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CHARACTERIZATIONS OF PARACOMPACTNESS BY INCREASING COVERS AND NORMALITY OF PRODUCT SPACES

By

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1. Introduction. For paracomparactness and μ-paracompactness (μ—an infinite cardinal), many characterizations have been obtained until now. In particular, for countable paracompactness the following simple characterization is known:

A normal space (a topological space) X is countably paracompact if and only if, for each countable increasing open cover $\{U_n\}$, there exists a countable closed (open) cover $\{V_n\}$ such that $V_n \subset U_n$ ($\bar{V}_n \subset U_n$) for each n (Dowker [4], Ishikawa [5]).

In this paper we shall give new characterizations of paracompactness and μ-paracompactness in terms of "well-ordered increasing cover", and using these characterizations we shall obtain some results with respect to normality of product spaces.

Here a space X is *paracompact* (*μ-paracompact*), if each open cover (with cardinality $\leq \mu$) of X has a locally finite open refinement.

Let λ be an ordinal. We say that a space X has the *property* $P(\lambda)$, if for each open cover $\{U_\alpha | \alpha < \lambda\}$ of X with length λ satisfying

(1) $U_\alpha \subset U_{\alpha+1}$,

(2) $U_\beta = \bigcup_{\alpha<\beta} U_\alpha$ for each limit ordinal $\beta < \lambda$,

there exists an open cover $\{V_{\alpha,n} | \alpha < \lambda, n = 0, 1, 2, \cdots\}$ of X such that

(3) $V_{\alpha,n} \subset V_{\alpha+1,n}$,

(4) $V_{\beta,n} = \bigcup_{\alpha<\beta} V_{\alpha,n}$ for each limit ordinal $\beta < \lambda$,

(5) $\bar{V}_{\alpha,n} \subset V_{\alpha,n+1}$,

(6) $V_{\alpha,n} \subset U_\alpha$.

Our characterizations for paracompactness and μ-paracompactness are as follows:

THEOREM 1.1. *Consider the following statements about a space X:*

(a) *X has the property $P(\lambda)$ for each regular ordinal λ.*

(b) *Each well-ordered increasing open cover $^c\mathcal{U}$ of X has an open refinement $^c\mathcal{V} = \bigcup_{n=0}^{\infty} {}^c\mathcal{V}_n$ such that $^c\mathcal{V}_n$ is cushioned in $^c\mathcal{V}_{n+1}$ (in the sense of Michael* [8]) *for*

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each n.

(c) *X is paracompact.*

Then (a) *implies* (b) *and* (b) *implies* (c). *If X is a normal (regular or Hausdorff) space, then* (a), (b) *and* (c) *are equivalent.*

THEOREM 1.2. *Consider the following statements about a space X and an infinite cardinal (=an initial ordinal) μ:*

(a) *X has the property $P(\lambda)$ for each regular ordinal λ such that $\lambda \leq \mu$.*

(b) *Each well-ordered increasing open cover $\mathcal{U}$ of X with length $\leq \mu$ has an open refinement $\mathcal{V} = \bigcup_{n=0}^{\infty} \mathcal{V}_n$ such that $\mathcal{V}_n$ is cushioned in $\mathcal{V}_{n+1}$ for each n.*

(c) *X is μ-paracompact.*

Then (a) *implies* (b) *and* (b) *implies* (c). *If X is a normal space, then* (a), (b) *and* (c) *are equivalent.*

Theorem 1.1 follows immediately from Theorem 1.2; Theorem 1.2 will be proved in section 2.

In sections 3 and 4, we shall have some applications of Theorems 1.1 and 1.2; each of them is related to the normality of the product space of a normal space with a compact Hausdorff space. Specifically, in section 4, the following is proved:

If the product $X \times Y$ of a space X with a compact Hausdorff space Y is normal, then X is μ-paracompact for each infinite cardinal μ such that $\mu < t(Y)$. If, furtheremore, $t(Y)$ is not weakly inaccessible, then X is $t(Y)$-paracompact.

Here $t(Y)$ denotes the tightness of Y [1].

Theorems 1.1, 1.2 and the contents of section 3 were announced at the Fourth Prague Topological Symposium in 1976.

2. Proof of Theorem 1.2. Throughout this paper, the Greek letters $\alpha, \beta, \cdots$, $\lambda, \mu, \cdots$ denote ordinal numbers, and each ordinal is the set of its predecessors. Thus, $\alpha \in \lambda \Longleftrightarrow \alpha < \lambda$. The *cofinality* of λ, $cf(\lambda)$, is defined by $cf(\lambda) = \min\{\mu \mid \lambda$ has a cofinal subset of order type $\mu\}$. As is well-known, for each ordinal λ, $cf(\lambda)$ is a cardinal (=an initial ordinal). An ordinal λ is *regular* if $cf(\lambda) = \lambda$. Hence each regular ordinal is a cardinal, and $cf(\lambda)$ is regular for each ordinal λ. The successor of λ is denoted by $\lambda+1$; namely, $\lambda+1 = \lambda \cup \{\lambda\} = \{\alpha \mid \alpha \leq \lambda\}$. As usual, ω denotes the first infinite ordinal. An element of ω (=a natural number) is denoted by n or m.

First, for convenience, we introduce the following terminology. An indexed cover $\{U_\alpha \mid \alpha \in \lambda\}$ of a space X is a *λ-increasing cover*, if it satisfies the following conditions

(1) $U_\alpha \subset U_{\alpha+1}$,

(2) $U_\beta = \bigcup_{\alpha<\beta} U_\alpha$ for each limit $\beta \in \lambda$.

Of course, a λ-increasing cover is a monotone increasing cover of length λ, and a $(\lambda+1)$-increasing cover of X contains X as the last member.

