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ON D-PARACOMRACT σ-SPACES

By

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1. Introduction.

Throughout this paper, all spaces are T_1 topological spaces and mappings are continuous and onto. The letter N denotes the set of natural numbers.

By a well-known theorem of Dowker, a Hausdorff space X is paracompact if and only if for every open cover $\mathcal{A}$ of X there exists an $\mathcal{A}$-mapping of X onto a metrizable space. On the other hand, developable spaces are a nice generalization of metrizable spaces. Pareek [P] called a space X is *d-paracompact* if for every open cover $\mathcal{A}$. there exists an $\mathcal{A}$-mapping of X onto a developable space. Another nice generalization is σ-spaces in the sense of [O]. Especially, paracompact σ-spaces have important features in generalized metric spaces and dimension theories. We notice that the following properties of the class $\mathcal{C}$ of paracompact σ-spaces: (1) $\mathcal{C}$ is closed under any countable product and any subspace. (2) $\mathcal{C}$ is closed under any image under perfect or closed mappings. (3) $\mathcal{C}$ is closed under the domination.

In this paper, we call a space X is a *σ-space* if X has a σ-locally finite "closed" network, which is slightly different from the original definition in [O]. For regular spaces, both coincide with each other. The purpose of this paper is to study the class of d-paracompact σ-spaces, comparing with that of paracompact σ-spaces. We show that this class behaves well as to the subspaces and perfect images, but not as to the others. We show that d-paracompact spaces and s-paracompact spaces do not coincide, answering the question of Brandenburg [B_1, Question 2].

2. *D*-paracompact σ-spaces.

DEFINITION 1. A space X is called *d-paracompact* if for each open cover $\mathcal{V}$ of X, ther exists a *$\mathcal{V}$-mapping* f X onto a developable space Y, where a *$\mathcal{V}$-mapping* f means that there exists an open cover $\mathcal{V}$ of Y such that $f^{-1}(\mathcal{V}) < \mathcal{V}$.

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DEFINITION 2. A family $\mathcal{U}$ of open subsets of a space X is called *dissectable in* X [B_1], if there exists a function $D: \mathcal{U}\times N\to\{$closed subsets of $X\}$, called the *dissection of* $\mathcal{U}$ *in* X, satisfying the following:

(1) $U=\cup\{D(U, n): n\in N\}$ for every $U\in\mathcal{U}$.

(2) For every $n\in N$, $\{D(U, n): U\in\mathcal{U}\}$ is a closure-preserving family of closed subsets of X and if $p\in\cup\{D(U, n): U\in\mathcal{U}\}$, then

$$\cap\{U\in\mathcal{U}: p\in D(U, n)\}$$

is a neighborhood of p in X.

DEFINITION 3. A space X is called *d-expandable* [B_2] if for each discrete family $\mathcal{F}$ of closed subsets and for each family $\mathcal{U}=\{U(F): F\in\mathcal{F}\}$ of open subsets of X such that $F\subset U(F)$ and $U(F)\cap F'=\emptyset$ if $F\neq F'$, $F, F'\in\mathcal{F}$, there exists a dissectable family $\{V(F): F\in\mathcal{F}\}$ of X such that $F\subset V(F)\subset U(F)$ for every $F\in\mathcal{F}$.

We call the pair $\langle\mathcal{F}, \mathcal{U}\rangle$ a *d-pair* of X.

DEFINITION 4. A space X is called *semistratifiable* if there exists a function $S: \mathcal{T}\times N\to\{$closed subsets of $X\}$, where $\mathcal{T}$ is the topology of X, such that:

(1) $U=\cup\{S[U, n]: n\in N\}$ for every $U\in\mathcal{T}$.

(2) If $U, V\in\mathcal{T}$ and $U\subset V$, then $S[U, n]\subset S[V, n]$ for every n.

The function S is called the *semistratification of* X.

As seen easily, every σ-space is semistratifiable. If we use the argument in [SN], then it is obvious that a space X is a σ-space if and only if X has a σ-discrete closed network if and only if X has a σ-closure-preserving closed network. We list up the facts as to d-paracompact spaces and developable spaces, which are known already and used later in our proofs.

FACT 1 ([B_1]). A space X is developable if and only if X has a σ-dissectable base.

2 ([G]). A space X is developable if and only if there exists a sequence $\{\mathcal{U}_n: n\in N\}$ of open covers of X such that if $x\in U$ for a point x of X and an open subset U of X, then there exists $n\in N$ such that ord $(x, \mathcal{U}_n)=1$ and $S(x, \mathcal{U}_n)\subset U$.

3 ([B_2, Theorem 1]). A space X is d-paracompact if and only if X is θ-refinable and d-expandable if and only if every open cover of X has a σ-dis-

sectable refinement.

4. Let X be a semistratifiable space and $\mathcal{F}$ a closure-preserving family of closed subsets of X. Then there exists a σ-discrete closed cover $\mathcal{H}$ of X such that $H\cap F\neq\emptyset$, $H\in\mathcal{H}$ and $F\in\mathcal{F}$ imply $H\subset F$. (The construction of $\mathcal{H}$ is essentially stated in [SN].)

5 ([B_1, Theorem 2.3]). Every family of open subsets of a developable space is dissectable in it.

Before stating Lemma 1, we give definitions of (P_i), $i=1, \cdots, 5$. For a space X, let (P_i) $(i=1, \cdots, 5)$ be the following statements:

(P_1) X is d-paracompact.

(P_2) For each d-pair $\langle\mathcal{F}, \mathcal{U}\rangle$ of X, there exists a $\mathcal{V}$-mapping of X onto a developable space, where

$$\mathcal{V}=\mathcal{U}\cup\{X-\cup\mathcal{F}\}.$$

(P_3) For each d-pair $\langle\mathcal{F}, \mathcal{U}\rangle$ of families of X, there exists a family $\{V(F): F\in\mathcal{F}\}$ of open subsets of X and a sequence $\{\mathcal{U}_n: u\in N\}$ of open covers of X such that for each $F\in\mathcal{F}$, $F\subset V(F)\subset U(F)$ and such that if $p\in V(F)$, then there exists $n\in N$ such that $\operatorname{ord}(p, \mathcal{U}_n)=1$ and $S(p, \mathcal{U}_n)\subset V(F)$.

(P_4) For each d-pair $\langle\mathcal{F}, \mathcal{U}\rangle$ of families of X, there exists a pair collection $\mathcal{P}=\cup\{\mathcal{P}_n: n\in N\}$ of X and a family $\{V(F): F\in\mathcal{F}\}$ of open subsets of X such that $F\subset V(F)\subset U(F)$ for each $F\in\mathcal{F}$ and such that $\mathcal{P}$ satisfies the following two conditions:

(1) For each n, $\{P_1: P=(P_1, P_2)\in\mathcal{P}_n\}$ is a discrete family of closed subsets of X and $\{P_2: P\in\mathcal{P}_n\}$ is a family of open subsets of X.

(2) If $p\in V(F)$, then there exists $P\in\mathcal{P}$ such that $p\in P_1\subset P_2\subset V(F)$.

(P_5) X is d-expandable.

For the later use, we give the term to such a sequence of open covers as in (P_3). Let $\{\mathcal{U}_n: n\in N\}$ be a sequence of open covers of a space X and $\mathcal{V}$ a family of open subsets of X. Then we call $\{\mathcal{U}_n\}$ the *d-development for* $\mathcal{V}$ if for each point $p\in X$ and each $V\in\mathcal{V}$ with $p\in V$, there exists $n\in N$ such that $\operatorname{ord}(p, \mathcal{U}_n)=1$ and $S(p, \mathcal{U}_n)\subset V$. If $\{\mathcal{U}_n: n\in N\}$ is a sequence of families of open subsets of X with this property for $\mathcal{V}$, then we call $\{\mathcal{U}_n\}$ the *d-quasi-development for* $\mathcal{V}$.

LEMMA 1. *For a space,* $(P_1)\to(P_2)\to(P_3)\to(P_4)\to(P_5)$ *holds. Moreover, if* X *is* θ*-refinable, then all* (P_i) *are equivalent.*

PROOF. $(P_1)\to(P_2)$ is straightforward from Definition 1. $(P_2)\to(P_3)$ follows from the Fact 2. $(P_3)\to(P_4)$: Let $\{\mathcal{U}_n : n\in N\}$ be the sequence of open covers in (P_3). For each n, let

$$\mathcal{P}_n=\{(H(U), U): U\in\mathcal{U}_n,\ H(U)\neq\emptyset\},$$

where

$$H(U)=U-\cup\{U'\in\mathcal{U}_n : U'\neq U\}.$$

Then it is easy to see that $\{\mathcal{P}_n : n\in N\}$ has the required property. $(P_4)\to(P_5)$: For each d-pair $\langle\mathcal{F}, \mathcal{U}\rangle$, take $\mathcal{P}=\cup\{\mathcal{P}_n : n\in N\}$ and $\{V(F): F\in\mathcal{F}\}$ by (P_4). We define a function $D: \mathcal{V}\times N\to\{\text{closed subsets of } X\}$ with $\mathcal{V}=\{V(F): F\in\mathcal{F}\}$ by

$$D(V(F), n)=\cup\{P_1 : P\in\mathcal{P}_n \text{ and } P_2\subset V(F)\}.$$

Then D is the dissection of $\mathcal{V}$ in X. If X is θ-refinable, then $(P_5)\to(P_1)$ follows from Fact 3.

