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AN ACCESSIBILITY PROOF OF ORDINAL DIAGRAMS IN INTUITIONISTIC THEORIES FOR ITERATED INDUCTIVE DEFINITIONS

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Let $(I, \prec)$ be a non-empty well-ordered system with the least element 0, and $\tilde{I}$ be $I \cup \{\infty\}$ with the largest element ∞. Let A be a non-empty well-ordered set. Then $O(I, A)$ denotes the system of ordinal diagrams (o.d.'s) based on I and A. (cf. [9, § 26].) The accessibility proof for $O(I, A)$ in [9, pp. 298-309] shows that *every* o.d. from $O(I, A)$ is accessible with respect to $<_i$ for *every* i in $\tilde{I}$.

The central notions in this proof are i-fans and i-accessibility for i in $\tilde{I}$. Roughly speaking, an o.d. μ is an i-fan if for every $j \prec i$ and every j-section ν of μ, ν is j-accessible, and an o.d. is i-accessible if it is accessible in i-fans with respect to $<_i$.

Consider the case when the order type of $(I, \prec)$ is a successor ordinal $\xi+1$. If we formalize this accessibility proof for $O(\xi+1, 1)$ $(=O(I, 1))$ naturally, then this proof can be done in the intuitionistic theory $\mathrm{ID}^i_{\xi+1}$ for $\xi+1$-times iterated inductive definitions.

The purpose of this paper is to show the following fact: the accessibility of *each* o.d. from $O(\xi+1, 1)$ with respect to $<_0$ is derivable in ID^i_ξ. (Theorem)

In the case when ξ equals ω, this theorem will complement the consistency proof in [1] in the following sense. We will give in [1] a consistency proof for the subsystem $(\Pi^1_1-\mathrm{CA})+(\mathrm{BI})$ of classical analysis by the accessibility of $O(\omega+1, 1)$ with respect to $<_0$. It follows from the well-known equivalence between the classical version ID_ω of ID^i_ω and $(\Pi^1_1-\mathrm{CA})+(\mathrm{BI})$ that this consistency proof is optimal.

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Let $\prec$ be a primitive recursive well-ordering with the least element 0 and the largest element ξ, and I be the primitive recursive domain of $\prec$. Let $\lambda x.\ x\oplus 1$ and $\lambda x.\ x\ominus 1$ be primitive recursive successor and predecessor function with respect to $\prec$, respectively. And we will assume throughout this paper that the above facts except the well-orderedness of $\prec$ are all derivable in the primitive recursive arithmetic PRA, that is to say, we will assume that the following formulae are all derivable in PRA:

$$x\prec y \longrightarrow I(x)\wedge I(y)\,,$$

$$I(x) \longrightarrow \neg(x\prec x)\,,$$

$$x\prec y\wedge y\prec z \longrightarrow x\prec z\,,$$

$$I(x)\wedge I(y) \longrightarrow x\prec y\vee x=y\vee y\prec x\,,$$

$$I(0),\ I(x) \longrightarrow 0\leqq x\,, \qquad (x\leqq y:=x\prec y\vee x=y)$$

$$I(x) \longrightarrow x\leqq\xi\,,$$

$$I(x) \longrightarrow x\leqq x\oplus 1\,,$$

$$x\prec\xi \longrightarrow x\prec x\oplus 1\,,$$

$$y\prec x \longrightarrow y\oplus 1\leqq x\,,$$

$$I(x) \longrightarrow x\ominus 1\leqq x\,,$$

$$x\prec\xi \longrightarrow (x\oplus 1)\ominus 1=x\,,$$

$$x\ominus 1\prec x \longrightarrow x=(x\ominus 1)\oplus 1\,.$$

Then the following formulae are also derivable in PRA:

$$x\prec\xi \longrightarrow (y\prec x\oplus 1 \longleftrightarrow y\leqq x)\,,$$

$$x\prec\xi \longrightarrow (x\leqq y\leqq x\oplus 1 \longrightarrow y=x\vee y=x\oplus 1)\,,$$

$$y\prec\xi \longrightarrow (y\oplus 1=x \longrightarrow x\ominus 1\prec x)\,.$$

Further let Suc and Lim be unary predicate constants with their defining axioms:

