TSUKUBA J. MATH.
Vol. 17 No. 1 (1993), 71–76

# IDEMPOTENT RINGS WHICH ARE EQUIVALENT TO RINGS WITH IDENTITY

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Let $A$ be a ring such that $A=A^2$, but which does not necessarily have an identity element. In studying properties of the ring $A$ through properties of its modules, it is pointless to consider the category $A$-MOD of all the left $A$-modules: for instance, every abelian group -with trivial multiplication- is in $A$-MOD. The natural choice for an interesting category of left $A$-modules seems to be the following: if a left $A$-module ${}_AM$ is *unital* when $AM=M$, and is *$A$-torsionfree* when the annihilator $\iota_M(A)$ is zero, then $A$-mod will be the full subcategory of $A$-MOD whose objects are the unital and $A$-torsionfree left $A$-modules. The category $A$-mod appears in a number of papers (for instance, [7-9]) and when $A$ has local units [1, 2] or is a left $s$-unital ring [6, 12], then the objects of $A$-mod are the unital left $A$-modules. $A$-mod is a Grothendieck category and we study here the question of finding necessary and sufficient conditions on the ring $A$ for $A$-mod to be equivalent to a category $R$-mod of modules over a ring with 1. This was already considered for rings with local units in [1], [2] or [3], and for left $s$-unital rings in [6]. Our situation is therefore more general.

In this paper, all rings will be associative rings, but we do not assume that they have an identity. A ring $A$ has local units [2] when for every finite family $a_1, \cdots, a_n$ of elements of $A$ there is an idempotent $e\in A$ such that $ea_j=a_j=a_je$ for all $j=1, \cdots, n$. A left $A$-module $M$ is said to be unital if $M$ has a spanning set (that is, if $AM=M$); and $M$ has a finite spanning set when $M=\sum Ax_i$ for a finite family of elements $x_1, \cdots, x_n$ of $M$. The module ${}_AM$ will be called $A$-torsionfree when $\iota_M(A)=0$. A ring $A$ is said to be left nondegenerate if the left module ${}_AA$ is $A$-torsionfree, and $A$ is nondegenerate when it is both left and right nondegenerate (see [10, p. 88]). Clearly, a ring with local units is nondegenerate. The ring $A$ will be called (left) $s$-unital [12] in case for each $a\in A$ (equivalently, for every finite family $a_1, \cdots, a_n$ of elements

Received March 25, 1991, Revised November 1, 1991.
[1] With partial support from the D.G.I.C.Y.T. of Spain (PB 87-0703)

of $A$) there is some $u\in A$ such that $ua=a$ (respectively, $ua_i=a_i$, for all i): see [12, Theorem 1]. Any left $s$-unital ring is idempotent and left nondegenerate.

We will say that a ring $A$ is generated by the element $a\in A$ in case $A=AaA$. The above mentioned results of Abrams and Ánh-Márki [1], [2], and Komatsu [6] may be stated as follows: if $A$ has local units, then $A$-mod is equivalent to a category of modules over a ring with 1 if and only if $A$ is generated by an idempotent $e$ [2, Proposition 3.5]; if $A$ is left $e$-unital and $A$-mod is equivalent to the category of left modules over a ring with 1, then $A$ is generated by some element $a$ [6, Proposition 4.7].

In the sequel, we will be dealing with left modules, and so we follow the convention of denoting the composition $g\circ f$ of two module homomorphisms as the product $fg$. On the other hand, if $R$ is a ring with 1, ${}_RM$ is a left $R$-module and $E=\mathrm{End}({}_RM)$ is its endomorphism ring, then we will denote by $E_0=f\,\mathrm{End}({}_RM)$ the following subring -in general, without identity- of $E$: $E_0=\{f\in E \mid f: M\to M$ factors through a finitely generated free module$\}$.

We now state and prove the following result.

THEOREM. *Let $A$ be an idempotent ring. Then the category $A$-mod is eqivalent to the category $R$-mod of left modules over a ring $R$ with* 1 *if and only if there is some integer $n\geqq 1$ such that the matrix ring $M_n(A)$ is generated by an idempotent.*

PROOF. We divide the proof in several steps.