We weaken the statement (P_3) to the following:
(P_3') For each d-pair $\langle\mathcal{F}, \mathcal{U}\rangle$ of families of X, there exist a family $\mathcal{V}=\{V(F): F\in\mathcal{F}\}$ of open subsets of X and the d-quasidevelopment $\{\mathcal{U}_n : n\in N\}$ for $\mathcal{V}$ such that $F\subset V(F)\subset U(F)$ for every $F\in\mathcal{F}$.

LEMMA 2. *If X is a perfect space, i.e., every closed subset is G_δ, then (P_3') implies that X is d-expandable.*

PROOF. Suppose we are given $\langle\mathcal{F}, \mathcal{U}\rangle$, $\{V(F): F\in\mathcal{F}\}$ and $\{\mathcal{U}_n : n\in N\}$ in (P_3'). For each n, let

$$\cup\mathcal{U}_n=\cup\{E_{nm} : m\in N\},$$

where each E_{nm} is closed in X. For each $n, m\in N$, define

$$\mathcal{V}_{nm}=\mathcal{U}_n\cup\{X-E_{nm}\}.$$

Then it is easy to see that $\{\mathcal{V}_{nm} : n, m\in N\}$ is the d-development for $\{V(F)\}$ in X. Therefore, by the above, X is d-expandable.

LEMMA 3. *Let X be a semistratifiable space. Then a family $\mathcal{U}$ of open subsets of X is dissectoble in X if and only if there exists a d-development for $\mathcal{U}$ in X.*

PROOF. The only if part: Let $D: \mathcal{U}\times N\to\{\text{closed subsets of } X\}$ be the dissection of $\mathcal{U}$. Since for each n, $\{D(U, n): U\in\mathcal{U}\}$ is closure-preserving family of closed subsets of X, by Fact 4, there exists a closed cover $\mathcal{F}=\cup\{\mathcal{F}_{nm} : m\in N\}$ of X such that each $\mathcal{F}_{nm}$ is discrete in X and for each $F\in\mathcal{F}$ and each $U\in\mathcal{U}$,

$D(U, n)\cap F\neq\emptyset$ implies $F\subset D(U, n)$. For each $F\in\mathcal{F}_{nm}$, $n, m\in N$, choose an open subset V_F of X such that

$$F\subset V_F\subset\cap\{U\in\mathcal{U}: F\subset D(U, n)\}$$

and $V_F\cap F'=\emptyset$ for $F'\in\mathcal{F}_{nm}$ with $F\neq F'$. Let

$$\mathcal{V}_{nm}=\{V_F: F\in\mathcal{F}_{nm}\}\cup\{X-\cup\mathcal{F}_{nm}\},\ n, m\in N.$$

Then ti is easy to see that $\{\mathcal{V}_{nm}: n, m\in N\}$ forms the d-development for $\mathcal{U}$ in X. The if part is similar to the proof of $(P_3)\to(P_4)\to(P_5)$ in Lemma 1.

LEMMA 4. *If $\mathcal{U}$ is a σ-dissectable family of a space X, then $\mathcal{U}$ is dissectable in X.*

PROOF. Let $\mathcal{U}=\cup\{\mathcal{U}_n: n\in N\}$, where each $\mathcal{U}_n$ is dissectable in X. Let $D_n: \mathcal{U}_n\times N\to\{\text{closed subsets of } X\}$ be the dissection of $\mathcal{U}_n$ in X. Let $\phi: N\to N^2$ be a bijection. As a dissection T of $\mathcal{V}$, we define T as follows:

$$T(U, n)=\begin{cases} D_m(U, k) & \text{if } \phi(n)=(m, k) \text{ and } U\in\mathcal{U}_m \\ \emptyset & \text{otherwise.}\end{cases}$$

Obviously T is the dissection of $\mathcal{V}$ in X.

LEMMA 5. *If $\mathcal{U}$ is a dissectable family of a space X, then $\mathcal{V}=\{\cup\mathcal{U}_0: \mathcal{U}_0\subset\mathcal{U}\}$ is also dissectable in X.*

PROOF. Let $D: \mathcal{U}\times N\to\{\text{closed subsets of } X\}$ be the dissection of $\mathcal{U}$ in X. For each $V=\cup\mathcal{U}_0$, $\mathcal{U}_0\subset\mathcal{U}$, and each $n\in N$, let

$$T(V, n)=\cup\{D(U, n): U\in\mathcal{U}_0\}.$$

Then T is obviously the dissection of $\mathcal{V}$ in X.

LEMMA 6. *Let X be a semistratifiable space and $\mathcal{U}=\{U_\alpha: \alpha\in A\}$ a point-finite family of open subsets of X. If for each $\alpha\in A$, $\mathcal{V}_\alpha$ is a dissectable family of the subspace U_α, then $\cup\{\mathcal{V}_\alpha: \alpha\in A\}$ is dissectable in X.*

PROOF. Let Δ be the totality of

$$\delta(p)=\{\alpha\in A: p\in U_\alpha\},\qquad p\in\cup\mathcal{U}.$$

Then $\Delta=\cup\{\Delta_n: n\in N\}$, where

$$\Delta_n=\{\delta\in\Delta: |\delta|=n\},\qquad n\in N.$$

Let $\delta\in\Delta_n$, $n\in N$. Since by Lemma 4

$$\mathcal{V}(\delta)=\cup\{\mathcal{V}_\alpha:\alpha\in\delta)\mid\cap\{U_\alpha:\alpha\in\delta\}$$

is dissectable in $\cap\{U_\alpha:\alpha\in\delta\})$ by Lemma 3, there exists the d-development $\{\mathcal{V}_{nm}'(\delta):m\in N\}$ for $\mathcal{V}(\delta)$ in the subspace $\cap\{U_\alpha:\alpha\in\delta\}$. For each n, m, $k\in N$, let

$$\mathcal{V}_{nmk}(\delta)=\mathcal{V}_{nm}'(\delta)\mid(X-S[\cup\{U_\alpha:\alpha\in A-\delta\},k]),$$

where $S[\phi, k]=\emptyset$, $k\in N$, and let

$$\mathcal{V}_{nmk}=\cup\{\mathcal{V}_{nmk}(\delta):\delta\in\Delta_n\}.$$

We shall show that $\{\mathcal{V}_{nmk}:n, m, k\in N\}$ is the d-quasidevelopment for $\cup\{\mathcal{V}_\alpha:\alpha\in A\}$ in X. Let $p\in V\in\mathcal{V}_\alpha$, $\alpha\in A$. Since $\mathcal{U}$ is point-finite at p, $\delta(p)\in\Delta_n$ for some n. There exists $k\in N$ such that

$$p\in S[\cap\{U_\alpha:\alpha\in\delta(p)\},k].$$

Take $m\in N$ such that $\operatorname{ord}(p,\mathcal{V}_{nm}'(\delta(p)))=1$ and $S(p,\mathcal{V}_{nm}'(\delta(p)))\subset V$. Suppose $\delta\in\Delta_n$. If $\delta-\delta(p)\neq\emptyset$, then $p\notin\cup\mathcal{V}_{nm}'(\delta)$ because $\cup\mathcal{V}_{nm}'(\delta)\subset U_\alpha$ for each $\alpha\in\delta-\delta(p)$. If $\delta(p)-\delta\neq\emptyset$, then

$$p\in S[\cup\{U_\alpha:\alpha\in A-\delta\},k].$$

From these observations, we have

$$p\notin\cup[\cup\{\mathcal{V}_{nmk}(\delta):\delta\neq\delta(p)\text{ and }\delta\in\Delta_n\}].$$

Therefore $\operatorname{ord}(p,\mathcal{V}_{nmk})=1$ and $S(p,\mathcal{V}_{nmk})\subset V$. This completes the proof.