A double indexed cover $\{V_{\alpha,n}|\alpha\in\lambda, n\in\omega\}$ of X is a $(\lambda,\bar{\omega})$-*increasing cover* (resp. a (λ,ω)-*increasing cover*), if it satisfies the following conditions

(3) $V_{\alpha,n}\subset V_{\alpha+1,n}$,

(4) $V_{\beta,n}=\bigcup_{\alpha<\beta} V_{\alpha,n}$ for each limit $\beta\in\lambda$,

(5) $\overline{V}_{\alpha,n}\subset V_{\alpha,n+1}$ (resp. $V_{\alpha,n}\subset V_{\alpha,n+1}$).

A double indexed cover $\{V_{\alpha,n}|\alpha\in\lambda, n\in\omega\}$ is called an *indexed refinement* of an indexed cover $\{U_\alpha|\alpha\in\lambda\}$, if for each $\alpha\in\lambda$ and each $n\in\omega$

(6) $V_{\alpha,n}\subset U_\alpha$.

Thus, a space X has the property $P(\lambda)$ if and only if each λ-increasing open cover of X has a $(\lambda,\bar{\omega})$-increasing open indexed refinement. An arbitrary space X has the property $P(\lambda+2)$ for each ordinal λ; indeed, the cover $\{V_{\alpha,n}|\alpha\in\lambda+2, n\in\omega\}$, given by $V_{\alpha,n}=\phi$ for $\alpha\leq\lambda, n\in\omega$ and $V_{\lambda+1,n}=X$ for $n\in\omega$, is a $(\lambda+2,\bar{\omega})$-increasing open indexed refinement of any $(\lambda+2)$-increasing open cover of X.

LEMMA 2.1. *For a limit ordinal* λ,

$$P(\lambda+1)\Rightarrow P(\lambda).$$

PROOF. Let X be a space with the property $P(\lambda+1)$, and let $\mathcal{U}=\{U_\alpha|\alpha\in\lambda\}$ be a λ-increasing open cover of X. We put $U_\lambda=\bigcup_{\alpha\in\lambda} U_\alpha(=X)$ and let $\mathcal{U}^+=\{U_\alpha|\alpha\in\lambda+1\}$. Then $\mathcal{U}^+$ is a $(\lambda+1)$-increasing open cover of X, and hence $\mathcal{U}^+$ has a $(\lambda+1,\bar{\omega})$-increasing open indexed refinement $\mathcal{V}^+=\{V_{\alpha,n}|\alpha\in\lambda+1, n\in\omega\}$. Since λ is a limit, $V_{\lambda,n}=\bigcup_{\alpha\in\lambda} V_{\alpha,n}$ for $n\in\omega$. Hence the subcollection $\mathcal{V}=\{V_{\alpha,n}|\alpha\in\lambda, n\in\omega\}$ of $\mathcal{V}^+$ is a cover of X, and so $\mathcal{V}$ is a $(\lambda,\bar{\omega})$-increasing open indexed refinement of $\mathcal{U}$. Thus X has the property $P(\lambda)$.

LEMMA 2.2. $P(\lambda)+P(\omega)\Rightarrow P(\lambda+1)$.

PROOF. Since any space has the property $P(\lambda+1)$ for a non-limit ordinal λ, we may assume that λ is a limit ordinal. Let X be a space with the properties $P(\lambda)$ and $P(\omega)$, and let $\mathcal{U}=\{U_\alpha|\alpha\in\lambda+1\}$ be a $(\lambda+1)$-increasing open cover of X. Since λ is a limit ordinal, the subcollection $\mathcal{U}'=\{U_\alpha|\alpha\in\lambda\}$ is a λ-increasing open cover of X. Hence $\mathcal{U}'$ has a $(\lambda,\bar{\omega})$-increasing open indexed refinement $\mathcal{U}'=\{V'_{\alpha,n}|\alpha\in\lambda, n\in\omega\}$. If we put $V_n'=\bigcup_{\alpha\in\lambda} V'_{\alpha,n}$ for each $n\in\omega$, then $\{V_n'|n\in\omega\}$ is an ω-increasing open cover of X. Since X has the property $P(\omega)$, there exists an open cover $\{W_{m,n}|m,n\in\omega\}$ of X such that

$$W_{m,n}\subset W_{m+1,n},\ \overline{W}_{m,n}\subset W_{m,n+1} \quad\text{and}\quad W_{m,n}\subset V_m'.$$

Let us define

$$V_{\alpha,n}=\begin{cases} V'_{\alpha,n}\cap W_{n,n} & \text{if} \quad \alpha<\lambda, \\ W_{n,n} & \text{if} \quad \alpha=\lambda. \end{cases}$$

It is easily seen that the collection $\{V_{\alpha,n}|\alpha\in\lambda+1, n\in\omega\}$ is a $(\lambda+1,\bar{\omega})$-increasing open indexed refinement of the given $(\lambda+1)$-increasing open cover $\mathcal{U}$. Therefore X has the property $P(\lambda+1)$.

LEMMA 2.3. *Let λ and μ be two limit ordinals with the same cofinality. Then we have the following equivalences*:

$$P(\lambda)\Longleftrightarrow P(\mu) \quad \textit{and} \quad P(\lambda+1)\Longleftrightarrow P(\mu+1).$$

PROOF. Let $cf(\lambda)=cf(\mu)=\nu$. (Since λ is a limit, the ordinal ν is also a limit). Then the set $\lambda=\{\alpha|\alpha<\lambda\}$ has a cofinal subset $A=\{\alpha_\xi|\xi\in\nu\}$ such that $\alpha_\xi<\alpha_\zeta$ provided $\xi<\zeta<\nu$. Without loss of generality, we may assume

$$\begin{cases} \alpha_0=0, \\ \alpha_\xi \text{ is a non-limit ordinal, if } \xi \text{ is a non-limit ordinal,} \\ \alpha_\xi=\sup\{\alpha_\zeta|\zeta<\xi\}, \text{ if } \xi \text{ is a limit ordinal.} \end{cases}$$

Similarly, there exists a cofinal subset $B=\{\beta_\xi|\xi\in\nu\}$ in μ with the same property.