*Step* 1. For any idempotent ring $A$, let us put $\mathrm{ann}(A)=\{x\in A \mid AxA=0\}$ and $A':=A/\mathrm{ann}(A)$. Then $A'$ is a nondegenerate idempotent ring and $A$-mod and $A'$-mod are equivalent categories.

The fact that $A'$ is nondegenerate is easy to verify. On the other hand, if $\varepsilon$: $A\to A'$ is the canonical projection, then one may see that the restriction of scalars functor $\varepsilon_*$ gives indeed a functor from $A'$-mod to $A$-mod. Now, if ${}_AM$ belongs to $A$-mod and $a\in\mathrm{ann}(A)$, then $AaM=AaAM=0$, so that $aM\subseteqq t_M(A)$, and $aM=0$, because $M$ is $A$-torsionfree. As a consequence, there is a functor $F$: $A$-mod$\to A'$-mod which views each ${}_AM$ of $A$-mod as a left $A'$-module. Then $F$ and $\varepsilon_*$ are inverse equivalences and hence $A$-mod and $A'$-mod are equivalent categories.

*Step* 2. For each $n\geqq 1$, let $\Delta$ be the matrix ring $M_n(A)$. Then $A$-mod and $\Delta$-mod are also equivalent categories.

To see this, consider the bimodules ${}_A(A^n)_\Delta$ and ${}_\Delta(A^n)_A$, and the natural mappings $\Phi$: $A^n\otimes_A A^n\to\Delta$, $\Psi$: $A^n\otimes_\Delta A^n\to A$. It is clear that they are bimodule homomorphisms which give a Morita context between $A$ and $\Delta$ (if we represent elements in ${}_A(A^n)_\Delta$ in row form, and elements of ${}_\Delta(A^n)_A$ in column form, then $\Phi$ and $\Psi$ are induced by products of matrices). Also, the fact that $A$ is idempotent allows us to deduce that $\Phi$ and $\Psi$ are surjective. Then, by [7, Theorem], $A$-mod and $\Delta$-mod are equivalent categories.

*Step* 3. We prove now the sufficiency of the condition of the Theorem. Assume that $\Delta=M_n(A)$ is generated by an idempotent. By step 1, $\Delta$ is equivalent to $\Delta'=\Delta/\text{ann}(\Delta)$. But $\Delta=\Delta e\Delta$ for the idempotent $e$ implies that $\Delta'=\Delta'e'\Delta'$ for the idempotent $e'=e+\text{ann}(\Delta)$; so, we can assume that $\Delta$ is a nondegenerate ring. Then $\Delta$ belongs to the category $\Delta$-mod and is a generator of this category. But ${}_\Delta(\Delta e)$ generates $\Delta$, so that it is also a generator of $\Delta$-mod. $\Delta e$, being finitely spanned, is clearly a finitely generated object of $\Delta$-mod [11, p. 121]. Finally, let $p: Y\to X$ be an epimorphism in $\Delta$-mod, and put $U=\text{Im }p$, $V=X/U$, $W=V/t_V(A)$. Then $W$ belongs to $\Delta$-mod and hence the canonical projection from $X$ to $W$ must be 0; thus, $\Delta V=0$ and $X=U$, so that $p$ is a surjective homomorphism. If $f:\Delta e\to X$ is now a homomorphism, then $f(e)=ea$ for some $a\in X$, and $\alpha(e):=ey$, with $y$ such that $p(y)=ea$, gives a morphism $\alpha$ with $f=\alpha\cdot p$. This shows that $\Delta e$ is projective. It follows that $\Delta$-mod is equivalent to the category of left modules over the ring $\text{End}_\Delta(\Delta e)\cong e\Delta e$. By step 2, $A$ is equivalent to a ring with 1.

*Step* 4. Let us now suppose that $A$ is an idempotent and left nondegenerate ring and that there is an equivalence $F$: $A$-mod$\to R$-mod, $R$ being a ring with 1. We are to show that $M_n(A)$ is generated by an idempotent, for some $n\geqq 1$.

