satisfy* (3.3) *and such that* $\nabla_{E_2}E_1=\nabla_{E_2}E_3=0$. *Finally,* (∇h) *is orthogonal to* $\mathrm{im}(h)$.

PROOF. Let $p\in M$. First, we will prove that $(\nabla h)\perp h$. In order to do so, it is sufficient to prove that (∇h) is orthogonal to g_1, g_2 and g_3. In order to do so, we first extend e_1, e_2, e_3 to local orthonormal vector fields U_1, U_2, U_3 such that $U_i(p)=e_i$ and $\nabla_{e_j}U_i=0$, where $i, j\in M$. Since M is constant isotropic, we know that

$$\langle h(U_i, U_i), h(U_i, U_i)\rangle=\lambda^2,$$

where $i\in\{1, 2, 3\}$. By differentiating this we obtain that

$$\langle(\nabla h)(e_j, e_i, e_i), h(e_i, e_i)\rangle=0. \tag{3.4}$$

Similarly, using the previous equations, we find from $\langle h(U_i, U_j), h(U_i, U_i)\rangle=0$, for different i and j, that

$$\langle(\nabla h)(e_k, e_i, e_j), h(e_i, e_i)\rangle+\langle(\nabla h)(e_k, e_i, e_i), h(e_i, e_j)\rangle=0. \tag{3.5}$$

So, if we take $i=1$, $j=3$ and $k=2$, we find that

$$\langle(\nabla h)(e_2, e_1, e_3), g_1\rangle=0. \tag{3.6}$$

Then (3.4) together with (3.6) implies that $(\nabla h)\perp g_1$. Therefore, (3.5) becomes

$$\langle(\nabla h)(e_k, e_i, e_i), h(e_i, e_j)\rangle=0,$$

where $i, j, k\in\{1, 2, 3\}$, $i\neq j$. Since (3.3) holds at the point p this implies that

$$\langle(\nabla h)(e_k, e_2, e_2), g_2\rangle=\langle(\nabla h)(e_k, e_1, e_1), g_2\rangle=0,$$

$$\langle(\nabla h)(e_k, e_2, e_2), g_3\rangle=\langle(\nabla h)(e_k, e_3, e_3), g_3\rangle=0.$$

Then, if we take the local orthonormal frame $\{E_1, E_2, E_3\}$ previously defined, we find that

$$\langle h(E_1, E_2), h(E_1, E_2)\rangle=\lambda^2.$$

Hence by deriving this with respect to E_3, we obtain that

$$0=\langle(\nabla h)(E_3, E_1, E_2), h(E_1, E_2)\rangle+\langle h(\nabla_{E_3}E_1, E_2), h(E_1, E_2)\rangle$$
$$+\langle h(E_1, \nabla_{E_3}E_2), h(E_1, E_2)\rangle .$$

Thus $(\nabla h)(E_3, E_1, E_2)$ is also orthogonal to g_2. Similarly, starting from $\langle h(E_2, E_3), h(E_2, E_3)\rangle=\lambda^2$, we obtain that $(\nabla h)(E_1, E_2, E_3)$ is also orthogonal to g_3. Using similar arguments, we also can prove that

$$\langle(\nabla h)(E_1, E_3, E_3), g_2\rangle=\langle(\nabla h)(E_2, E_3, E_3), g_2\rangle=0 ,$$
$$\langle(\nabla h)(E_1, E_1, E_3), g_3\rangle=\langle(\nabla h)(E_2, E_1, E_1), g_3\rangle=0 .$$

Therefore in order to prove that $(\nabla h)\perp h$, it only remains to prove that $(\nabla h)(E_3, E_3, E_3)$ is orthogonal to g_2 and that $(\nabla h)(E_1, E_1, E_1)$ is orthogonal to g_3. In particular, we already know that $(\nabla h)(E_k, E_1, E_3)$ is orthogonal to $\mathrm{im}(h)$ for every k. But on the other hand, we have that

$$\begin{aligned}(\nabla h)(E_k, E_1, E_3)&=-h(\nabla_{E_k}E_1, E_3)-h(E_1, \nabla_{E_k}E_3)\\&=-\langle\nabla_{E_k}E_1, E_2\rangle\lambda g_3-\langle E_2, \nabla_{E_k}E_3\rangle\lambda g_2\\&\quad-(\langle\nabla_{E_k}E_1, E_3\rangle+\langle\nabla_{E_k}E_3, E_1\rangle)\lambda g_1\end{aligned}$$

Thus $\nabla_{E_k}E_2=0$, where $k=1, 2, 3$. But then if follows by deriving $\langle h(E_3, E_3), h(E_1, E_2)\rangle=0$ and $\langle h(E_1, E_1), h(E_2, E_3)\rangle=0$ that also $(\nabla h)(E_3, E_3, E_3)$ is orthogonal to g_2 and that $(\nabla h)(E_1, E_1, E_1)$ is orthogonal to g_3. So $(\nabla h)\perp h$.