We prove first the implication $P(\lambda)\Rightarrow P(\mu)$. Let X be a $P(\lambda)$-space. (Hereafter, a space with the property $P(\lambda)$ is also called a $P(\lambda)$-*space*.) Let $\mathcal{U}=\{U_\beta|\beta\in\mu\}$ be a μ-increasing open cover of X. For each $\alpha\in\lambda$, let us put $U_\alpha'=U'_{\beta_\xi}$, where ξ is the unique element of ν such that $\alpha_\xi\leq\alpha<\alpha_{\xi+1}$. Then $\mathcal{U}'=\{U_\alpha'|\alpha\in\lambda\}$ is a λ-increasing open cover of X. Consequently, $\mathcal{U}'$ has a $(\lambda,\bar{\omega})$-increasing open indexed refinement $\{V'_{\alpha,n}|\alpha\in\lambda, n\in\omega\}$. If we define $V_{\beta,n}=V'_{\alpha_\zeta,n}$ for $\beta\in\mu$, where ζ is the unique element of ν such that $\beta_\zeta\leq\beta<\beta_{\zeta+1}$, then we can show without difficulty that the collection $\{V_{\beta,n}|\beta\in\mu, n\in\omega\}$ is a $(\mu,\bar{\omega})$-increasing open indexed refinement of the given cover $\mathcal{U}$. Hence X is a $P(\mu)$-space, and so the implication $P(\lambda)\Rightarrow P(\mu)$ is proved. The proof of $P(\mu)\Rightarrow P(\lambda)$ is the same to that of $P(\lambda)\Rightarrow P(\mu)$. Thus we have the equivalence $P(\lambda)\Longleftrightarrow P(\mu)$.

The proof of the equivalence $P(\lambda+1)\Longleftrightarrow P(\mu+1)$ is omitted; it is a slight modification of the proof of $P(\lambda)\Longleftrightarrow P(\mu)$. Consequently the proof of the lemma is concluded.

Here, furthermore, we introduce the following two terms. We say that a space X has the *property* $Q(\lambda)$ or X is a $Q(\lambda)$-*space*, if each λ-increasing open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}=\bigcup_{n\in\omega}\mathcal{V}_n$ such that $\mathcal{V}_n$ is cushioned in $\mathcal{V}_{n+1}$ for each $n\in\omega$; and we say that X has the *property* $R(\lambda)$ or X is an $R(\lambda)$-*space*, if for each λ-increasing open cover $\mathcal{U}$ of X there exists a σ-locally finite open cover $\mathcal{V}$ of X such that $\overline{\mathcal{V}}=\{\bar{V}|V\in\mathcal{V}\}$ refines $\mathcal{U}$. If λ is a non-limit ordinal, then it is obvious

that any space has the properties $Q(\lambda)$ and $R(\lambda)$.

LEMMA 2.4. $P(\lambda+1) \Rightarrow Q(\lambda)$.

PROOF. We may assume that λ is a limit ordinal. Let X be a $P(\lambda+1)$-space, and let $\mathcal{U}=\{U_\alpha|\alpha\in\lambda\}$ be a λ-increasing open cover of X. As has been mentioned in the proof of Lemma 2.1, there exists a $(\lambda+1,\bar{\omega})$-increasing open cover $\mathcal{C}\mathcal{V}^+=\{V_{\alpha,n}|\alpha\in\lambda+1, n\in\omega\}$ of X such that the subcollection $\mathcal{C}\mathcal{V}=\{V_{\alpha,n}|\alpha\in\lambda, n\in\omega\}$ is an indexed refinement of $\mathcal{U}$. Let $\mathcal{C}\mathcal{V}_n=\{V_{\alpha,n}|\alpha\in\lambda\}$ for $n\in\omega$, then $\mathcal{C}\mathcal{V}=\bigcup_{n\in\omega}\mathcal{C}\mathcal{V}_n$. To prove that $\mathcal{C}\mathcal{V}_n$ is cushioned in $\mathcal{C}\mathcal{V}_{n+1}$, it is sufficient to prove that

$$\overline{\bigcup_{\alpha\in A} V_{\alpha,n}} \subset \bigcup_{\alpha\in A} V_{\alpha,n+1}$$

for each subset A of λ. Let us put $\beta=\sup A$, then $\beta\leq\lambda$ (i.e., $\beta\in\lambda+1$). In case $\beta\in A$, β is the largest element of A, and hence $\bigcup_{\alpha\in A} V_{\alpha,m}=V_{\beta,m}$. In case $\beta\notin A$, β is a limit, and hence $\bigcup_{\alpha\in A} V_{\alpha,m}=\bigcup_{\alpha<\beta} V_{\alpha,m}=V_{\beta,m}$. In either case, $\bigcup_{\alpha\in A} V_{\alpha,m}=V_{\beta,m}$ for each $m\in\omega$. Therefore, we have

$$\overline{\bigcup_{\alpha\in A} V_{\alpha,n}}=\bar{V}_{\beta,n}\subset V_{\beta,n+1}=\bigcup_{\alpha\in A} V_{\alpha,n+1}.$$

Thus X is a $Q(\lambda)$-space.

LEMMA 2.5. $P(\lambda+1) \Rightarrow R(\lambda)$.