$$Suc(x) \longleftrightarrow x\ominus 1\prec x\,,$$

$$Lim(x) \longleftrightarrow I(x)\wedge x\neq 0\wedge \neg Suc(x)\,.$$

Then the following formulae are also derivable in PRA:

$$I(x) \longrightarrow (x=0\vee Suc(x)\vee Lim(x))\,.$$

$$Lim(x)\wedge y\prec x \longrightarrow y\oplus 1\prec x\,.$$

Next, we will consider the system of o.d.'s $O^*(I, 1)$. $O^*(I, 1)$ is an inessential modification of $O(I, 1)$. In contrast with $O(I, 1)$, $O^*(I, 1)$ has an identity 0 with respect to $\#$. For the precise definition of $O^*(I, 1)$, we refer to Levitz [7].

We will assume an arithmetization of the o.d.'s in $O^*(I, 1)$. Thus we have the following predicate constants for primitive recursive predicates:

'$*$ is an o.d.', '$*_1$ is a component of $*_2$',
$*_1 \equiv *_2$ for '$*_1$, $*_2$ are o.d.'s and $*_1$ is equal to $*_2$.',
$*_1 \subset_{*_3} *_2$ for '$*_1$, $*_2$ are o.d.'s, $*_3 \leqq \xi$ and $*_1$ is a $*_3$-section of $*_2$.',
$*_1 <_{*_3} *_2$ for '$*_1$, $*_2$ are o.d.'s, $*_3 \leqq \xi$ and $*_1$ is smaller than $*_2$ with respect to $<_{*_3}$.'.

In the following, we will employ the following syntactical variables:

i, j, k vary through the elements in I,
μ, ν, ρ, λ vary through o.d.'s.

Following Kreisel [6], we will define the notion of i-accessibility for $i \prec \xi$ in $\mathrm{ID}^i_\xi(\mathfrak{A})$ for some positive operator form $\mathfrak{A}$. Let $\mathfrak{A}(X, Y, i, \mu)$ be the following positive operator form:

$$\mathfrak{F}(i, \mu, Y) \wedge \forall \nu <_i \mu (\mathfrak{F}(i, \nu, Y) \longrightarrow X(\nu))$$

where $\mathfrak{F}(i, \mu, Y)$ is the formula $\forall k \prec i \, \forall \rho \subset_k \mu \, Y(k, \rho)$.

Let $\mathrm{Prog}[X, R, Y]$ be the formula

$$\forall \mu (X(\mu) \wedge \forall \nu (R(\nu, \mu) \wedge X(\nu) \longrightarrow Y(\nu)) \longrightarrow Y(\mu)).$$

If we write A for the set constant $P^{\mathfrak{A}}$, and $F_i(\mu)$ for $\forall j \prec i \, \forall \nu \subset_j \mu \, A_j(\nu)$, then the axioms $(P^{\mathfrak{A}}.1)_\xi$ and $(P^{\mathfrak{A}}.2)_\xi$ in [4, p. 307] become the following $(A.1)_\xi$ and $(A.2)_\xi$, respectively:

$(A.1)_\xi$ $\forall i \prec \xi \, \mathrm{Prog}[F_i, <_i, A_i]$,
$(A.2)_\xi$ $\forall i \prec \xi \, (\mathrm{Prog}[F_i, <_i, Q] \longrightarrow A_i \subseteqq Q)$,

for each formula Q in $\mathrm{ID}^i_\xi(\mathfrak{A})$.

And further $\mathrm{ID}^i_\xi(\mathfrak{A})$ has the following $(\mathrm{TI})_\xi$ going beyond the Heyting's arithmetic:

$(\mathrm{TI})_\xi$ $\forall i \prec \xi \, (\forall j \prec i Q(j) \longrightarrow Q(i)) \longrightarrow \forall i \prec \xi Q(i)$

for each formula Q in $\mathrm{ID}^i_\xi(\mathfrak{A})$.

The intended meanings of $A_i(\mu)$ and $F_j(\nu)$ are that μ is i-accessible and ν is a j-fan in the sense of introduction.