Since $\nabla_{E_k}E_2=0$, $k=1, 2, 3$, it follows that $R(E_1, E_2)E_2=0$. Hence from the Gauss equation we find that $0=1-2\lambda^2$. Thus $\lambda=\frac{1}{\sqrt{2}}$.

Now, we can define two orthogonal distributions T_1 and T_2 by

$$T_1: p\longmapsto T_1(p)=\mathrm{vect}\{E^2(p)\} ,$$
$$T_2: p\longmapsto T_2(p)=\mathrm{vect}\{E_1(p), E_3(p)\} .$$

Since $\nabla_{E_k}E_2=0$, $k=1, 2, 3$, we find that $\nabla_{T_2}T_1\subset T_1$, $\nabla_{T_1}T_1\subset T_1$. Since T_1 and T_2 are orthogonal distributions, we find from the de Rham decomposition theorem ([K-N]) that M is locally isometric with $\boldsymbol{R}\times M'$, where T_1 is tangent to $\boldsymbol{R}$ and T_2 is tangent to M'. Since M' has constant Gaussian curvature $\frac{3}{2}$, we also have that M' is locally isometric with a sphere of radius $\frac{\sqrt{2}}{\sqrt{3}}$.

Finally, since M is locally isometric with a product, it is clear that we can choose locally vector fields E_1 and E_3, orthogonal to T_1, such that $\nabla_{E_2}E_1=\nabla_{E_2}E_3=0$. ■

In the following lemmas, we will compute the normal connection on M and prove that M lies linearly full in a 7-dimensional sphere. Let $p\in M$ and let $\{E_1, E_2, E_3\}$ the local orthonormal basis given by Lemma 3.2. Then, if we define local functions α and β on M, by

$$\nabla_{E_1}E_3=\alpha E_1,$$
$$\nabla_{E_3}E_1=\beta E_3,$$

we have the following lemmas.

LEMMA 3.4. *If we denote the corresponding orthonormal normal vector fields by g_1, g_2 and g_3 we obtain that*

$$\nabla^{\perp}_{E_1}g_1=\nabla^{\perp}_{E_3}g_1=\nabla^{\perp}_{E_2}g_2=\nabla^{\perp}_{E_2}g_3=0,$$
$$\nabla^{\perp}_{E_1}g_3=\alpha g_2,$$
$$\nabla^{\perp}_{E_3}g_2=\beta g_3,$$
$$\nabla^{\perp}_{E_2}g_1=f,$$
$$\nabla^{\perp}_{E_1}g_2=-\alpha g_3+f,$$
$$\nabla^{\perp}_{E_3}g_3=-\beta g_2+f,$$

where f is a normal vector field to M which is also normal to g_1, g_2 and g_3.

PROOF. First, we notice that

$$(\nabla h)(E_k, E_1, E_3)=-h(\nabla_{E_k}E_1, E_3)-h(E_1, \nabla_{E_k}E_3)=0.$$

Therefore, if we put $k=1$ and apply the Codazzi equation, we obtain that

$$0=\lambda\nabla^{\perp}_{E_3}g_1-2h(\nabla_{E_3}E_1, E_1)=\lambda\nabla^{\perp}_{E_3}g_1.$$

Similarly, if we put $k=3$, we obtain that $\nabla^{\perp}_{E_1}g_1=0$. Finally, if we put $k=2$, we find from the Codazzi equations that

$$0=\lambda\nabla^{\perp}_{E_1}g_3-\alpha h(E_1, E_2),$$
$$0=\lambda\nabla^{\perp}_{E_3}g_2-\beta h(E_2, E_3).$$