PROOF. We may assume that λ is a limit ordinal. Let X be a $P(\lambda+1)$-space, and let $\mathcal{U}=\{U_\alpha|\alpha\in\lambda\}$ be a λ-increasing open cover of X. After the proof of Lemma 2.4, we have a $(\lambda+1,\bar{\omega})$-increasing open cover $\mathcal{C}\mathcal{V}^+=\{V_{\alpha,n}|\alpha\in\lambda+1, n\in\omega\}$ whose subcollection $\mathcal{C}\mathcal{V}=\{V_{\alpha,n}|\alpha\in\lambda, n\in\omega\}$ is an indexed refinement of $\mathcal{U}$. We need a σ-locally finite open cover $\mathcal{G}$ such that $\bar{\mathcal{G}}$ refines $\mathcal{U}$. The collection $\mathcal{G}$ is constructed as follows: $\mathcal{G}=\bigcup_{n\in\omega}\mathcal{G}_n$, $\mathcal{G}_n=\{G_{\alpha,n}|\alpha\in\lambda\}$ for $n\in\omega$, and

$$G_{\alpha,n}=\begin{cases} V_{0,n} & \text{if } \alpha=0, \\ \phi & \text{if } \alpha \text{ is a limit}, \\ V_{\alpha,n}-\bar{V}_{\alpha-1,n+1} & \text{otherwise.}\end{cases}$$

Here $\alpha-1$ denotes the predecessor of α, in case that α is a non-limit, non-zero ordinal. Obviously, each member $G_{\alpha,n}$ of $\mathcal{G}$ is open in X. Since $\bar{G}_{\alpha,n}\subset\bar{V}_{\alpha,n}\subset V_{\alpha,n+1}\subset U_\alpha$ for $\alpha\in\lambda$ and $n\in\omega$, $\bar{\mathcal{G}}=\{\bar{G}_{\alpha,n}|\alpha\in\lambda, n\in\omega\}$ refines $\mathcal{U}$. Hence, to complete the proof of the lemma, it is sufficient to prove the following two assertions (i) and (ii).

(i) $\mathcal{G}$ is a cover of X: Let $x\in X$, and let

$$\alpha(x)=\min\{\alpha\in\lambda|x\in V_{\alpha,n} \text{ for some } n\in\omega\}.$$

Since $\mathcal{V}$ is a cover of X, $\alpha(x)$ is well-defined. Then there exists an element $n(x)\in\omega$ such that $x\in V_{\alpha(x),n(x)}$. From the condition (4) in the definition of a $(\lambda,\bar{\omega})$-increasing cover, it is seen that $\alpha(x)$ is a non-limit ordinal. In case $\alpha(x)\neq 0$, from the definition of $\alpha(x)$, we have $x\notin V_{\alpha(x)-1,n(x)+2}$ and so $x\notin\bar{V}_{\alpha(x)-1,n(x)+1}$. Hence, in this case, $x\in G_{\alpha(x),n(x)}$. In case $\alpha(x)=0$, $x\in V_{0,n(x)}=G_{0,n(x)}$. Thus $\mathcal{G}$ is a cover of X.

(ii) $\mathcal{G}_n$ is locally finite (more precisely, discrete) for each $n\in\omega$: For each $n\in\omega$ and for each $x\in X$, let us construct an open neighborhood $N(x)$ of x which intersects at most only one member of $\mathcal{G}_n$. The neighborhood $N(\alpha)$ is defined by

$$N(x)=\begin{cases} X-\bar{V}_{\lambda,n} & \text{if } x\notin V_{\lambda,n+1},\\ V_{0,n+1} & \text{if } x\in V_{0,n+1},\\ V_{\beta_n(x),n+1}-\bar{V}_{\beta_n(x)-1,n} & \text{otherwise},\end{cases}$$

where $\beta_n(x)=\min\{\alpha\in\lambda+1 \mid x\in V_{\alpha,n+1}\}$. In case $x\in V_{\lambda,n+1}$, $\beta_n(x)$ is well-defined, and $\beta_n(x)$ is a non-limit ordinal by the condition (4). Moreover, if $x\notin V_{0,n+1}$, then we have the predecessor $\beta_n(x)-1$ of $\beta_n(x)$ and $x\notin V_{\beta_n(x)-1,n+1}$. Since $\bar{V}_{\alpha,n}\subset V_{\alpha,n+1}$ for each $\alpha\in\lambda+1$, $N(x)$ is surely an open neighborhood of x in any case. From the definition of $G_{\alpha,n}$, we have $G_{\alpha,n}\cap V_{\beta,n+1}=\phi$ for $0\leq\beta<\alpha<\lambda$, and $G_{\alpha,n}\subset\bar{V}_{\beta,n}$ for $0\leq\alpha<\beta\leq\lambda$. Hence it follows that $N(x)$ intersects at most one member of $\mathcal{G}_n$; indeed, $G_{\alpha,n}\cap N(x)=\phi$ for all $\alpha\in\lambda$ provided $x\notin V_{\lambda,n+1}$, and $G_{\alpha,n}\cap N(x)=\phi$ for all $\alpha\neq\beta_n(x)$ provided $x\in V_{\lambda,n+1}$. Therefore $\mathcal{G}_n$ is discrete and so locally finite for each $n\in\omega$.

Thus the proof of the lemma is completed.

LEMMA 2.6. *If* $cf(\lambda)>\omega$, *then*

$$Q(\lambda)\Rightarrow P(\lambda+1).$$

PROOF. Let X be a $Q(\lambda)$-space, and let $\mathcal{U}=\{U_\alpha \mid \alpha\in\lambda+1\}$ be a $(\lambda+1)$-increasing open cover of X. Since λ is a limit ordinal, the subcollection $\mathcal{U}'=\{U_\alpha \mid \alpha\in\lambda\}$ is a cover of X and so it is a λ-increasing open cover of X. Hence $\mathcal{U}'$ has an open refinement $\mathcal{W}=\bigcup_{n\in\omega}\mathcal{W}_n$ such that $\mathcal{W}_n$ is cushioned in $\mathcal{W}_{n+1}$ for each $n\in\omega$. Let us $f_n\colon \mathcal{W}_n\to\mathcal{W}_{n+1}$ be the cushioned function for $n\in\omega$. For each $n\in\omega$ and for each $W\in\mathcal{W}_n$, we define ordinals $\alpha_n(W)$ and $\beta_n(W)$ as follows:

$$\alpha_n(W)=\min\{\alpha\in\lambda \mid W\subset U_\alpha\},\quad \beta_n(W)=\sup\{\alpha_{n+m}(f_{n,m}(W)) \mid m\in\omega\},$$

where $f_{n,m}\colon \mathcal{W}_n\to\mathcal{W}_{n+m}$, $n,m\in\omega$, is the function given by

$$f_{n,m}=\begin{cases} f_{n+m-1}\circ\cdots\circ f_n & \text{if } m>0,\\ \text{the identity} & \text{if } m=0.\end{cases}$$