The following proposition is easily verified:

PROPOSITION 1. *The following formulae are all derivable in* $\mathrm{ID}_{\xi}^{i}(\mathfrak{A})$:

1.1. $\forall i \prec \xi\,(A_i \subseteqq F_i)$;

1.2. $\forall i \prec \xi\, \forall \mu\,(A_i(\mu) \rightarrow \forall \nu <_i \mu (F_i(\nu) \rightarrow A_i(\nu)))$;

1.3. $\forall i \preceq \xi\, \forall \mu\, \forall \nu\,(\mu \equiv \nu \wedge F_i(\mu) \rightarrow F_i(\nu))$;

1.4. $\forall i \prec \xi\, \forall \mu\, \forall \nu\,(\mu \equiv \nu \wedge A_i(\mu) \rightarrow A_i(\nu))$;

1.5. $\forall i \prec \xi\, \forall \mu\,(\forall \nu($'$\nu$ *is a component of* μ'$\rightarrow A_i(\nu)) \rightarrow A_i(\mu))$.

LEMMA 2. *Let* $\cap_{k \prec i} A_k(\mu)$ *be the formula* $\forall k \prec i\; A_k(\mu)$. *Then* $\forall i \preceq \xi\,(\forall j \prec i (A_j \subseteqq \cap_{k \prec j} A_k) \rightarrow \mathrm{Prog}\,[F_i, <_i, \cap_{k \prec i} A_k])$ *is derivable in* $\mathrm{ID}_{\xi}^{i}(\mathfrak{A})$.

PROOF.

2.1. The case $i=0$. Trivial.

2.2. The case $Suc(i)$.

Put

$$i_0 = i \ominus 1\,,$$

then

$$i = i_0 \oplus 1 \quad \text{and} \quad i_0 \prec i$$

Assume that

$$\forall j \prec i\,(A_j \subseteqq \cap_{k \prec j} A_k)\,,$$

then we have

$$\cap_{k \prec i} A_k = A_{i_0}\,.$$

Now we have to show

$$\mathrm{Prog}\,[F_{i_0 \oplus 1}, <_{i_0 \oplus 1}, A_{i_0}]\,.$$

But the proof of lemma 26.32 in [9] can be regarded as the proof of $\mathrm{Prog}\,[F_{i_0 \oplus 1}, <_{i_0 \oplus 1}, A_{i_0}]$ in $\mathrm{ID}_{\xi}^{i}(\mathfrak{A})$.

2.3. The case $Lim(i)$.

We can read the proof of lemma 26.33 in [9] as the proof of this case in $\mathrm{ID}_{\xi}^{i}(\mathfrak{A})$.

LEMMA 3. *Let* $\bar{A}$ *be* $\cap_{i \prec \xi} A_i$. *Then* $\mathrm{Prog}\,[F_\xi, <_\xi, \bar{A}]$ *is derivable in* $\mathrm{ID}_{\xi}^{i}(\mathfrak{A})$.

PROOF.

From $(A.2)_\xi$ we have

$$\forall j \prec \xi\,(\mathrm{Prog}\,[F_j, <_j, \cap_{k \prec j} A_k] \longrightarrow A_j \subseteqq \cap_{k \prec j} A_k)\,.$$

Hence it follows from lemma 2 that

$$\forall i \preceq \xi\,(\forall j \prec i\, \mathrm{Prog}\,[F_j, <_j, \cap_{k \prec j} A_k] \longrightarrow \mathrm{Prog}\,[F_i, <_i, \cap_{k \prec i} A_k])\,.$$

It follows from this and $(\mathrm{TI})_\xi$ that

$$\forall i \prec \xi\, \mathrm{Prog}\,[F_i, <_i, \cap_{k \prec i} A_k]\,,$$

and

$$\forall i \prec \xi \, \mathrm{Prog}\,[F_i, <_i, \cap_{k \prec i} A_k] \longrightarrow \mathrm{Prog}\,[F_\xi, <_\xi, \bar{A}]\,.$$

Therefore the assertion follows.

LEMMA 4. $\forall \mu <_\xi (\xi, 0)(F_\xi(\mu) \rightarrow \bar{A}(\mu))$ *is derivable in* $\mathrm{ID}_\xi^1(\mathfrak{A})$.

PROOF.