From the Codazzi equation $(\nabla h)(E_2, E_1, E_2)=(\nabla h)(E^1, E_2, E_2)$, we then obtain that

$$\lambda\nabla^{\perp}_{E_2}g_2=-\lambda\nabla^{\perp}_{E_1}g_1=0.$$

Similarly, we obtain from the Codazzi equation $(\nabla h)(E_2, E_3, E_2)=(\nabla h)(E_3, E_2, E_2)$ that $\nabla^{\perp}_{E_2}g_3=0$. Then, we define a normal vector field f by

$$f=\nabla^{\perp}_{E_2}g_1.$$

It follows from the fact that $(\nabla h)(E_2, E_1, E_1)$ is orthogonal to $\mathrm{im}(h)$ that the normal vector field f is orthogonal to g_1, g_2 and g_3. Then, it follows from the Codazzi equations $(\nabla h)(E_2, E_1, E_1)=(\nabla h)(E_1, E_2, E_1)$ and $(\nabla h)(E_2, E_3, E_3)=(\nabla h)(E_3, E_2, E_3)$ that

$$\nabla^{\perp}_{E_1} g_2=-\alpha g_3+f\,,$$

$$\nabla^{\perp}_{E_3} g_3=-\beta g_2+f\,.$$

This completes the proof of this lemma.

LEMMA 3.5. *Let $p\in M$ and let $E_1, E_2, E_3, g_1, g_2, g_3$ and f be local vector fields (normal or tangent) as defined above. Then*

(i) $\langle f, f\rangle=1$

(ii) $\nabla^{\perp}_{E_1} f=-g_2\,,$

$\nabla^{\perp}_{E_3} f=-g_3\,.$

PROOF. First, we take a local normal vector field η which is orthogonal to $\mathrm{im}(h)$, i.e. which is orthogonal to g_1, g_2 and g_3. Since η is orthogonal to $\mathrm{im}(h)$, it follows from the Ricci equation that

$$\langle R^{\perp}(E_1, E_2)g_1, \eta\rangle=0\,, \tag{3.7}$$

$$\langle R^{\perp}(E_2, E_3)g_1, \eta\rangle=0\,, \tag{3.8}$$

$$\langle R^{\perp}(E_2, E_1)g_2, \eta\rangle=0\,. \tag{3.9}$$

On the other hand, using Lemma 3.4, we find that

$$\begin{aligned} R^{\perp}(E_1, E_2)g_1 &= \nabla^{\perp}_{E_1}\nabla^{\perp}_{E_2} g_1-\nabla^{\perp}_{E_2}\nabla^{\perp}_{E_1} g_1-\nabla^{\perp}_{[E_1, E_2]} g_1 \\ &= \nabla^{\perp}_{E_1} f\,. \end{aligned}$$

Combining this with (3.7) and Lemma 3.4, we find that

$$\nabla^{\perp}_{E_1} f=-\langle f, f\rangle g_2\,.$$

Similarly, we find from (3.8) and (3.9) that

$$\nabla^{\perp}_{E_2} f=-\langle f, f\rangle g_1\,,$$

$$\nabla^{\perp}_{E_3} f=-\langle f, f\rangle g_3\,.$$

From these formulas, it is immediately clear that f has constant length. But, we have first that

$$\langle R^{\perp}(E_1, E_2)g_1, g_2\rangle=\langle \nabla^{\perp}_{E_1} f, g_2\rangle=-\langle f, f\rangle\,.$$

On the other hand, by applying the Ricci identity, we obtain that

$$\begin{aligned}\langle R^{\perp}(E_1, E_2)g_1, g_2\rangle &= \langle [A_{g_1}, A_{g_2}]E, E_2\rangle \\ &= \langle A_{g_2}E_1, A_{g_1}E_2\rangle - \langle A_{g_1}E_1, A_{g_2}E_2\rangle \\ &= -\lambda^2 - \lambda^2 = -1 .\end{aligned}$$

This completes the proof of this lemma. ■

From these lemmas, it is clear that (∇h) only has a component in the direction of f. Furthermore, by applying all these lemmas, we see that the space spanned by $\mathrm{im}(h)$ and $\mathrm{im}(\nabla h)$ is parallel with respect to the normal connection and has constant dimension 4. Therefore, by the reduction theorem of J. Erbacher [E], there exists a totally geodesic S^7 of S^n, such that M is contained in S^7. The following example then shows that this case is possible.