Since $\mathcal{W}$ refines $\mathcal{U}'$, $\alpha_n(W)$ is well-defined and $\alpha_n(W)<\lambda$. By the assumption $cf(\lambda)>\omega$, we have $\beta_n(W)<\lambda$ for $W\in\mathcal{W}_n$. Now, we put

$$V_{\alpha,n}=\cup\{W\in W_n|\beta_n(W)<\alpha\}$$

for $\alpha\in\lambda+1$ and $n\in\omega$, and let $\mathcal{V}=\{V_{\alpha,n}|\alpha\in\lambda+1, n\in\omega\}$. Let us show that $\mathcal{V}$ is a $(\lambda+1,\bar{\omega})$-increasing open indexed refinement of $\mathcal{U}$. It is obvious that $\mathcal{V}$ is an open cover of X. Since $\alpha_n(W)\leq\beta_n(W)$ and $W\subset U_{\alpha_n(W)}$ for $W\in\mathcal{W}_n$, we have $V_{\alpha,n}\subset U_\alpha$ for $\alpha\in\lambda$ and $n\in\omega$. As a matter of course, $V_{\lambda,n}\subset U_\lambda(=X)$ for $n\in\omega$. Hence $\mathcal{V}$ is an indexed refinement of $\mathcal{U}$. It is obvious that $\mathcal{V}$ satisfies the conditions (3) and (4). So it is remained to examine $\bar{V}_{\alpha,n}\subset V_{\alpha,n+1}$ for $\alpha\in\lambda+1$ and $n\in\omega$ (the condition (5)). Since f_n is a cushioned function, we have

$$\overline{\cup\{W\in W_n|\beta_n(W)<\alpha\}}\subset\cup\{f_n(W)|W\in W_n,\beta_n(W)<\alpha\}.$$

And the inclusion

$$\cup\{f_n(W)|W\in W_n,\beta_n(W)<\alpha\}\subset\cup\{W'\in W_{n+1}|\beta_{n+1}(W')<\alpha\}$$

follows from the fact $\beta_{n+1}(f_n(W))\leq\beta_n(W)$ for $W\in\mathcal{W}_n$, which is directly proved from the definition of $\beta_n(W)$. Hence $\bar{V}_{\alpha,n}\subset V_{\alpha,n+1}$ for $\alpha\in\lambda+1$ and $n\in\omega$. Thus X is a $P(\lambda+1)$-space and the proof is completed.

LEMMA 2.7. *Let λ and μ be two ordinals with the same cofinality, then we have the following equivalences*:

$$Q(\lambda)\Longleftrightarrow Q(\mu)\quad\text{and}\quad R(\lambda)\Longleftrightarrow R(\mu).$$

This lemma is more easily proved than Lemma 2.3.

LEMMA 2.8. $Q(\lambda)\Rightarrow R(\lambda)$.

PROOF. If $cf(\lambda)<\omega$ (i.e., λ is a non-limit ordinal), the lemma is obvious. The implication $Q(\omega)\Rightarrow R(\omega)$ is easily proved. Hence, by Lemma 2.7, the implication $Q(\lambda)\Rightarrow R(\lambda)$ is true for any ordinal λ with $cf(\lambda)=\omega$. If $cf(\lambda)>\omega$, then, from Lemmas 2.6 and 2.5, we have the implications $Q(\lambda)\Rightarrow P(\lambda+1)\Rightarrow R(\lambda)$. Therefore the lemma holds for all ordinals λ.

LEMMA 2.9. *In countably paracompact normal spaces,*

$$R(\lambda)\Rightarrow P(\lambda+1).$$

PROOF. We may assume that λ is a limit ordinal. Let X be a countably paracompact normal space with the property $R(\lambda)$. We shall show that X has the following property which is stronger than $P(\lambda+1)$: Each λ-increasing open cover $\{U_\alpha|\alpha\in\lambda\}$ of X has a $(\lambda,\bar{\omega})$-increasing open indexed refinement $\{V_{\alpha,n}|\alpha\in\lambda, n\in\omega\}$ such that $\{V_{\alpha,n}|\alpha\in\lambda\}$ is a cover of X for each $n\in\omega$. Let $\mathcal{U}=\{U_\alpha|\alpha\in\lambda\}$ be a λ-increasing

open cover of X. Since X is an $R(\lambda)$-space, $\mathcal{U}$ has a σ-locally finite open refinement. Each σ-locally finite open cover of a countably paracompact space has a locally finite open refinement, and each locally finite open cover of a normal space is shirinkable. Therefore we have a locally finite open cover $\mathcal{G}=\{G_\alpha|\alpha\in\lambda\}$ and a closed cover $\mathcal{F}=\{F_\alpha|\alpha\in\lambda\}$ of X such that $F_\alpha\subset G_\alpha\subset U_\alpha$ for each $\alpha\in\lambda$. Furthermore, by the normality of X, there exists a sequence $\{W_{\alpha,n}|n\in\omega\}$ of open sets of X for each $\alpha\in\lambda$ such that

$$F_\alpha\subset W_{\alpha,0}\subset\overline{W}_{\alpha,0}\subset W_{\alpha,1}\subset\cdots\subset W_{\alpha,n}\subset\overline{W}_{\alpha,n}\subset W_{\alpha,n+1}\subset\cdots\subset G_\alpha\,.$$