PROPOSITION 1. *Let X be a d-paracompact semistratifiable space and let $\mathcal{F}$ be a locally finite family of closed subsets of X with its open expansion $\{U(F):F\in\mathcal{F}\}$. Then there exists a dissectable family $\{V(F):F\in\mathcal{F}\}$ of X such that $F\subset V(F)\subset U(F)$ for each $F\in\mathcal{F}$.*

PROOF. By Fact 4, from the cover $\mathcal{F}\cup\{X\}$ of X we can construct a closed cover $\mathcal{H}=\cup\{\mathcal{H}_n:n\in N\}$ of X such that each $\mathcal{H}_n$ is discrete in X and such that if $H\cap F\neq\emptyset$, $F\in\mathcal{F}$ and $H\in\mathcal{H}$, then $H\subset F$. Since for each $H\in\mathcal{H}$,

$$\mathcal{F}(H)=\{F\in\mathcal{F}:H\subset F\}$$

is finite,

$$G(H)=\cap\{U(F):F\in\mathcal{F}(H)\}$$

is open in X. Since X is d-paracompact, for each n there exists a dissectable family $\mathcal{W}_n=\{W(H):H\in\mathcal{H}_n\}$ of X such that $H\subset W(H)\subset G(H)$ for each $H\in\mathcal{H}_n$. For each $F\in\mathcal{F}_n$. For each $F\in\mathcal{F}$, let

$$\mathcal{H}(F)=\{H\in\mathcal{H} : H\subset F\}.$$

Then obviously $F=\cup\mathcal{H}(F)$. For each $F\in\mathcal{F}$, set

$$V(F)=\cup\{W(H) : H\in\mathcal{H}(F)\}.$$

Then $\{V(F) : F\in\mathcal{F}\}$ is dissectable in X by Lemmas 4 and 5. This completes the proof.

LEMMA 7. *Let $\mathcal{U}$, $\mathcal{V}$ be dissectable families of X, Y, respectively. Then $\mathcal{U}\times\mathcal{V}$ is dissectable in the product space $X\times Y$.*

PROOF. Let D, D' be the dissections of $\mathcal{U}$, $\mathcal{V}$ in X, Y, respectively. Let $f : N\rightarrow N^2$ be a bijection. Define a function $T : (\mathcal{U}\times\mathcal{V})\times N\rightarrow\{$closed subsets of $X\times Y\}$ by

$$T(U\times V,\ k)=D(U,\ n)\times D'(V,\ n)$$

for $U\in\mathcal{U}$, $V\in\mathcal{V}$, $k\in N$, where $f(k)=(n,\ m)$. Then it is easy to see that T is the dissection of $\mathcal{U}\times\mathcal{V}$ in $X\times Y$.

Let $\mathcal{U}$, $\mathcal{V}$ be families of subsets of a space. Then we call that $\mathcal{U}$ is a *weak refinement of* $\mathcal{V}$ if for every $U\in\mathcal{U}$ there exists $V\in\mathcal{V}$ such that $U\subset V$.

DEFINITION 5. A space X is called a *P-space* [M_1] if for any family

$$\{G(\alpha_1,\ \cdots,\ \alpha_i) : \alpha_1,\ \cdots,\ \alpha_i\in A,\ i\in N\}$$

of open subsets of X such that

$$G(\alpha_1,\ \cdots,\ \alpha_i)\subset G(\alpha_i,\ \cdots,\ \alpha_i,\ \alpha_{i+1})$$

for each $\alpha_1,\ \cdots,\ \alpha_i,\ \alpha_{i+1}\in A$, $i\in N$, there exists a family

$$\{C(\alpha_1,\ \cdots,\ \alpha_i) : \alpha_1,\ \cdots,\ \alpha_i\in A,,\ i\in N\}$$

of closed subsets of X satisfying the following conditions:

(1) $C(\alpha_1,\ \cdots,\ \alpha_i)\subset G(\alpha_1,\ \cdots,\ \alpha_i)$ for each $\alpha_1,\ \cdots,\ \alpha_i\in A$, $i\in N$.

(2) For each sequence $\{\alpha_i : i\in N\}$ such that $X=\bigcup_i G(\alpha_1,\ \cdots,\ \alpha_i)$, then $X=\bigcup_i C(\alpha_1,\ \cdots,\ \alpha_i)$.

Obviously, every perfect space is a P-space. As for the product theorem of d-paracompact spaces, we can settle the following theorem.

THEOREM 1. *Let X be a d-paracompact P-space and Y a metacompact developable space. Then $X\times Y$ is d-paracompact.*

PROOF. Though the procedurc is due to the stereotyped method, we describe it to see how the properties of Y are used.

Let $\mathcal{F}=\{F(\alpha):\alpha\in A\}$ be a σ-discrete closed network for Y. Since Y is metacompact, there exists a σ-point-finite family $\{H(\alpha):\alpha\in A\}$ of open subsets of X such that $F(\alpha)\subset H(\alpha)$ for each $\alpha\in A$. Let $\mathcal{G}$ be an open cover of $X\times Y$. For each $\alpha_1, \cdots, \alpha_k\in A$, $k\in N$, let $\mathcal{G}(\alpha_1, \cdots, \alpha_k)$ be the frmily of open rectangles $U_\lambda\times V_\lambda$ such that $U_\lambda\times V_\lambda\subset G$ for some $G\in\mathcal{G}$ and

$$\cap\{F(\alpha_i):i=1, \cdots, k\}\subset V_\lambda\subset\cap\{H(\alpha_i):i=1, \cdots, k\}.$$

Write

$$\mathcal{G}(\alpha_1, \cdots, \alpha_k)=\{U_\lambda\times V_\lambda:\lambda\in\Lambda(\alpha_1, \cdots, \alpha_k)\}.$$

For each $\alpha_1, \cdots, \alpha_k\in A$, $k\in N$, let

$$U(\alpha_1, \cdots, \alpha_k)=\cup\{U_\lambda:\lambda\in\Lambda(\alpha_1, \cdots, \alpha_k)\}.$$

Then $U(\alpha_1, \cdots, \alpha_k)$ is an open subset of X such that

$$U(\alpha_1, \cdots, \alpha_k)\subset U(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}).$$

Since X is a P-space, there exists a family

$$\{C(\alpha_1, \cdots, \alpha_k):\alpha_1, \cdots, \alpha_k\in A,\ k\in N\}$$

of closed subsets of X, stated in Definition 5. By the d-paracompactness of X, there exists a dissectable family $\mathcal{W}(\alpha_1, \cdots, \alpha_k)$ of open subsets of X covering $C(\alpha_1, \cdots, \alpha_k)$ such that $\mathcal{W}(\alpha_1, \cdots, \alpha_k)$ is a weak refinement of $\{U_\lambda:\lambda\in\Lambda(\alpha_1, \cdots, \alpha_k)\}$. Without loss of generality, we can write $\mathcal{W}(\alpha_1, \cdots, \alpha_k)$ as the indexed family such that

$$\mathcal{W}(\alpha_1, \cdots, \alpha_k)=\{W_\lambda:\lambda\in\Lambda(\alpha_1, \cdots, \alpha_k)\},$$

where $W_\lambda\subset U_\lambda$ for each λ. For each $\alpha_1, \cdots, \alpha_k\in A$, $k\in N$, set

$$\mathcal{B}(\alpha_1, \cdots, \alpha_k)=\{W_\lambda\times V_\lambda:\lambda\in\Lambda(\alpha_1, \cdots, \alpha_k)\},$$

$$\mathcal{B}_k=\cup\{\mathcal{B}(\alpha_1, \cdots, \alpha_k):\alpha_1, \cdots, \alpha_k\in A\},$$

$$\mathcal{B}=\cup\{\mathcal{B}_k:k\in N\}.$$

Then we can show that $\mathcal{B}$ is a σ-dissectable refinement of $\mathcal{G}$. To show that $\mathcal{B}$ covers $X\times Y$, let $(x, y)\in X\times Y$.

Let $\{\alpha_i:i\in N\}$ be a sequence of A such that $\{F(\alpha_1)\cap\cdots\cap F(\alpha_k):k\in N\}$ is a local network at y in Y. For this sequence, we easily see that $X=\cup\{U(\alpha_1, \cdots, \alpha_k):k\in N\}$. This implies $X=\cup\{C(\alpha_1, \cdots, \alpha_k):k\in N\}$. Therefore $x\in C(\alpha_1, \cdots, \alpha_k)$ for some k. Since $\{W_\lambda:\lambda\in\bar{\Lambda}(\alpha_1, \cdots, \alpha_k)\}$ covers $C(\alpha_1, \cdots, \alpha_k)$,

$x\in W_\lambda$ for some $\lambda\in\Lambda(\alpha_1, \cdots, \alpha_k)$. Hence we have $(x, y)\in W_\lambda\times V_\lambda\in\mathcal{B}$. Let $k\in N$ be fixed. For each $\alpha_1, \cdots, \alpha_k\in A$, $\mathcal{B}(\alpha_1, \cdots, \alpha_k)$ is dissectable in $X\times Y$ because $\{V_\lambda : \lambda\in\Lambda(\alpha_1, \cdots, \alpha_k)\}$ is dissectable in Y by Fact 5, and we can use Lemma 7. Observe that

$$\cup\mathcal{B}(\alpha_1, \cdots, \alpha_k)\subset X\times(\cap\{H(\alpha_i) : i=1, \cdots, k\})$$

and that

$$\{X\times(\cap\{H(\alpha_i) : i=1, \cdots, k\}) : \alpha_1, \cdots, \alpha_k\in A\}$$

is a σ-point-finite in $X\times Y$. Hence by Lemma 6, $\mathcal{B}_k$ is σ-dissectable in $X\times Y$, which means that $\mathcal{B}$ is σ-dissectable in $X\times Y$. This completes the proof.