By [4, Theorem 2.4], there exists a generator ${}_RM$ of $R$-mod with the property that, if $E=\text{End}({}_RM)$, and $E_0=f\,\text{End}({}_RM)$, then $A$ is isomorphic to some right ideal $T$ of $E_0$ such that $E_0T=E_0$.

We now point out that we can further assume that there is an epimorphism of left $R$-modules $\pi: M\to R$. Indeed, this is true for some $M^k$, and we put $S:=\text{End}({}_RM^k)$, $S_0:=f\,\text{End}({}_RM^k)$, so that there is an isomorphism $S\cong M_k(E)$. We assert that, in this isomorphism, $S_0\cong M_k(E_0)$; in fact, the inclusion $S_0\subseteqq M_k(E_0)$ is obvious, and the inclusion $M_k(E_0)\subseteqq S_0$ depends on the easily verified fact that morphisms $M^r\to M$ or $M\to M^s$ factor through free modules of finite type whenever they are induced by endomorphisms of ${}_RM$ belonging to $E_0$. By substituting $M^k$, $S$ and $S_0$ for $M$, $E$ and $E_0$, we have that the matrix ring $M_k(A)$ is still (isomorphic to) a right ideal of $S_0$ in such a way that -assuming the obvious identification- $S_0\cdot M_k(A)=S_0$. So, by replacing $A$ by $M_k(A)$ if necessary (note that $M_k(A)$ is again idempotent and left nondegenerate), we may indeed assume that $\pi: M\to R$ is an epimorphism.

Let $x\in M$ be such that $\pi(x)=1$. Since $E_0A=E_0$ and $\sum_{\sigma\in E_0}\text{Im }\sigma=M$ we deduce that $\sum_{\sigma\in A}\operatorname{Im}\sigma=M$. Therefore there exists a homomorphism $\alpha: M^n\to M$ such that $x\in\operatorname{Im}\alpha$; and each component $\alpha_j:=\mu_j\cdot\alpha$, with $\mu_j: M\to M^n$ being the canonical inclusion, satisfies $\alpha_j\in A$. So we have that $\alpha\cdot\pi: M^n\to R$ is an epimorphism and hence there is $g: R\to M^n$ with $g\alpha\pi=1_R$ and $\alpha\pi g=e$ an idempotent in the ring $\operatorname{End}(_RM^n)\cong M_n(E)$. Moreover, each of the components of $e$, when considered as a matrix, consists of $\mu_j\alpha\pi g p_k=\alpha_j(\pi g p_k)\in\alpha_jE\subseteqq A$ (where the $p_k$ are the canonical projections $M^n\to M$). This means that $e\in M_n(A)$.

As before, we may put $S:=\operatorname{End}(_RM^n)\cong M_n(E)$, $S_0:=f\operatorname{End}(_RM^n)\cong M_n(E_0)$ so that $M_n(A)$ is an idempotent right ideal in $S_0$ which satisfies $S_0M_n(A)=S_0$. Thus, $e$ is an idempotent element in $M_n(A)\subseteqq S_0$ and is an endomorphism of $M^n$ such that $\operatorname{Im}e$ is a direct aummand of $M^n$ isomorphic to $R$. Consequently, $\operatorname{Im}e$ generates $M^n$ and hence, if we let $t$ range over all the elements in $eS_0$, we have $\sum_t\operatorname{Im}t=M^n$. This shows that $eS_0$ is a right ideal of $S$ which satisfies $M^n\cdot(eS_0)=M^n$. If we apply now [5, Proposition 2.5], we see that this implies $S_0eS_0=S_0$.

Since $A=A^2$, $M_n(A)\cdot S_0=M_n(A)$ and so we have: $M_n(A)\cdot e\cdot M_n(A)=M_n(A)\cdot S_0e\cdot S_0=M_n(A)\cdot S_0=M_n(A)$. This proves that $M_n(A)$ is generated by an idempotent element.