Let $R_i(\nu)$ be the formula:

$$\forall \mu <_\xi (i, \nu)(F_\xi(\mu) \longrightarrow \bar{A}(\mu))\,.$$

Firstly we will prove the following 4.1. :

4.1. $\forall i \prec \xi \,(R_i(0) \longrightarrow \mathrm{Prog}\,[F_i, <_i, R_i])\,.$

For this, suppose that $i \prec \xi$, $R_i(0)$, $F_i(\rho)$, $\forall \nu <_i \rho (F_i(\nu) \rightarrow R_i(\nu))$, $\mu <_\xi (i, \rho)$ and $F_\xi(\mu)$.

Now we want to show that $\bar{A}(\mu)$. We may assume μ is connected by proposition 1.5.

Furthermore we may assume

$$(i, 0) \leqq_\xi \mu <_\xi (i, \rho)$$

by the assumptions $R_i(0)$ and $\mu <_\xi (i, \rho)$. Therefore μ must be of the form (i, μ'). $i \prec \xi$ and $(i, \mu') <_\xi (i, \rho)$ imply $\mu' <_i \rho$. $F_\xi((i, \mu'))$ implies $A_i(\mu')$. It follows from proposition 1.1. that $F_i(\mu')$. It follows from these and the assumption $\forall \nu <_i \rho (F_i(\nu) \rightarrow R_i(\nu))$ that $R_i(\mu')$, i.e.,

$$\forall \lambda <_\xi \mu (F_\xi(\lambda) \longrightarrow \bar{A}(\lambda))\,.$$

It follows from this and lemma 3 that $\bar{A}(\mu)$.

4.1. and $(A.2)_\xi$ imply that

$$\forall i \prec \xi \,(R_i(0) \longrightarrow A_i \subseteqq R_i)\,.$$

Since for some primitive recursive function f, we have:

$$\forall i \prec \xi \, \forall \mu (\mu <_\xi (i \oplus 1, 0) \wedge F_\xi(\mu) \longrightarrow \mu <_\xi (i, f(i, \mu)) \wedge A_i(f(i, \mu)))$$

we have the following 4.2. :

4.2. $\forall i \prec \xi \,(R_i(0) \longrightarrow R_{i \oplus 1}(0))\,.$

On the other hand, $R_0(0)$ and $\forall i \prec \xi \,(Lim(i) \wedge \forall j \prec i \, R_j(0) \rightarrow R_i(0))$ clearly hold. Hence from $(\mathrm{TI})_\xi$ we have:

4.3. $\forall i \prec \xi \, R_i(0)\,.$

If $Lim(\xi)$ holds, then the assertion follows from 4.3. Assume that $Suc(\xi)$, i.e.,

$\xi=(\xi\ominus 1)\oplus 1$. By 4.3. and 4.2. we have $R_{\xi\ominus 1}(0)$, $R_{\xi\ominus 1}(0)\rightarrow R_{\xi}(0)$, hence also $R_{\xi}(0)$.

TI $[X, R, Y, \mu]$ abbreviates the formula:

$$X(\mu)\wedge(\text{Prog}[X, R, Y]\longrightarrow\forall\nu(R(\nu, \mu)\wedge X(\nu)\longrightarrow Y(\nu))$$

and TI $[X, R, \mu]$ denotes the schema $\{\text{TI}[X, R, Q, \mu]\}_{Q}$. Namely, 'TI $[X, R, \mu]$ is derivable in $\text{ID}^{i}_{\xi}(\mathfrak{A})$' means that TI $[X, R, Q, \mu]$ is derivable in $\text{ID}^{i}_{\xi}(\mathfrak{A})$ for every formula Q in $\text{ID}^{i}_{\xi}(\mathfrak{A})$.

LEMMA 5. TI $[F_{\xi}, <_{\xi}, (\xi, 0)]$ *is derivable in* $\text{ID}^{i}_{\xi}(\mathfrak{A})$.

PROOF.