REMARK. The proof assures that the following is true: Let X be a P-space and Y a metacompact developable space. If X has the property that every family $\mathcal{U}$ of open subsets of X has a dissectable family $\mathcal{V}$ of X such that $\cup\mathcal{U}=\cup\mathcal{V}$ and $\mathcal{V}$ is a weak refinement of $\mathcal{U}$, then $X\times Y$ is d-paracompact.

The properties of Y used actually in the above proof is just that Y is an almost expandable space with a σ-discrete closed network $\mathcal{F}$ such that each $F\in\mathcal{F}$ has a dissectable outer base in Y, where a space Y is called *almost expandable* if for every locally finite family $\mathcal{H}$ of closed subsets of Y there exists a point-finite family $\{G(H) : H\in\mathcal{H}\}$ of open subsets of Y such that $H\subset G(H)$ for every $H\in\mathcal{H}$. But these properties give a sufficient condition for Y to be a metacompact developable space.

PROPOSITION 2. *A space Y is a metacompact developable space if and only if Y is an almost expandable σ-space with the property that every closed subset of Y has a dissectable outer base in Y.*

PROOF. The if part: Let $\cup\{\mathcal{F}_i : i\in N\}$ be a closed network for Y, where each $\mathcal{F}_i$ is discrete in Y. For each i, there exists a point-finite family $\{U(F) : F\in\mathcal{F}_i\}$ of open subsets of Y such that $F\subset U(F)$ for each $F\in\mathcal{F}_i$. Let $\mathcal{U}(F)$ be a dissectable outer base of F in Y such that $\cup\mathcal{U}(F)\subset U(F)$. Then by Lemma 6, $\cup\{\mathcal{U}(F) : F\in\cup_i\mathcal{F}_i)$ is a σ-dissectable base for Y. By Fact 1, Y is developable. Since an almost expandable σ-space is metacompact, Y has the required properties. The only if part is trivial.

COROLLARY. *A space X is metrizable if and only if X is a paracompact σ-space with the property that every closed subset of X has a dissectable outer base in X.*

We do not know whether a similar characterization is obtained for developable spaces, removing the terms metacompact and almost expandable from Proposition 2. That is, we do not know whether every σ-space (or even d-paracompact σ-space) with the same outer base property as in Proposition 2 is developable.

The metacompactness of Y cannot be dropped from Theorem 1. In fact, there exist a Lašnev space (i.e., a closed image of a metric space) and a non-metacompact developable space Y such that $X\times Y$ is not d-paracompact, as seen in Example 1. It is shown that a space which is dominated by paracompact σ-spaces is also a paracompact σ-space [M_2 and O]. But this is not true for the case of d-paracompact σ-spaces. To state the counterexample, we sketch the space $Y(\kappa)$. Lət κ be a cardinal number and let $Y(\kappa)$ be a set

$$Y(\kappa)=N\cup[0, \kappa),$$

which is topologized as follows: All points of N are isolated and basic neighborhoods of a point $\alpha\in[0, \kappa)$ are sets of the form:

$$\{\alpha\}\cup(N-F),$$

where F is a finite subset of N. The space $Y(\kappa)$ is a developable space. In fact, if $\{F_k : k\in N\}$ be the totality of finite subsets of N, then

$$\mathcal{U}_k=\{\{n\} : n\in F_k\}\cup[\{\alpha\}\cup(N-F_k) : \alpha\in[0, \kappa)], k\in N,$$

is a development for $Y(\kappa)$.

We should remark that this space $Y(\kappa)$ is just T_1, but unfortunately not Hausdorff. This leads that our examples stated here are T_1 but not Hausdorff since they contain $Y(\kappa)$ as the subspace. To simplify the examples, we prepare the following proposition:

PROPOSITION 3. *Let z be a point of a space Z with the uncountable character τ. If $\kappa\geqq\tau$, then the product space $Y(\kappa)\times Z$ is not d-paracompact.*

PROOF. Assume the contrary to get a contradiction. Let $\{W_\alpha : \alpha<\tau)$ be a local base at z in Z. It is easily observed that

$$\{(\alpha, z) : \alpha\in[0, \kappa)\}$$

is a discrete closed subset of $Y(\kappa)\times Z$ and that for each $\alpha<\tau$, $(\{\alpha\}\cup N)\times W_\alpha$ is an open neighborhood of (α, z) in $Y(\kappa)\times Z$ such that

$$(\alpha', z)\notin(\{\alpha\}\cup N)\times W_\alpha$$

if $\alpha\neq\alpha'$, α, $\alpha'<\tau$. By the assumption that $Y(\kappa)\times Z$ is d-paracompact and by

Lemma 1, there exist a family $\mathcal{V}=\{V_\alpha : \alpha<\tau\}$ of open subsets of $Y(\kappa)\times Z$ and the d-development $\{\mathcal{U}_n : n\in N\}$ for $\mathcal{V}$ such that

$$(\alpha, z)\in V_\alpha\subset(\{\alpha\}\cup N)\times W_\alpha, \qquad \alpha<\tau.$$

Let $\Pi : Y(\kappa)\times Z\to Z$ be the projection. We show that

$$\{\Pi(S((n, z), \mathcal{U}_k)) : n, k\in N\}$$

is a local base at z in Z. Suppose $\alpha<\tau$. We can take $n\in N$ such that $(n, z)\in V_\alpha$. Since $\{\mathcal{U}_n\}$ is the d-development for $\mathcal{V}$, there exists $k\in N$ such that $S((n, z), \mathcal{U}_k)\subset V_\alpha$. This implies

$$\Pi(S((n, z), \mathcal{U}_k))\subset W_\alpha,$$

which is a contradiction to the uncountability of the character τ of z in Z, This completes the proof.

For each $n\in N$, let S_n be the copy of the subspace

$$S=\{0\}\cup\{1, 1/2, 1/3, \cdots\}$$

of the real line with the usual topology and $S_n\cap S_m=\emptyset$ if $n\neq m$. We write by S_ω the quotient space obtained from $\bigoplus\{S_n : n\in N\}$ by identifying all limit points with a single point, which we denote by 0 again. Then S_ω is known to be a non-metrizable Lašnev space. Obviously 0 has a character c less than or equal to c, where c is the cardinality of the continuum.

Example 1. There exist a non-metacompact developable space X and a Lašnev space Y such that $Z=X\times Y$ is not d-paracompact.

Construction. We take $X=Y(c)$ and $Y=S_\omega$. Then by Proposition 3, $X\times Y$ is not d-paracompact. X is not metacompact because the open cover

$$\{\{\alpha\}\cup N : \alpha\in[0, c)\}$$

has no point-finite open refinement.

Example 2. There exists a non-d-paracompact σ-space which is dominated by d-paracompact σ-spaces.

Construction. Let $\rho : \bigoplus\{S_n : n\in N\}\to S_\omega$ be the natural mapping. For each $n\in N$, let $Z_n=Y(c)\times\rho(S_n)$. Since both $Y(c)$ and $\rho(S_n)$ are developable spaces, so is Z_n. Let Z be the same space $X\times Y$ as above. Then Z is a non-d-paracompact σ-space, and is easily seen to be dominated by $\{Z_n : n\in N\}$.