We put $V_{\alpha,n}=\bigcup_{\beta<\alpha}W_{\beta,n}$ for $\alpha\in\lambda, n\in\omega$, and let $\mathcal{V}=\{V_{\alpha,n}|\alpha\in\lambda,n\in\omega\}$. It is easily proved that $\mathcal{V}$ is a $(\lambda,\bar{\omega})$-increasing open cover of X; in particular, from the local finiteness of $\mathcal{G}$, we have

$$\bar{V}_{\alpha,n}=\overline{\bigcup_{\beta<\alpha}W_{\beta,n}}=\bigcup_{\beta<\alpha}\overline{W}_{\beta,n}\subset\bigcup_{\beta<\alpha}W_{\beta,n+1}=V_{\alpha,n+1}\,.$$

Since

$$V_{\alpha,n}=\bigcup_{\beta<\alpha}W_{\beta,n}\subset\bigcup_{\beta<\alpha}G_\beta\subset\bigcup_{\beta<\alpha}U_\beta\subset U_\alpha\,,$$

$\mathcal{V}$ is an indexed refinement of $\mathcal{U}$. Finally, since $\mathcal{F}$ is a cover of X, $\{V_{\alpha,n}|\alpha\in\lambda\}$ is a cover of X for each $n\in\omega$. This completes the proof.

The following lemma is essentially due to Mack [7].

LEMMA 2.10. *Let μ be an infinite cardinal. Then the following are equivalent for a space X:*

(i) *X is μ-paracompact.*

(ii) *For each well-ordered increasing open cover $\mathcal{U}$ of X with length $\leq\mu$, there exists a σ-locally finite open cover $\mathcal{V}$ of X such that $\overline{\mathcal{V}}=\{\bar{V}\,|\,V\in\mathcal{V}\}$ refines $\mathcal{U}$.*

(iii) *X has the property $R(\lambda)$ for each ordinal $\lambda\leq\mu$.*

(iv) *X has the property $R(\lambda)$ for each regular ordinal $\lambda\leq\mu$.*

PROOF. The equivalence (i)$\Longleftrightarrow$(ii) was proved by Mack [7]. Since each λ-increasing open cover is a well-ordered increasing open cover with length λ, the implication (ii)$\Rightarrow$(iii) is obvious. The implication (iii)$\Rightarrow$(ii) is also obvious, since each well-ordered increasing open cover with length λ has a λ-increasing open refinement. Finally the equivalence (iii)$\Longleftrightarrow$(iv) follows from Lemma 2.7.

PROOF OF THEOREM 1.2. The statement (b) in Theorem 1.2 is equivalent to that X has the property $Q(\lambda)$ for each (regular) ordinal $\lambda\leq\mu$ (cf. (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) in Lemma 2.10). Therefore the implications (a)$\Rightarrow$(b) and (b)$\Rightarrow$(c) follow from Lemmas 2.2, 2.4 and Lemmas 2.8, 2.10, respectively. It remains only to show the im-

plication (c)⇒(a) under the normality of X. Assume that X is μ-paracompact and normal. Of course, X is countably paracompact, since μ is an infinite cardinal. By Lemma 2.10, X has the property $R(\lambda)$ for each (regular) ordinal $\lambda\leq\mu$. Hence, by Lemmas 2.9 and 2.1, X has the property $P(\lambda)$ for each (regular) ordinal $\lambda\leq\mu$. This completes the proof of Theorem 1.2.

3. Application (I). First we prove a lemma used not only in this section but also in the next section.

LEMMA 3.1. *Let X be a space and Y be a compact space. Let $\{C_\alpha|\alpha\in\lambda+1\}$ and $\{G_n|n\in\omega\}$ be respectively a decreasing sequence of length $\lambda+1$ by closed sets of Y and an increasing sequence of length ω by open sets of the product $X\times Y$ such that*

$$\begin{cases} C_\beta=\bigcap_{\alpha<\beta} C_\alpha \text{ for each limit ordinal } \beta\in\lambda+1, \\ \bar{G}_n\subset G_{n+1} \text{ for each } n\in\omega, \text{ and} \\ X\times C_\lambda\subset\bigcup_{n\in\omega} G_n . \end{cases}$$

Then the collection $\{V_{\alpha,n}|\alpha\in\lambda+1, n\in\omega\}$, defined by

$$V_{\alpha,n}=\{x\in X|\{x\}\times C_\alpha\subset G_n\} \quad \text{for} \quad \alpha\in\lambda+1, n\in\omega ,$$

is a $(\lambda+1,\bar{\omega})$-increasing open cover of X.

PROOF. The lemma follows directly from the following five assertions.

(i) $V_{\alpha,n}$ is open in X: This assertion is obvious, because C_α is compact in Y and G_n is open in $X\times Y$.

(ii) $V_{\alpha,n}\subset V_{\beta,n}$ provided $\alpha<\beta$: This follows from the fact $C_\alpha\supset C_\beta$ provided $\alpha<\beta$.

(iii) $V_{\beta,n}\subset\bigcap_{\alpha<\beta} V_{\alpha,n}$ for each limit ordinal $\beta\in\lambda+1$: Let $x\in V_{\beta,n}$, then $\{x\}\times C_\beta\subset G_n$. By assumption, $C_\beta=\bigcap_{\alpha<\beta} C_\alpha$, and so $\bigcap_{\alpha<\beta}(\{x\}\times C_\alpha)\subset G_n$. Since $\{\{x\}\times C_\alpha|\alpha<\beta\}$ is a decreasing sequence of compact closed sets and G_n is open in $X\times Y$, as is easily shown, there exists an element $\alpha_0<\beta$ such that $\{x\}\times C_{\alpha_0}\subset G_n$. Hence $x\in V_{\alpha_0,n}\subset\bigcup_{\alpha<\beta} V_{\alpha,n}$. Thus the assertion is verified.

(iv) $\bar{V}_{\alpha,n}\subset V_{\alpha,n+1}$: From the definition of $V_{\alpha,n}$, we have $V_{\alpha,n}\times C_\alpha\subset G_n$. Hence

$$\bar{V}_{\alpha,n}\times C_\alpha\subset\overline{V_{\alpha,n}\times C_\alpha}\subset\bar{G}_n\subset G_{n+1} ,$$

and hence $\bar{V}_{\alpha,n}\subset V_{\alpha,n+1}$.