*Step* 5. Now we complete the proof of the Theorem. Let $A$ be an idempotent ring (but not necessarily left nondegenerate), and assume that there is an equivalence of categories between $A$-mod and $R$-mod for $R$ a ring with 1. Put $\iota_A(A)=\{a\in A\mid Aa=0\}$, and $A^*=A/\iota_A(A)$. In a way analogous to that of Step 1, we may show that $A$ and $A^*$ are equivalent rings, so that we can deduce from stea 4, that for some $n\geqq 1$, the matrix ring $M_n(A^*)$ is generated by an idempotent. Thus, all that is left to show is that this property can be lifted from $M_n(A^*)$ to $M_n(A)$. But we have that $M_n(A^*)=M_n(A/\iota_A(A))\cong(M_n(A))/(M_n(\iota_A(A)))$, and this last quotient is nothing else than $M_n(A)/\iota_{M_n(A)}(M_n(A))$, that is, $(M_n(A))^*$. Therefore, it will suffice to prove that if a ring of the form $A^*=A/\iota_A(A)$ is generated by an idempotent, then so is the ring $A$.

So, let us assume that $A^*=A^*\cdot e\cdot A^*$ for some idempotent $e$. There is $u\in A$ with $u+\iota_A(A)=e$, and then $u^2-u\in\iota_A(A)$, from which we see that $u^3=u^2=u^4$. Therefore, $w=u^2$ is an idempotent of $A$ such that $w+\iota_A(A)=e$. Now, let $a$, $b\in A$; by hypothesis, $b+\iota_A(A)=\sum\alpha_j\cdot e\cdot\beta_j$ in the ring $A^*$, so that $b-\sum a_j\cdot w\cdot b_j\in\iota_A(A)$, for some $a_j$ and $b_j$ in $A$. Then $ab=\sum aa_jwb_j$ and $ab\in AwA$. But since $A$ is idempotent, we have finally that $A=AwA$ and $A$ is generated by an idempotent.

REMARKS. 1) It follows from the Theorem that an idempotent ring $A$

which is equivalent to a ring with 1 must be finitely generated as a bimodule over $A$: the coordinates of the idempotent matrix $e$ in the adequate $M_n(A)$ give the family of generators. When $A$ is left $s$-unital this gives as a consequence the already mentioned result of Komatsu [6, Proposition 4.7]. If $A$ has local units, we get [2, Proposition 3.5].

2) However, the condition that $A$ be finitely generated as a bimodule over itself is not sufficient for $A$ to be equivalent to a ring with 1. To see this, take a ring $A$ such that $A=A^2$, $A$ is finitely generated as an $A-A$-bimodule, is nondegenerate and coincides with its Jacobson radical (Sasiada's example [10, p. 314] of a simple radical ring fulfills these requirements). It is not difficult to show that the Jacobson radical of such a ring is the intersection of all the subobjects of $A$ in $A$-mod which give a simple quotient of $A$ in $A$-mod, so that $A$ has no simple quotients in $A$-mod. Suppose that the category $A$-mod were equivalent to $R$-mod for $R$ a ring with 1. Then, if ${}_RM$ corresponds to $A$ in this equivalence, we would have that ${}_RM$ is a generator of $R$-mod without simple quotients. But this is absurd, since $R$ is isomorphic to a summand of some $M^k$.

3) It may happen that $A$ be an idempotent ring such that $A$-mod is equivalent to a category $R$-mod for a ring $R$ with 1 but, nevertheless, $A$ is not generated by an idempotent. For instance, let $R$ be a simple domain which is not a division ring and let $I$ be a right ideal of $R$ such that $I\neq 0$, $I\neq R$. Then $RI=R$, $I=IR=I^2$ and $I$ is a faithful right ideal of $R$, so that we can view $I$ as a left nondegenerate and idempotent ring contained in $R=f\,\mathrm{End}({}_RR)$. By [4, Theorem 2.4], we see that $I$-mod is equivalent to the category $R$-mod. But $I$ contains no idempotent other than 0, so that $I$ is not generated by an idempotent.

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