For the proof of next lemma, we introduced the following notations: Let $\mathcal{W}$ be an open cover of a space X. For each $W\in\mathcal{W}$, let

$$H(W)=W-\cup\{W'\in\mathcal{W}: W\neq W'\}.$$

Then it is easy to see that

$$\mathcal{H}(\mathcal{W})=\{H(W): W\in\mathcal{W}\}$$

is a discrete family of closed subsets of X. We define the subset $H(\mathcal{W})$ and the family $\mathcal{W}^{(1)}$ by

$$H(\mathcal{W})=\cup\mathcal{H}(\mathcal{W})$$

and

$$\mathcal{W}^{(1)}=\{W\in\mathcal{W}: H(W)\neq\emptyset\}.$$

If f is a closed mapping of a space X onto a space Y, for each open subset U of X we define an open subset $f^*(U)$ of Y by

$$f^*(U)=Y-f(X-U).$$

LEMMA 8. *Let $f: X\to Y$ be a perfect mapping. If X is a perfect d-paracompact space, then so is Y.*

PROOF. Obviously Y is perfect. By Fact 3, X is θ-refinable. Since it is well known that θ-refinability is preserved by perfect mappings, Y is θ-refinable. By Fact 3 again, it suffices to show that Y is d-expandable. Let $\langle\mathcal{F}, \mathcal{U}\rangle$ be a d-pair of families of Y, where

$$\mathcal{F}=\{F_\lambda: \lambda\in\Lambda\}, \qquad \mathcal{U}=\{U_\lambda: \lambda\in\Lambda\}.$$

Since X is d-expanadable, for the d-pair $\langle f^{-1}(\mathcal{F}), f^{-1}(\mathcal{U})\rangle$ there exists a dissectable family $\mathcal{V}=\{V_\lambda: \lambda\in\Lambda\}$ of X and the d-development for $\mathcal{V}$ in X such that

$$f^{-1}(F_\lambda)\subset V_\lambda\subset f^{-1}(U_\lambda), \qquad \lambda\in\Lambda.$$

By the same method as [Bu$_2$, Lemma 3.1], we can construct a sequence $\{\mathcal{W}_n: n\in N\}$ of families of open covers of X such that if $C\subset V_\lambda$ with C compact and $\lambda\in\Lambda$, then there exists $n\in N$ such that $C\cap H(\mathcal{W}_n)\neq\emptyset$ and

$$C\cap H(\mathcal{W}_n)\subset\mathcal{W}\subset V_\lambda$$

for some finite $\mathcal{W}\subset\mathcal{V}_n^{(1)}$. For each $t, s\in N^r$, $r\in N$, with $t=(t_1, \cdots, t_r)$, $s=(s_1, \cdots, s_r)$, define a family $\mathcal{W}(t, s)$ of subsets of X by the following:

$$\mathcal{W}(t, s)=\{W(\mathcal{W}_i', \cdots, \mathcal{W}_r'): \mathcal{W}_i'\subset\mathcal{W}_{s_i}^{(1)}$$

and

$$|\mathcal{W}_i'|=t_i \qquad \text{for each } i=1, \cdots, r\},$$

where

$$W(\mathcal{W}_1', \cdots, \mathcal{W}_r')=\cup[\cup\mathcal{W}_i'-\cup\{H(\mathcal{W}_{s_j}{}^{(1)}: j<i\} ; i\leqq r]$$

Then $\{\mathcal{W}(t, s): t, s\in N^r, r\in N\}$ has the following properties:

(1) $\mathcal{W}(t, s)$ is a family of open subsets of X.

(2) If $C\subset V_\lambda$ with C compact and $\lambda\in\Lambda$, then there exist $t, s\in N^r$, $r\in N$, such that C is contained in only one element $W_C\in\mathcal{W}(t, s)$ and $W_C\subset V_\lambda$.

We show (2). Let s_1 be the first number such that there exists a finite minimal subfamily $\mathcal{W}_1'$ of $\mathcal{W}_{s_1}{}^{(1)}$ such that

$$\emptyset\neq C\cap H(\mathcal{W}_{s_1})\subset\cup\mathcal{W}_1'\subset V_\lambda .$$

Let $|\mathcal{W}_1'|=t_1$ and

$$C_2=C-\cup\mathcal{W}_1' .$$

Let s_2 be the first number such that there exists a finite minimal subfamily $\mathcal{W}_2'$ of $\mathcal{W}_{s_2}{}^{(1)}$ such that

$$\emptyset\neq C_2\cap H(\mathcal{W}_{s_2})\subset\cup\mathcal{W}_2'\subset V_\lambda .$$

Let $|\mathcal{W}_2|=t_2$. Repeating this process and using the compactness of C, we can obtain two finite systems

$$s=(s_1, \cdots, s_r), \qquad t=(t_1, \cdots, t_r)\in N^r$$

for some $r\in N$ such that

$$C\subset W(\mathcal{W}_1', \cdots, \mathcal{W}_r')=W_C\subset V_\lambda \quad\text{and}\quad W_C\in\mathcal{W}(t, s).$$

Then W_C is seen to be the required one by the sεme argument as in [Bu, Lemmas 4.2 and 4.3]. Thus (2) is satisfied. Set

$$\mathcal{G}(t, s)=\{f^*(W): W\in\mathcal{W}(t, s)\}$$

for each $t, s\in N^r$, $r\in N$. It is easy to see that

$$\{\mathcal{G}(t, s): t, s\in N^r, r\in N\}$$

forms a d-quasidevelopment for $\{f^*(V_\lambda): \lambda\in\Lambda\}$ in Y. Since Y is perfect, Y is d-expanadable by Lemma 2. This completes the proof.

THEOREM 2. *Let f be a perfect mapping of a space X onto a space Y. If X is a d-paracompact σ-space, then so is Y.*

But closed mappings do not have this property.

EXAMPLE 3. There exists a closed mapping of d-paracompact σ-space $\hat{X}$ onto a non-d-paracompact σ-space Z.

CONSTRUCTION. We show that the same space Z as in Example 1 is the image of a d-paracompact σ-space $\hat{X}$ under a closed mapping. For each $n \in N$, let S_n be the same as in the preceding section to Example 1, and let Z_n' be the set $Y(c) \times S_n$. Set

$$\hat{X} = \cup \{Z_n' : n \in N\} \cup Y(c).$$

Topology of $\hat{X}$ is defined as follows: For each n, each point $p \in Z_n'$ has a neighborhood V in $\hat{X}$ if and only if $V \cap Z_n'$ is a neighborhood of p in Z_n'. Each $n \in N \subset Y(c)$ is isolated. For each $\alpha \in [0, c)$ has a neighborhood base

$$\{\{\alpha\} \cup (N-F) \cup (\cup \{(\{\alpha\} \cup (N-F)) \times W_k : k \geqq m\}) :$$
$$W_k \text{ is a neighborhood of } 0 \text{ in } S_k \text{ for each } k \geqq m,$$
$$F \text{ is a finite subset of } N \text{ and } m \in N\}.$$

It is easy to see that $Y(c)$ is a σ-discrete closed subset of $\hat{X}$ and each Z_n' is a developable clopen subspace of $\hat{X}$. Therefore Z_n', $n \in N$, has a σ-ciscrete closed (in $\hat{X}$) network $\mathcal{F}_n$ for Z_n'. Thus we have a σ-discrete closed network

$$\cup \{\mathcal{F}_n : n \in N\} \cup \{\{p\} : p \in Y(c)\}.$$

for $\hat{X}$, proving that $\hat{X}$ is a σ-space. To see that $\hat{X}$ is d-paracompact, let $\mathcal{U}$ be an open cover of X. For each $n \in N$, $\mathcal{V}_n = \mathcal{U} | Z_n'$ is a dissectable (in X) weak refinement of $\mathcal{U}$ because Z_n' is a clopen developable subspace of $\hat{X}$. For each $p \in Y(c)$, we take a basic neighborhood $V(p)$ in $\hat{X}$, as defined just above, such that $V(p) \subset U$ for some $U \in \mathcal{U}$. Since for each n the family $\{V(p) : p \in Y(c)\} | Z_n'$ is dissectable in $\hat{X}$ and since $q \notin V(p)$ if $p \neq q$ and $p, q \in Y(c) - N$, $\mathcal{V}_0 = \{V(p) : p \in Y(c)\}$ is dissectable in $\hat{X}$. Hence

$$\mathcal{V}_0 \cup \cup \{\mathcal{V}_n : n \in N\}$$

is a σ-didissectable refinement of $\mathcal{U}$. Let $g : \hat{X} \to Z$ be a mapping defined by

$$g | (\cup \{Z_n' : n \in N\}) = f$$

and

$$g(p) = (p, 0) \qquad \text{if } p \in Y(c),$$

where f is a natural mapping of $\oplus \{Z_n' : n \in N\}$ onto Z. g is obviously continuous and onto. We show that g is a closed mapping. For the purpose, it suffices to show that for each point $p \in Y(c)$ and each open set V of $\hat{X}$, if $g^{-1}((p, 0)) \subset V$, then there exists a neighborhood O of $(p, 0)$ in Z such that

$g^{-1}(O)\subset V$. If $p=n\in N$, then by the definition of the topology of $\hat{X}$, we can easily take neighborhoods W_k of 0 in S_k, $k\in N$, such that

$$g^{-1}((n, 0))\subset\cup\{\{n\}\times W_k : k\in N\}\cup\{n\}\subset V,$$

Let

$$O=f(\cup\{\{n\}\times W_k : k\in N\}).$$

Then O is a neighborhood of $(n, 0)$ in Z such that $g^{-1}(O)\subset V$. Let $p=\alpha\in[0, c)$. Then there exist a finite subset F of N and neighborhoods W_k of 0 in S_k, $k\in N$, such

$$g^{-1}((\alpha, 0))\subset\{\alpha\}\cup(N-F)\cup(\cup\{(\{\alpha\}\cup(N-F))\times W_k : k\in N\})\subset V,$$

Letting

$$O=f(\cup\{(\{\alpha\}\cup(N-F))\times W_k : k\in N\}),$$

we obtain a neighborhood O of $(\alpha, 0)$ in Z such that $g^{-1}(O)\subset V$. This completes the proof of the closedness of g.