(v) $\{V_{\lambda,n}|n\in\omega\}$ is a cover of X: Let $x\in X$. Then $\{x\}\times C_\lambda\subset X\times C_\lambda\subset\bigcup_{n\in\omega} G_n$. Hence $\{x\}\times C_\lambda\subset G_{n_0}$ for some $n_0\in\omega$, because $\{G_n|n\in\omega\}$ is an increasing sequence of open sets

which covers the compact set $\{x\}\times C_\lambda$. Hence $x\in V_{\lambda,n_0}$.

Thus the lemma is completed.

A subset G of a space X is said to be *perfectly open*, if there exists a sequence $\{G_n|n\in\omega\}$ of open sets of X such that $G=\bigcup_{n\in\omega} G_n$ and $\bar{G}_n\subset G_{n+1}$ for each $n\in\omega$. Obviously,

$$\text{cozero} \Rightarrow \text{perfectly open} \Rightarrow \text{open } F_\sigma,$$

and the converses are true in normal spaces.

Let X be a subspace of a compact Hausdorff space Y. For each open cover $\mathcal{U}$ of X, we define a subset $C(\mathcal{U})$ of Y (more precisely, $C_Y(\mathcal{U})$) by

$$C(\mathcal{U})=\bigcap_{U\in\mathcal{U}} \mathrm{Cl}_Y(X-U).$$

(For a subset A of X, $\mathrm{Cl}_Y A$ denotes the closure of A in Y; on the other hand, the closure of A in X is denoted by $\bar{A}$ as usual.) Obviously, $C(\mathcal{U})$ is a closed set of Y such that $C(\mathcal{U})\cap X=\phi$. Hence, in the product $X\times Y$, $X\times C(\mathcal{U})$ and the diagonal Δ $(=\{(x,x)|x\in X\})$ are disjoint closed sets. (Since Y is a Hausdorff space, Δ is closed in $X\times Y$.)

THOREM 3.2. *Let X be a subspace of a compact Hausdorff space Y. If, for each infinite regular cardinal λ and for each λ-increasing open cover $\mathcal{U}$ of X, there exists a perfectly open set G of the product $X\times Y$ such that*

$$X\times C(\mathcal{U})\subset G, \quad G\cap\Delta=\phi,$$

then X is paracompact.

PROOF. By Theorem 1.1 (together with Lemma 2.1), it is sufficient to show that X has the property $P(\lambda+1)$ for each infinite regular cardinal λ. Let $\mathcal{U}=\{U_\alpha|\alpha\in\lambda+1\}$ be a $(\lambda+1)$-increasing open cover of X, and let us construct a $(\lambda+1,\bar{\omega})$-increasing open indexed refinement $\mathcal{V}$ of $\mathcal{U}$. Since λ is a limit ordinal, the subcollection $\mathcal{U}'=\{U_\alpha|\alpha\in\lambda\}$ of $\mathcal{U}$ is a λ-increasing open cover of X. By assumption, there exists a perfectly open set G of $X\times Y$ such that $X\times C(\mathcal{U}')\subset G$ and $G\cap\Delta=\phi$. Put

$$C_\alpha=\bigcap_{\beta<\alpha} \mathrm{Cl}_Y(X-U_\beta) \quad \text{for} \quad \alpha\in\lambda+1,$$

then $\{C_\alpha|\alpha\in\lambda+1\}$ is a decreasing sequence of closed sets of Y. Moreover, $C_\beta=\bigcap_{\alpha<\beta} C_\alpha$ for each limit ordinal $\beta\in\lambda+1$. By definition, we have a sequence $\{G_n|n\in\omega\}$ of open sets of $X\times Y$ such that $G=\bigcup_{n\in\omega} G_n$ and $\bar{G}_n\subset G_{n+1}$ for each $n\in\omega$. In particular, $X\times C_\lambda=X\times C(\mathcal{U}')\subset\bigcup_{n\in\omega} G_n$. Therefore, by Lemma 3.1, the collection $\mathcal{V}=\{V_{\alpha,n}|\alpha\in\lambda+1, n\in\omega\}$, defined by

$$V_{\alpha,n}=\{x\in X|\{x\}\times C_\alpha\subset G_n\} \quad \text{for} \quad \alpha\in\lambda+1, n\in\omega,$$

is a $(\lambda+1,\bar{\omega})$-increasing open cover of X. It remains only to see that $\mathcal{V}$ is an indexed refinement of $\mathcal{U}$. To do this, assume that there is a point x in X such that $x\in V_{\alpha_0,n_0}$ and $x\notin U_{\alpha_0}$ for some $\alpha_0\in\lambda+1$ and some $n_0\in\omega$. Then $x\notin U_\beta$ for each $\beta<\alpha_0$, and so $x\in\bigcap_{\beta<\alpha_0}\mathrm{Cl}_Y(X-U_\beta)=C_{\alpha_0}$. Since $x\in V_{\alpha_0,n_0}$, we have $\{x\}\times C_{\alpha_0}\subset G_n\subset G$. Hence $(x,x)\in G$. This is contradictory to $G\cap\Delta=\phi$. Therefore $V_{\alpha,n}\subset U_\alpha$ for each $\alpha\in\lambda+1$ and $n\in\omega$; that is, $\mathcal{V}$ is an indexed refinement of $\mathcal{U}$. Thus the proof is completed.

COROLLARY 3.3. *Let X be a subspace of a compact Hausdorff space Y. If, for each closed set C of Y with $C\cap X=\phi$, there exists a perfectly open set G of the product $X\times Y$ such that*

$$X\times C\subset G, \quad G\cap\Delta=\phi,$$

then X is paracompact.