EXAMPLE 3.1. Let us consider S^7 as a hypersurface in $\boldsymbol{C}^4$. Then, it is well-known that starting from the complex structure on S^7, one can induce a Sasakian structure ϕ on S^7 with structure vector field ζ. Then, we consider the following immersion j from $\boldsymbol{R}\times S^2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$ into S^7:

$$j(u, y_1, y_2, y_3) = (j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8),$$

where

$$j_1 = \frac{\sqrt{2}}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{2}}u\right)y_1 + \frac{1}{3}\cos(\sqrt{2}\,u)$$

$$j_2 = -\frac{\sqrt{2}}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{2}}u\right)y_1 + \frac{1}{3}\sin(\sqrt{2}\,u)$$

$$j_3 = \frac{1}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{2}}u\right)y_1 + \frac{\sqrt{2}}{3}\sin(\sqrt{2}\,u)$$

$$j_4 = \frac{1}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{2}}u\right)y_1 - \frac{\sqrt{2}}{3}\cos(\sqrt{2}\,u)$$

$$j_5 = \cos\left(\frac{1}{\sqrt{2}}u\right)y_2$$

$$j_6 = -\sin\left(\frac{1}{\sqrt{2}}u\right)y_2$$

$$j_7 = \cos\left(\frac{1}{\sqrt{2}}u\right)y_3$$

$$j_8 = -\sin\left(\frac{1}{\sqrt{2}}u\right)y_3$$

where $y_1^2+y_2^2+y_3^2=\frac{2}{3}$. Then, a straightforward computation shows that ζ is a normal vector field on $\boldsymbol{R}\times S^2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$. Hence the immersion is a totally real immersion. Furthermore, if we choose coordinates in such a way that

$$y_1=\frac{\sqrt{2}}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{\sqrt{2}}v\right)\cos\left(\frac{\sqrt{3}}{\sqrt{2}}w\right)$$

$$y_2=\frac{\sqrt{2}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{\sqrt{2}}v\right)\cos\left(\frac{\sqrt{3}}{\sqrt{2}}w\right)$$

$$y_3=\frac{\sqrt{2}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{\sqrt{2}}w\right),$$

it is easy to check that j is an isotropic immersion.

THEORFM 3.1. *Let $x: M\to S^n$ be a constant isotropic immersion such that* $\dim(\mathrm{im}(h))\leqq 3$. *Then, one of the following holds:*

(a) *M is totally geodesic in S^n,*

(b) *There exists a totally geodesic S^4 in S^n, such that the image of M is an open part of a small hypersphere of S^4,*

(c) *There exists a totally geodesic S^7 in S^n, such that the image of M congruent with an open part of $j\left(\boldsymbol{R}\times S^2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\right)$ in S^7.*

PROOF. From Lemma 3.1 and Lemma 3.2 it follows that either (3.1) or (3.2) or (3.3) holds on M. If (3.1) or (3.2) holds on M, M is totally umbilical. Therefore, by the classification of totally umbilical submanifolds of spheres ([C]), we obtain (a) and (b). Therefore, we may assume that (3.3) holds on M. Then, we know from Lemma 3.3 that M is locally isometric with $\boldsymbol{R}\times S^2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$. Further from the remark following Lemma 3.5, we know that there exists a totally geodesic S^7 in S^n such that the image of M is contained in S^7. Now, let $p\in M$ and let U be a neighbourhood of p on which M is isometric with $\boldsymbol{R}\times S^2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$. Then, on U, we can consider the two following immersions

$$\begin{tikzcd}[row sep=small] & M \arrow[r, "x"] & S^7 \\ U \arrow[ur] \arrow[dr] & & \\ & \boldsymbol{R}\times S^2\left(\frac{\sqrt{2}}{\sqrt{2}}\right) \arrow[r, "j"'] & S^7 . \end{tikzcd}$$

From Lemma 3.3, Lemma 3.4 and Lemma 3.5, it then follows that we can apply the uniqueness theorem ([Sp], volume 4). Thus there exists an isometry

A of S^7 such that on U

$$A\circ x=j.$$

Since the immersion j is an analytic immersion, it follows immediately that the isometry A is indepenent of the point p. ∎

THEOREM 3.2. *Let M be a 3-dimensional, minimal, isotropic submanifold in $S^n(1)$. Then, M has constant sectional curvature.*

PROOF. Let $p\in M$ and let us assume that p is not a totally geodesic point. Then, it follows by combining Lemma 3.1 with the minimality of M that $\dim(\mathrm{im}(h))$ is greater than or equal to 4. On the other hand, it follows from the minimality of M that $\dim(\mathrm{im}(h))$ is less than or equal to 5. Let us choose, as indicated at the beginning of this section, an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM. Clearly $\mathrm{im}(h)$ is spanned by $h(e_1, e_1)$, $h(e_1, e_2)$, $h(e_2, e_2)$, $h(e_1, e_3)$ and $h(e_2, e_3)$.