We do not know whether the perfectness of X can be dropped from Lemma 8. That is, it is still open whether perfect mappings preserve d-paracompactness [C, 181p], [B_2, Question 1]. The next gives a sufficient condition for a closed image of a d-paracompact σ-space to be a d-paracompact σ-space.

THEOREM 3. *Let $f: X\to Y$ be a closed mapping and let Y be a first countable space. If X is a d-paracompact σ-space, then so is Y.*

PROOF. Since Y is obviously a σ-space, we show that Y is d-expandable. Let $\langle\mathcal{F}, \mathcal{U}\rangle$ be a d-pair of families of Y. Then for the d-pair $\langle f^{-1}(\mathcal{F}), f^{-1}(\mathcal{U})\rangle$ of families of a d-paracompact space X, by Lemma 1, there exist families

$$\mathcal{V}=\{V(F): F\in\mathcal{F}\},\quad \mathcal{H}=\{H_\alpha : \alpha\in A\},\quad \mathcal{W}=\{W_\alpha : \alpha\in A\}$$

of subsets of X satisfying the following:

(1) For each $F\in\mathcal{F}$, $V(F)$ is an open subset of X such that

$$f^{-1}(F)\subset V(F)\subset f^{-1}(U(F)).$$

(2) $A=\cup\{A_n : n\in N\}$ and for each n, $A_n\subset A_{n+1}$,

$\mathcal{H}_n=\{H_\alpha : \alpha\in A_n\}$ is a locally finite family of closed subsets and $\mathcal{W}_n=\{W_\alpha : \alpha\in A_n\}$ a family of open subsets of X such that $H_\alpha\subset W_\alpha$, $\alpha\in A_n$.

(3) For each $F\in\mathcal{F}$ and each point $p\in X$, if $p\in V(F)$, then there exists $\alpha\in A$ such that

$$p\in H_\alpha\subset W_\alpha\subset V(F).$$

Moreover, since X is a σ-space, without loss of generality we can assume that

(4) $\{H_\alpha : \alpha \in A\}$ satisfies that for each $F \in \mathcal{F}$ and each poitn $p \in V(F)$, the family $\{H_\alpha : p \in H_\alpha \subset W_\alpha \subset V(F),\ \alpha \in A\}$ is a local network at p in X.

For each $n \in N$, let Y_n' be the set of all points $y \in Y$ such that $\operatorname{ord}(y, f(\mathcal{H}_n))$ is infinite. Then each Y_n' is a σ-discrete closed subset of Y because Y is a first countable space and $f(\mathcal{H}_n)$ is a hereditarily closure-preserving family of closed subsets of Y. Set

$$Y_1 = \cup\{Y_n' : n \in N\}, \qquad Y_0 = Y - Y_1 .$$

For each n, let Δ_n be the totality of finite subsets δ of A_n such that $H(\delta) \subset \operatorname{Int} W(\delta)$, where

$$H(\delta) = \cap\{f(H_\alpha) : \alpha \in \delta\},$$

$$W(\delta) = \cup\{f(W_\alpha) : \alpha \in \delta\}.$$

Claim 1: For each point $p \in Y_0$ and each $F \in \mathcal{F}$, if $p \in f^*(V(F))$, then there exists $\delta \in \Delta_n$, $n \in N$, such that

$$p \in H(\delta) \subset \operatorname{Int} W(\delta) \subset U(F) .$$

Proof of the claim: Let $p \in Y_0$ and for each n, let

$$\delta_n = \{\alpha \in A_n : f^{-1}(p) \cap H_\alpha \neq \emptyset \quad \text{and} \quad W_\alpha \subset V(F)\}.$$

Then obviously $p \in H(\delta_n) \subset W(\delta_n) \subset U(F)$ for each n. First we show the following:

(5) $$p \in \operatorname{Int} W(\delta_n) \qquad \text{for some } n.$$

Throughout the proof of the theorem, for each $y \in Y$ let $\{O_n(y) : n \in N\}$ be the decreasing local base of y in Y. Assume the contrary to (5). Then

$$O_n(p) - W(\delta_n) \neq \emptyset, \qquad n \in N.$$

Take a sequence $\{p_n : n \in N\}$ of points of Y such that

$$p_n \in O_n(p) - W(\delta_n), \qquad n \in N.$$

Since f is a closed mapping, $(f^{-1}(p_n) : n \in N\}$ clusters at a point of $f^{-1}(p)$. Hence by (3) there exists $\alpha \in \delta_n$, $n \in N$ such that $p \in f(H_\alpha)$ and $f(W_\alpha)$ contains infinitely many p_n. But this is a contradiction to the fact that $p_k \notin W(\delta_n)$, $k \geqq n$. Thus we have $p \in \operatorname{Int} W(\delta_n)$ for some n.

Next, we show the following:

(6) $$H(\delta_n) - \{p\} \subset \operatorname{Int} W(\delta_n) \qquad \text{for some } n.$$

Assume the contrary. If $H(\delta_n)-\{p\}-\mathrm{Int}\, W(\delta_n)$ is finite for some n, then by (4) we easily have

$$H(\delta_m)-\{p\}\subset \mathrm{Int}\, W(\delta_m)$$

for some $m>n$. Therefore we can assume that

$$H(\delta_n)-\{p\}-\mathrm{Int}\, W(\delta_n)$$

is infinite for each n. Take a sequence $\{p_n : n\in N\}$ of points of Y such that for each n

$$p_n\in H(\delta_n)-\{p\}-\mathrm{Int}\, W(\delta_n)-\{p_1, \cdots, p_{n-1}\}.$$

Since Y is a Fréchet space, for each n there exists a convergent sequence $Z(n)$ to p_n in Y such that

$$Z(n)\cap W(\delta_n)=\emptyset .$$

Since p has the decreasing local base $\{O_n(p): n\in N\}$ in Y, by (4) $p_n\to p$ as $n\to\infty$. Therefore by Fréchet-ness of Y, we can take a sequence $Z\subset\cup\{Z(n): n\in N\}$ such that $Z\to p$. Since $p_n\neq p$, $n\in N$, $Z\cap Z(n)\neq\emptyset$ for infinitely many n. The closedness of f implies that there exists $\alpha\in\delta_n$, $n\in N$, such that $f(W_\alpha)$ contains infinitely many points of Z, but this is a contradiction, proving (6).

We observe by (2) that $\{H(\delta_n): n\in N\}$, $\{W(\delta_n): n\in N\}$ are decreasing, increasing, respectively, families of subsets of Y. By (5) and (6), we can conclude Claim 1.

Claim 2: There exists a pair collection

$$\mathcal{P}_1'=\{(F_\beta, U_\beta): \beta\in B_1\}$$

of Y satisfying the following conditions:

(7) $\{F_\beta : \beta\in B_1\}$ is a σ-discrete family of closed subsets of Y and for each $\beta\in B_1$, U_β is an open subset of Y such that $F_\beta\subset U_\beta$.

(8) For each $p\in Y$ and each $F\in\mathcal{F}$, if $p\in f^*(V(F))$, then there exists $\beta\in B_1$ such that

$$p\in F_\beta\subset U_\beta\subset U(F).$$

The proof of the claim: For each $n, m\in N$, let $\mathcal{P}_{nm}$ be the pair collection of Y

$$\mathcal{P}_{nm}=\{(\{y\}, O_m(y)): y\in Y_n'\}$$

and set

$$\mathcal{P}'=\cup\{\mathcal{P}_{nm} : n, m\in N\}.$$

Obviously $\mathcal{P}'$ satisfies (7) and (8) for each point $p\in Y_1$. Using the fact that Y

is semistratifiable, by the method of Fact 4, from the closure-preserving family $\{H(\delta):\delta\in\Delta_n\}$ of closed subsets of Y, we can construct a σ-discrete closed cover $\{K_\lambda:\lambda\in\Lambda_n\}$ of Y such that $K_\lambda\cap H(\delta)\neq\emptyset$, $\lambda\in\Lambda_n$ and $\delta\in\Delta_n$ imply $K_\lambda\subset H(\delta)$.

Suppose that $\lambda\in\Lambda_n$ has the property that

$$\Delta_n(\lambda)=\{\delta\in\Delta_n: K_\lambda\subset H(\delta)\}$$

is finite. Take an open subset G_λ of Y such that

$$K_\lambda\subset G_\lambda\subset\cap\{\operatorname{Int} W(\delta):\delta\in\Delta_n(\lambda)\}.$$

Write

$$\begin{aligned}\mathcal{P}'\cup\{(K_\lambda, G_\lambda):\lambda\in\Lambda_n \text{ with } \Delta_n(\lambda) \text{ finite}, n\in N\}\\ =\mathcal{P}_1'\\ =\{(F_\beta, U_\beta):\beta\in B_1\}.\end{aligned}$$

Then by Claim 1, it is easy to see that $\mathcal{P}_1'$ satisfies the conditions (7) and (8). This proves Claim 2.