5.1. The case $Lim(\xi)$.

For each formula Q, let $Q_{i}(\mu)$ be the formula:

$$\mu<_{\xi}(i, 0)\longrightarrow Q(\mu).$$

Since $\mu<_{\xi}(i, 0)$ implies that μ has no j-section for all $j\geqq i$, the following is easily verified:

$$\mu<_{\xi}(i, 0)\longrightarrow(\nu<_{i}\mu\wedge F_{i}(\nu)\wedge\nu<_{\xi}(i, 0)\longleftrightarrow\nu<_{\xi}\mu\wedge F_{\xi}(\nu)).$$

It follows from this that:

$$\text{Prog}[F_{\xi}, <_{\xi}, Q]\longrightarrow\forall i\prec\xi\,\text{Prog}[F_{i}, <_{i}, Q_{i}].$$

This and $(A.2)_{\xi}$ imply that:

$$\text{Prog}[F_{\xi}, <_{\xi}, Q]\longrightarrow\forall i\prec\xi\,(A_{i}\subseteqq Q_{i}).$$

That is,

$$\text{Prog}[F_{\xi}, <_{\xi}, Q]\longrightarrow\forall i\prec\xi\,\forall\mu<_{\xi}(i, 0)(A_{i}(\mu)\longrightarrow Q(\mu$$

Thus by lemma 4 we have the assertion.

5.2. The case $Suc(\xi)$.

We have easily the following 5.2.1.:

5.2.1. $\forall\mu\,\forall\nu(\nu<_{\xi}\mu<_{\xi}(\xi, 0)\longrightarrow\nu<_{\xi\ominus 1}\mu)$.

Put

$$R(\mu):=\mu<_{\xi}(\xi, 0)\longrightarrow Q(\mu),$$

then we have the following 5.2.2. by 5.2.1.:

5.2.2. $\text{Prog}[F_{\xi}, <_{\xi}, Q]\longrightarrow\text{Prog}[F_{\xi\ominus 1}, <_{\xi\ominus 1}, R]$.

It follows from 5.2.2. and $(A.2)_{\xi}$ that:

$$\text{Prog}[F_{\xi}, <_{\xi}, Q]\longrightarrow\forall\mu<_{\xi}(\xi, 0)(A_{\xi\ominus 1}(\mu)\longrightarrow Q(\mu)).$$

Thus by lemma 4 we have the assertion.

Let $\bar{n}$ be the numeral corresponding to n for each natural number n. Let $\lambda x.\ \xi(x, 0)$ be the primitive recursive function defined by:

$$\xi(0, 0)=0, \qquad \xi(x+1, 0)=(\xi, \xi(x, 0)).$$

Next, we will show that $\mathrm{TI}[F_\xi, <_\xi, \xi(\bar{n}, 0)]$ implies $\mathrm{TI}[F_\xi, <_\xi, \xi(\overline{n+1}, 0)]$ for $n \geqq 1$, following Gentzen [5].

Let $\lambda\mu\nu.\ \mu+^\xi\nu$ be a primitive recursive function such that:

$$\mu\equiv 0 \longrightarrow \mu+^\xi\nu=\nu+^\xi\mu=\nu.$$

Suppose $\mu\not\equiv 0$, $\nu\not\equiv 0$ and

$$\mu\equiv\mu_1\#\cdots\#\mu_m, \qquad \mu_1 {}_\xi{\geqq} \cdots {}_\xi{\geqq}\mu_m\not\equiv 0,$$

$$\nu\equiv\nu_1\#\cdots\#\nu_n, \qquad \nu_1 {}_\xi{\geqq} \cdots {}_\xi{\geqq}\nu_n\not\equiv 0.$$

Let l be the number such that

$$0\leqq l\leqq m \quad \text{and} \quad \mu_l\ {}_\xi{\geqq}\nu_1\ {}_\xi{>}\mu_{l+1}.$$

Then

$$\mu+^\xi\nu=\mu_1\#\cdots\#\mu_l\#\nu_1\#\cdots\#\nu_n.$$

LEMMA 6. *For each formula Q, let $t[Q](\mu)$ be the formula $\forall\rho(F_\xi(\rho)\to(\forall\nu<_\xi\rho(F_\xi(\nu)\to Q(\nu))\to\forall\nu<_\xi\rho+^\xi\mu(F_\xi(\nu)\to Q(\nu))))$. Then $\mathrm{Prog}[F_\xi, <_\xi, Q]\to\mathrm{Prog}[F_\xi, <_\xi, t[Q]]$ is derivable in $\mathrm{ID}^1_\xi(\mathfrak{A})$.*

PROOF.
Obvious.