Since $h(e_1, e_3)$ is orthogonal to $h(e_1, e_1)$ and $h(e_3, e_3)$, it follows from the minimality M that $h(e_1, e_3)$ is orthogonal to $h(e_2, e_2)$. Furthermore, by the choice of our basis, we also know that $h(e_1, e_3)$ is orthogonal to $h(e_1, e_2)$. Similarly, we also obtain that $h(e_2, e_3)$ is orthogonal to $h(e_1, e_1)$, $h(e_2, e_2)$ and $h(e_1, e_2)$.

From (2.8) and the minimality of M, it than follows that

$$\begin{aligned}\langle h(e_1, e_3), h(e_2, e_3)\rangle &= -\frac{1}{2}\langle h(e_1, e_2), h(e_3, e_3)\rangle \\ &= \frac{1}{2}\langle h(e_1, e_2), h(e_1, e_1)\rangle+\frac{1}{2}\langle h(e_1, e_2), h(e_2, e_2)\rangle \\ &= 0.\end{aligned}$$

On the other hand, it follows from (2.7) and the Cauchy Schwartz inequality that $h(e_1, e_3)\neq 0\neq h(e_2, e_3)$. By the choice of our basis, this implies that $h(e_1, e_2)\neq 0$.

By combining these information, we see that there orthonormal normal vector fields g_1, g_2, g_3, g_4 and a normal vector field g_5, which is orthogonal to g_1, g_2, g_3 and g_4 such that

$$\begin{aligned}h(e_1, e_1)&=\lambda g_1,\\ h(e_1, e_2)&=\mu_1 g_2,\\ h(e_1, e_3)&=\mu_2 g_3,\\ h(e_2, e_3)&=\mu_3 g_4,\\ h(e_2, e_2)&=\mu_4 g_1+g_5.\end{aligned}$$

Then, we find from (2.7) that

$$2\mu_2^2=\lambda^2-\langle h(e_1, e_1), h(e_3, e_3)\rangle$$
$$=2\lambda^2+\langle h(e_1, e_1), h(e_2, e_2)\rangle$$
$$=2\lambda^2+\lambda^2-2\mu_1^2.$$

Similarly, we find that

$$2\mu_3^2=3\lambda^2-2\mu_1^2.$$

The minimality and (2.7) also imply that

$$\lambda^2=\langle h(e_3, e_3), h(e_3, e_3)\rangle$$
$$=2\lambda^2+2\langle h(e_1, e_1), h(e_2, e_2)\rangle$$
$$=4\lambda^2-4\mu_1^2.$$

Thus, we may assume that $\mu_1=\frac{\sqrt{3}}{2}\lambda$. Hence, we may assume that $\mu_2=\mu_3=\frac{\sqrt{3}}{2}$. From (2.7) it then follows that $\mu_4=-\frac{1}{2}\lambda$. Finally, it follows from $\langle h(e_2, e_2), h(e_2, e_2)\rangle=\lambda^2$ that $\langle g_5, g_5\rangle=\frac{3}{4}\lambda^2$. We can summarize this as follows. There exists orthonormal normal vector fields f_1, f_2, f_3, f_4 and f_5 such that:

$$h(e_1, e_1)=\lambda f_1,$$
$$h(e_1, e_2)=\frac{\sqrt{3}}{2}\lambda f_2,$$
$$h(e_2, e_2)=-\frac{\lambda}{2}f_1+\frac{\sqrt{3}}{2}\lambda f_3,$$
$$h(e_1, e_3)=\frac{\sqrt{3}}{2}\lambda f_4,$$
$$h(e_2, e_3)=\frac{\sqrt{3}}{2}\lambda f_5,$$
$$h(e_3, e_3)=-\frac{\lambda}{2}f_1-\frac{\sqrt{3}}{2}\lambda f_3.$$

Using the Gauss equation, we find from these formulas that

$$R(e_1, e_2)e_3=R(e_2, e_3)e_1=R(e_3, e_1)e_2=0,$$
$$\langle R(e_1, e_2)e_2, e_1\rangle=\langle R(e_1, e_3)e_3, e_1\rangle=\langle R(e_2, e_3)e_3, e_2\rangle=1-\frac{5}{4}\lambda^2.$$

Hence, M has constant sectional curvatures. ∎

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