Now, write $B_1=\cup\{B_{1n}:n\in N\}$, where for each n $\{F_\beta:\beta\in B_{1n}\}$ is discrete in Y. We apply countably many times the arguments of Claims 1 and 2 to the countable d-pairs

$$\langle\{F_\beta:\beta\in B_{1n}\}, \{U_\beta:\beta\in B_{1n}\}\rangle, \qquad n\in N,$$

of families of Y. Consequently, we get pair collections

$$\mathcal{P}_1=\{(F_\beta, V_\beta):\beta\in B_1\}$$

and

$$\mathcal{P}_2'=\{(F_\beta, U_\beta):\beta\in B_2\}$$

of Y satisfying the following conditions:

(9) For each $\beta\in B_1$, V_β is an open subset of Y such that $F_\beta\subset V_\beta\subset U_\beta$.

(10) $\{F_\beta:\beta\in B_2\}$ is a σ-discrete family of closed subsets of Y and for each $\beta\in B_2$, U_β is an open subset of Y such that $F_\beta\subset U_\beta$,

(11) For each point $p\in Y$ and each $\beta_1\in B_1$, if $p\in V_{\beta_1}$, there exists $\beta_2\in B_2$ such that

$$p\in F_{\beta_2}\subset U_{\beta_2}\subset U_{\beta_1}.$$

For each $F\in\mathcal{F}$, let $W_1(F)=f^*(V(F))$. Then $W_1(F)$ is an open subset of Y such that

$$F\subset W_1(F)\subset U(F), \qquad F\in\mathcal{F}.$$

For each $F\in\mathcal{F}$, set

$$W_2(F)=W_1(F)\cup(\cup\{V_\beta : \beta\in B_1, \\ F_\beta\cap W_1(F)\neq\emptyset \text{ and } U_\beta\subset U(F)\}.$$

Then $W_2(F)$ is an open subset of Y such that

$$F\subset W_1(F)\subset W_2(F)\subset U(F), \qquad F\in\mathcal{F}.$$

Moreover, by (8) and (9), it is obvious that:

(12) For each point $p\in Y$ and each $F\in\mathcal{F}$, if $p\in W_1(F)$, then there exists $\beta\in B_1$ such that

$$p\in F_\beta\subset V_\beta\subset W_2(F).$$

From the definition of $W_2(F)$ and (11) it follows that:

(13) For each point $p\in Y$ and each $F\in\mathcal{F}$, if $p\in W_2(F)$, then there exists $\beta\in B_2$ such that

$$p\in F_\beta\subset U_\beta\subset U(F).$$

Again, we apply countably many times the arguments of Claims 1 and 2 to the countable d-pairs contained in $\mathcal{P}_2'$ and get two pair collections

$$\mathcal{P}_2=\{(F_\beta, V_\beta): \beta\in B_2\}$$

and

$$\mathcal{P}_3'=\{(F_\beta, U_\beta): \beta\in B_3\}$$

of Y satisfying the conditions corresponding to (9). (10) and (11) with B_1, B_2 replaced by B_2, B_3, respectively. For each $F\in\mathcal{F}$, let

$$W_3(F)=W_2(F)\cup(\cup\{V_\beta : \beta\in B_2, F_\beta\cap W_2(F)\neq\emptyset \\ \text{and } U_\beta\subset U(F)\}).$$

Then for each $F\in\mathcal{F}$, $W_3(F)$ is an open subset of Y suɔh that

$$F\subset W_1(F)\subset W_2(F)\subset W_3(F)\subset U(F).$$

It is easily seen that:

(14) For each point $p\in Y$ and each $F\in\mathcal{F}$, if $p\in W_2(F)$, then there exists $\beta\in B_2$ such that

$$p\in F_\beta\subset V_\beta\subset W_3(F).$$

Repeating these processes, we can easily settle the following claim:

Claim 3: For each $F\in\mathcal{F}$, there exists a sequence $\{W_n(F): n\in N\}$ of open subsets of Y such that

$$F\subset W_1(F)\subset W_2(F)\subset \cdots \subset U(F)$$

and at the same time there exists a pair collection

$$\mathcal{P}_n=\{(F_\beta, V_\beta): \beta\in B_n\}$$

of Y satisfying the following conditions:

(15) For each point p by Y, each $F\in\mathcal{F}$ and each $n\in N$, if $p\in W_n(F)$, then there exists $\beta\in B_n$ such that

$$p\in F_\beta\subset V_\beta\subset W_{n+1}(F).$$

Set

$$W(F)=\cup\{W_n(F): n\in N\}, \qquad F\in\mathcal{F}$$

and

$$\mathcal{P}=\cup\{\mathcal{P}_n : n\in N\}$$
$$=\{(F_\beta, V_\beta): \beta\in B\},$$

where $B=\cup\{B_n : n\in N\}$. Then obviously, for each $F\in\mathcal{F}$, $W(F)$ is an open subset of Y such that $F\subset W(F)\subset U(F)$. By the construction, it is true that for each point $p\in Y$ and each $F\in\mathcal{F}$, if $p\in W(F)$, then there exists $\beta\in B$ such that

$$p\in F_\beta\subset V_\beta\subset W(F).$$

The family $\{F_\beta : \beta\in B\}$ is a σ-discrete one of closed subsets of Y. Therefore by Lemma 1, Y is d-expandable. This completes the proof of the theorem.

PROPOSITION 4. *Let $f: X\to Y$ be a closed mapping and Y a first countable space. If X is a d-paracompact semistratifiable space having the property that every closed subset of X has a dissectable outer base in X, then every closed subset of Y has a dissectable outer base in Y.*

PROOF. We proceed referring to the proof just above. Let M be a closed subset of Y. Then by the assumption $f^{-1}(M)$ has a dissectable outer base $\mathcal{V}$ in X. By the proof of Lemma 1, there exist families

$$\mathcal{H}=\{H_\alpha : \alpha\in A\}, \qquad \mathcal{W}=\{W_\alpha : \alpha\in A\}$$

of subsets of Y satisfying the following (3)′ besides (2) in the proof above:

(3)′ For each $V\in\mathcal{V}$ and point p of X, if $p\in V$, then there exists $\alpha\in A$ such that

$$p\in H_\alpha\subset W_\alpha\subset V.$$

Let Y_n', Y_1, Y_0 are the same as above. For each n, let Δ_n be the totality of finit subsets δ of A_n such that $H(\delta)\cap\operatorname{Int} W(\delta)\neq\emptyset$, where

$$H(\delta)=\cap\{f(H_\alpha): \alpha\in\delta\},$$

$$W(\delta)=\cup\{f(W_\alpha): \alpha\in\delta\}.$$

By the same argument as in the proof of (5) above, we can show the following:

(4) For each $p\in Y_0$ and each $V\in\mathcal{V}$, if $p\in f^*(V)$, then there exists $\delta\in\Delta_n$, $n\in N$, such that

$$p\in H(\delta)\cap \mathrm{Int}\, W(\delta)\subset f(V).$$

Claim 1: There exists a pair collection

$$\mathcal{P}_1'=\{(F_\beta, U_\beta): \beta\in B_1\}$$

of Y satisfying the following conditions:

(5) $\{F_\beta: \beta\in B_1\}$ is a σ-discrete family of closed subsets of Y and if $\beta\in B_1$, then U_β is an open subset of Y such that $F_\beta\subset U_\beta$.

(6) For each point $p\in Y$ and each $V\in\mathcal{V}$, if $p\in f^*(V)$, then there exists $\beta\in B_1$ such that

$$p\in F_\beta\subset U_\beta\subset f(V).$$

The proof of the claim: Since Y is semistratifiable, for each $\delta\in\Delta_n$, $n\in N$, $\mathrm{Int}\, W(\delta)$ is a countable union of closed subsets $F_m(\delta)$, $m\in N$. Note that

$$\mathcal{H}(n, m)=\{H(\delta)\cap F_m(\delta): \delta\in\Delta_n\}$$

is a closure-preserving family of closed subsets of Y. Therefore by the method of Fact 4, from $\mathcal{H}(n, m)$. $n, m\in N$, we can construct σ-discrete closed covers $\{K_\lambda: \lambda\in\Lambda_{nm}\}$, of Y, $n, m\in N$. For each $\lambda\in\Lambda_{nm}$, $n, m\in N$ with the property that

$$\Delta_{nm}(\lambda)=\{\delta\in\Delta_{nm}: K_\lambda\subset F_m(\delta)\}$$

is finite, take an open subset G_λ of Y such that

$$K_\lambda\subset G_\lambda\subset\cap\{\mathrm{Int}\, W(\delta): \delta\in\Delta_{nm}(\lambda)\}.$$

Let $\mathcal{P}'$ be the same pair collection of Y as in the proof of Claim 2 above. Then we can easily see that

$$\mathcal{P}_1'=\mathcal{P}'\cup\{(K_\lambda, G_\lambda): \lambda\in\cup\{\Lambda_{nm}: n, m\in N\}\}$$

is the required pair collection of Y.