LEMMA 7. *For each formula Q, let $s[Q](\mu)$ be the formula $t[Q]((\xi, \mu))$, i.e.,*

$$\forall\rho(F_\xi(\rho)\longrightarrow(\forall\nu<_\xi\rho(F_\xi(\nu)\longrightarrow Q(\nu))\longrightarrow\forall\nu<_\xi\rho+^\xi(\xi, \mu)(F_\xi(\nu)\longrightarrow Q(\nu)))).$$

Then

$$\mathrm{Prog}[F_\xi, <_\xi, Q]\longrightarrow\mathrm{Prog}[F_\xi, <_\xi, s[Q]]$$

is derivable in $\mathrm{ID}^1_\xi(\mathfrak{A})$.

PROOF.
By induction on x, we have:

7.1. $F_\xi(\lambda)\wedge s[Q](\lambda)\wedge F_\xi(\rho)\wedge\forall\nu<_\xi\rho(F_\xi(\nu)\longrightarrow Q(\nu))\longrightarrow$

$$\longrightarrow\forall x\,\forall\nu<_\xi\rho+^\xi(\xi, \lambda)\cdot x(F_\xi(\nu)\longrightarrow Q(\nu))$$

where $\mu\cdot x=\mu\#\cdots\#\mu(x$ times).

Since we can define primitive recursive functions f and g such that:

$$\mu\not\equiv 0\wedge\nu<_{\xi}\rho+^{\xi}(\xi,\ \mu)\wedge F_{\xi}(\nu)\longrightarrow F_{\xi}(f(\nu,\ \rho,\ \mu))\wedge f(\nu,\ \rho,\ \mu)<_{\xi}\mu\wedge$$
$$\wedge\nu<_{\xi}\rho+^{\xi}(\xi,\ f(\nu,\ \rho,\ \mu))\cdot g(\nu,\ \rho,\ \mu)\,,$$

it follows from 7.1. that:

7.2. $\mu\not\equiv 0\wedge F_{\xi}(\mu)\wedge\forall\lambda<_{\xi}\mu(F_{\xi}(\lambda)\longrightarrow \mathrm{s}[Q](\lambda))\longrightarrow \mathrm{s}[Q](\mu)$.

By lemmata 5 and 6, we have:

7.3. $\mathrm{Prog}\,[F_{\xi},\ <_{\xi},\ Q]\longrightarrow \mathrm{s}[Q](0)$.

7.2. and 7.3. imply that:

$$\mathrm{Prog}\,[F_{\xi},\ <_{\xi},\ Q]\longrightarrow \mathrm{Prog}\,[F_{\xi},\ <_{\xi},\ \mathrm{s}[Q]]\,.$$

From lemmata 5 and 7, we have the following lemma by metainduction on n.

LEMMA 8. $\mathrm{TI}\,[F_{\xi},\ <_{\xi},\ \xi(\bar{n},\ 0)]$ *is derivable in* $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$ *for each natural number* n.

THEOREM. $A_0(\ulcorner\mu\urcorner)$ *is derivable in* $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$ *for each o.d.* μ *from* $O^*(I,\ 1)$, *where* $\ulcorner\mu\urcorner$ *is the gödelnumber of* μ.

PROOF.

For some primitive recursive function f, we have in PRA $\nu\leqq_0\xi(f(\nu),\ 0)$. By lemmata 3 and 8 we have $\bar{A}(\xi(f(\ulcorner\mu\urcorner),\ 0))$ in $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$. In particular $A_0(\xi(f(\ulcorner\mu\urcorner),\ 0))$. Hence from proposition 1.2. $A_0(\ulcorner\mu\urcorner)$ is derivable in $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$.

REMARKS.

1. Let T^{i} be the theory $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$ and $Prov_{\mathrm{T}^{i}}$ be a canonical proof-predicate for T^{i}. Then we have constructed a primitive recursive function p such that:

PRA *proves that* 'x *is an o.d. from* $O^*(I,\ 1)$' $\longrightarrow Prov_{\mathrm{T}^{i}}(p(x),\ \ulcorner A_0(\dot{x})\urcorner)$,

where $\ulcorner A_0(\dot{x})\urcorner$ is a term whose value is the gödelnumber of $A_0(\bar{n})$ when the numeral $\bar{n}$ is substituted for the variable x.

2. Let the order type of $\prec$ be 2 or $\omega+1$, T be the classical version of T^{i} and T^* be the subsystem (BI) or (Π^1_1–CA)+(BI) of classical analysis, respectively. Then by the well-known translation * (cf. [4].), we have

$$\mathrm{T}\vdash A_0(\mu)\quad\text{implies}\quad \mathrm{T}^*\vdash A_0^*(\mu)$$

and also

$$\mathrm{T}^*\vdash A_0^*(\mu)\longrightarrow \mathrm{TI}_{<_0}[\mu]$$

where $\mathrm{TI}_{<_0}[\mu]$ is the formula

$$\forall X(\forall \nu <_0 \mu(\forall \rho <_0 \nu X(\rho) \longrightarrow X(\nu)) \longrightarrow \forall \nu <_0 \mu X(\nu)).$$

Hence from the remark 1, we have:

PRA *proves that 'x is an o.d. from* $O^*(I, 1)$' $\longrightarrow$ $Prov_{T^*}(p^*(x), \mathrm{TI}_{<_0}[\dot{x}])$

for some primitive recursive function p^*.

On the other hand, we will prove in [1] the consistency of (BI), $(\Pi^1_1\text{-CA})+$ (BI) by the accessibility of $O(2, 1)$, $O(\omega+1, 1)$ with respect to $<_0$, respectively.

3. From the remark 2, we have

$$|\mathrm{ID}^{\mathrm{i}}_\omega| = |\mathrm{ID}_\omega| = |(\Pi^1_1\text{-CA})+(\mathrm{BI})| = |O(\omega+1, 1)|_{<_0}$$

where $|\mathrm{ID}^{\mathrm{i}}_\omega|$ denotes the order type of the least unprovable recursive well-ordering in $\mathrm{ID}^{\mathrm{i}}_\omega$, etc., and $|O(\omega+1, 1)|_{<_0}$ denotes the order type of the system $O(\omega+1, 1)$ with respect to $<_0$.

Following Buchholz and Pohlers [2], and Pohlers [8] the common ordinal equals to $\theta\varepsilon_{\Omega_\omega+1}0$. Thus we have indirectly:

$$|O(\omega+1, 1)|_{<_0} = \theta\varepsilon_{\Omega_\omega+1}0 .$$

This is an analogue to the fact:

$$|O(n+1, 1)|_{<_0} = \theta\varepsilon_{\Omega_n+1}0 \quad \text{for every } n \text{ such that } 1 \leqq n < \omega .$$

But note that the latter was established directly in Levitz [7] and Buchholz and Schütte [3].

4. By [2] and [8]

$$|\mathrm{ID}^{\mathrm{i}}_\xi| = |\mathrm{ID}_\xi| = \theta\varepsilon_{\Omega_\xi+1}0 \quad \text{for} \quad \xi < \theta\Omega_{\Omega_1}0 ,$$

and

$$|\mathrm{ID}^{\mathrm{i}}_{<\xi}| = |\mathrm{ID}_{<\xi}| = \theta\Omega_\xi 0 = \sup_{\zeta<\xi}\theta\varepsilon_{\Omega_\zeta+1}0 \quad \text{for limit} \quad \xi \leqq \theta\Omega_{\Omega_1}0 .$$

On the other hand, for limit ξ and $\zeta<\xi$, the subsystem $\{\mu\in O(\xi, 1) : \mu <_0 (\zeta+1, 0)\}$ of $O(\xi, 1)$ is nothing but $O(\zeta+1, 1)$,

Hence we have:

$$|O(\xi, 1)|_{<_0} = \sup_{\zeta<\xi}|O(\zeta+1, 1)|_{<_0} \quad \text{for limit } \xi .$$

So one may conjecture that

$$|O(\xi+1, 1)|_{<_0} = \theta\varepsilon_{\Omega_\xi+1}0 ,$$

$$|O(\xi, 1)|_{<_0} = \theta\Omega_\xi 0 , \quad \xi ; \text{limit},$$

for appropriately small ξ.

But we have not verified this conjecture in any way.

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