Using the d-paracompactness and semistratifiability of Y and applying the argument of the proof above, we can get from $\mathcal{P}_1'=\{(F_\beta, U_\beta): \beta\in B_1\}$ two pair collections

$$\mathcal{P}_1=\{(F_\beta, V_\beta): \beta\in B_1\}$$

and

$$\mathcal{P}_2'=\{(F_\beta, U_\beta): \beta\in B_2\}$$

of V satisfying the same conditions (9), (10) and (11) of the proof above. For each $V\in\mathcal{V}$, set $W_1(V)=f^*(V)$ and

$$W_2(V)=W_1(V)\cup(\cup\{V_\beta: \beta\in B_1,\ F_\beta\cap W_1(V)\neq\emptyset \text{ and } U_\beta\subset f(V)\}),$$

Then for each $V\in\mathcal{V}$, $W_1(V)$, $W_2(V)$ are open subsets of Y such that

$$M\subset W_1(V)\subset W_2(V)\subset f(V)$$

and it is obvious that if $p\in W_1(V)$, then there exists $\beta\in B_1$ such that

$$p\in F_\beta\subset V_\beta\subset W_2(V).$$

Repeating these processes, we can get a sequence $\{W_n(F): n\in N\}$, $V\in\mathcal{V}$, of open subsets of Y such that

$$M\subset W_1(V)\subset W_2(V)\subset\cdots\subset f(V)$$

for each $V\in\mathcal{V}$ and at the same time there exists a pair collection

$$\mathcal{P}_n=\{(F_\beta, V_\beta): \beta\in B_n\}$$

of Y such that

(7) $\{F_\beta: \beta\in B_n\}$ is a σ-discrete family of closed subsets of Y and if $\beta\in B_n$, V_β is an open subset of Y such that $F_\beta\subset V_\beta$.

(8) For each point $p\in Y$, each $V\in\mathcal{V}$ and each $n\in N$, if $p\in W_n(V)$, then there exists $\beta\in B_n$ such that

$$p\in F_\beta\subset V_\beta\subset W_{n+1}(V).$$

For each $V\in\mathcal{V}$, set

$$W(V)=\cup\{W_n(V): n\in N\}.$$

Then it is easy to see that $\{W(V): V\in\mathcal{V}\}$ is an outer base of M in Y. By (8) and the proof of Lemma 1, it is dissectable in Y. This completes the proof.

The above proof assures the following: Let $f: X\to Y$ be a closed mapping of a d-paracompact semistratifiable space X onto a first countable space Y. If X has the property that every discrete family $\mathcal{F}$ of closed subsets of X has a dissectable family $\cup\{\mathcal{W}(F): F\in\mathcal{F}\}$ of X such that each $\mathcal{W}(F)$, $F\in\mathcal{F}$, is an outer base of F in X, then Y has the same property. On the other hand, it is obvious that a space X is developable if and only if X is a d-paracompact σ-space with this property.

From both observations, we can get the following as the corollary to Proposition 4:

COROLLARY. *Let $f: X \to Y$ be a closed mapping of a developable space X onto a space Y. Then Y is developable if and only if Y is first countable.*

This corresponds to the well known Hanai-Morita-Stone theorem that a closed image of a metric space is metrizable if and only if it is first countable.

THEOREM 4. *If X is a d-paracompact σ-space and $X_0 \subset X$, then X_0 is also a d-paracompact σ-space.*

PROOF. Let $\mathcal{V}$ be an open cover of X_0. We take a family $\mathcal{V}'$ of open subsets of X such that $\mathcal{V}|X_0 = \mathcal{V}$. Let $\mathcal{F}$ be a σ-discrete closed network for X. For each $F \in \mathcal{F}$, we choose $U(F) \in \mathcal{V}'$ such that $F \subset U(F)$, if possible. Since X is d-paracompact, there exists an open set $V(F)$ of X such that $F \subset V(F) \subset U(F)$ and such that $\{V(F): F \in \mathcal{F}\}$ is σ-dissectable in X. Then $\{V(F): F \in \mathcal{F}\}|X_0$ is a σ-dissectable refinement of $\mathcal{V}$. This proves the d-paracompactness of X_0.

In the above, the condition "σ-space" cannot be omitted [B_2, 23p].

3. The comparison with s-paracompact spaces

A space X is *semimetrizable* if there exists a distance function $d: X \times X \to \boldsymbol{R}$ such that $d(x, y) \geqq 0$, $d(x, y) = d(y, x)$, $d(x, y) = 0$ if and only if $x = y$ for all x, $y \in X$ and $\bar{A} = \{x \in X: d(x, A) = 0\}$ for each $A \subset X$, where

$$d(x, A) = \inf\{d(x, y): y \in A\}.$$

It is known that a space X is semimetrizable if and only if X is a first countable, semistratifiable space [Gr, Theorem 9.8]. Brandenburg called a space *s-paracompact* if for every open cover $\mathcal{A}$ of X, there exists an $\mathcal{A}$-mapping of X onto a semimetrizable space. Since every developable space is semimetrizable, every d-paracompact space is s-paracompact. He proposed the question whether every semimetrizable space is d-paracompact [B_2, Question 2]. If the positive answer would be given, both of d-paracompact spaces and s-paracompact spaces coincide. But we can give the negative answer to it. Thus, we can conclude that both are different.

To state Example 4, we propare the following:

PROPOSITION 5. *Let Z be a space such that Z has the weight and cardinality*

$\leqq \tau$. *If* $Y(\kappa)\times Z$ *is d-paracompact for some* $\kappa \geqq \tau$, *then* Z *is a developable space.*

PROOF. Let Z bas a base $\mathcal{B}$ with $|\mathcal{B}|\leqq\tau$. Let $\{(p_\alpha, O_\alpha): \alpha<\tau_1\}$ be the totality of the pairs (p_α, O_α) with $p_\alpha\in O_\alpha\in\mathcal{B}$, where $\tau_1\leqq\tau$. Note that

$$\{(\alpha, p_\alpha): \alpha<\tau_1\}$$

is a discrete closed subset of $Y(\kappa)\times Z$, and that $(\{\alpha\}\cup N(\times O_\alpha$ is an open neighborhood of (α, p_α) in $Y(\kappa)\times Z$ such that

$$(\beta, p_\beta)\notin(\{\alpha\}\cup N)\times O_\alpha,$$

if $\alpha\neq\beta$. Since $Y(\kappa)\times Z$ is d-paracompact, by Lemma 1 there exist a family $\mathcal{W}=\{W_\alpha: \alpha<\tau_1\}$ of open subsets of $Y(\kappa)\times Z$ and the d-development $\{\mathcal{U}_n: n\in N\}$ for $\mathcal{W}$ in $Y(\kappa)\times Z$ such that

$$(*) \qquad (\alpha, p_\alpha)\in W_\alpha\subset(\{\alpha\}\cup N)\times O_\alpha$$

for each $\alpha<\tau_1$.
Let $\pi: Y(\kappa)\times Z\to Z$ be the projection. For each $n, m\in N$, let

$$\mathcal{U}_{nm}=\pi(\mathcal{U}_n|\{m\}\times Z).$$

By (*), we can easily show that $\{\mathcal{U}_{nm}: n, m\in N\}$ is a development for Z. This completes the proof.

COROLLARY. *For a space* Z, *the following are equivalent:*
(1) Z *is a developable space.*
(2) $Z\times Y$ *is d-paracompact for every developable space* Y.

PROOF. (1)→(2) is obvious from the facts that the product of two developable and that every developable space is d-paracompact. (2)→(1) follows from the above proposition and the fact that $Y(\kappa)$ is developable.

EXAMPLE 4. There exists a semimetrizable space which is not d-paracompact.

CONSTRUCTION. Let $X=\boldsymbol{R}^2$ be the space with the bowtie topology. For each point $p=(x, y)\in X$, $\{B(p, \varepsilon, \delta): \varepsilon, \delta>0\}$ is a neighborhood base of p in X, where

$$B(p, \varepsilon, \delta)=\{p\}\cup\{(x', y')\in X: 0<|x'-x|<\varepsilon \text{ and } |(y'-y)/(x'-x)|<\delta\}.$$

Then X is a semimetrizable, non-developable space [Gr, Eemple 9.10]. Let

$Z=Y(c)\times X$. Then by Proposition 6, Z is not d-paracompact. But Z is semimetrizable because semimetrizable spaces have the countably productive property.

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