COROLLARY 3.4 (Morita [9]). *Let X be a subspace of a compact Hausdorff space Y. If the product $X\times Y$ is normal, then X is paracompact.*

The converse of Corollary 3.4 is true by Dieudonné [3]; consequently the converses of Theorem 3.2 and Corollary 3.3 are also true.

Next we give a characterization for the property $P(\lambda+1)$. For an ordinal λ, we denote by $W(\lambda)$ the set $\lambda=\{\alpha|\alpha<\lambda\}$ topologized with the order topology. As is well-known, $W(\lambda+1)$ is a compact Hausdorff space for each ordinal λ.

THEOREM 3.5. *A space X has the property $P(\lambda+1)$ if and only if, for each open set H of $X\times W(\lambda+1)$ containing $X\times\{\lambda\}$, there exists a perfectly open set G of $X\times W(\lambda+1)$ such that*

$$X\times\{\lambda\}\subset G\subset H.$$

PROOF. To apply Lemma 3.1 to the proof of the theorem, first, we define subsets $C_\alpha, \alpha\in\lambda+1$, of $W(\lambda+1)$ by

$$C_\alpha=\{\beta|\alpha\leq\beta\leq\lambda\}.$$

Then the collection $\{C_\alpha|\alpha\in\lambda+1\}$ is a decreasing sequence of closed sets of $W(\lambda+1)$ such that $C_\beta=\bigcap_{\alpha<\beta}C_\alpha$ for each limit ordinal $\beta\in\lambda+1$. In particular, $C_\lambda=\{\lambda\}$.

Necessity: Assume that X is a $P(\lambda+1)$-space, and let H be an open set of $X\times W(\lambda+1)$ such that $X\times\{\lambda\}\subset H$. If we put

$$U_\alpha=\{x\in X|\{x\}\times C_\alpha\subset H\} \quad \text{for} \quad \alpha\in\lambda+1,$$

then the collection $\mathcal{U}=\{U_\alpha|\alpha\in\lambda+1\}$ is a $(\lambda+1)$-increasing open cover of X; this is

more briefly proved than Lemma 3.1. By assumption, $\mathcal{U}$ has a $(\lambda+1, \bar{\omega})$-increasing open indexed refinement $\{V_{\alpha,n}|\alpha\in\lambda+1, n\in\omega\}$. Let us put

$$G_n=\bigcup_{\alpha\in\lambda+1}(V_{\alpha,n}\times C_\alpha) \quad \text{for} \quad n\in\omega .$$

Since $U_{\beta,n}=\bigcup_{\alpha<\beta} V_{\alpha,n}$ for each limit $\beta\in\lambda+1$, we have

$$G_n=\bigcup_{\alpha\in(\lambda+1)^*}(V_{\alpha,n}\times C_\alpha),$$

where $(\lambda+1)^*=\{\alpha\in\lambda+1|\alpha$ is a non-limit ordinal$\}$. In case that α is a non-limit ordinal, C_α is open in $W(\lambda+1)$. Hence G_n is open in $X\times W(\lambda+1)$ for each $n\in\omega$. To see that $\bar{G}_n\subset G_{n+1}$ for $n\in\omega$, let $(x,\alpha)\in\bar{G}_n$. For the nonce, we shall prove $x\in\bar{V}_{\alpha,n}$. Since $O(\alpha)=\{\beta|\beta\leq\alpha\}$ is an open neighborhood of α in $W(\lambda+1)$, for an arbitrary neighborhood $N(x)$ of x, $N(x)\times O(\alpha)\cap G_n\neq\phi$. Hence there exists an element $\beta\in\lambda+1$ such that $(N(x)\times O(\alpha))\cap(V_{\beta,n}\times C_\beta)\neq\phi$, i.e., $N(x)\cap V_{\beta,n}\neq\phi$ and $O(\alpha)\cap C_\beta\neq\phi$. From the definitions of $O(\alpha)$ and C_β, we have $\beta\leq\alpha$. Consequently, $N(x)\cap V_{\alpha,n}\neq\phi$, and hence $x\in\bar{V}_{\alpha,n}$. Then $(x,\alpha)\in\bar{V}_{\alpha,n}\times C_\alpha\subset V_{\alpha,n+1}\times C_\alpha\subset G_{n+1}$. This verifies $\bar{G}_n\subset G_{n+1}$. Therefore, if we put $G=\bigcup_{n\in\omega} G_n$, then G is a perfectly open set of $X\times W(\lambda+1)$. It is easily seen that $X\times\{\lambda\}\subset G\subset H$.

Sufficiency: Let $\mathcal{U}=\{U_\alpha|\alpha\in\lambda+1\}$ be a $(\lambda+1)$-increasing open cover of X. We put

$$H=\bigcup_{\alpha\in\lambda+1}(U_\alpha\times C_\alpha).$$

Then, as well as G_n above, H is open is $X\times W(\lambda+1)$, since $U_\beta=\bigcup_{\alpha<\beta} U_\alpha$ for each limit $\beta\in\lambda+1$. Obviously, $X\times\{\lambda\}\subset H$. Therefore, by assumption, we have a perfectly open set G of $X\times W(\lambda+1)$ satisfying $X\times\{\lambda\}\subset G\subset H$, and so we have a sequence $\{G_n|n\in\omega\}$ of open sets of $X\times W(\lambda+1)$ such that $X\times\{\lambda\}\subset\bigcup_{n\in\omega} G_n\subset H$ and $\bar{G}_n\subset G_{n+1}$ for $n\in\omega$. By Lemma 3.1, if we define $V_{\alpha,n}=\{x\in X|\{x\}\times C_\alpha\subset G_n\}$ for $\alpha\in\lambda+1$ and $n\in\omega$, then the collection $\mathcal{V}=\{V_{\alpha,n}|\alpha\in\lambda+1, n\in\omega\}$ is a $(\lambda+1,\bar{\omega})$-increasing open cover of X. To see that $\mathcal{V}$ is an indexed refinement of $\mathcal{U}$, let $x\in V_{\alpha,n}$ for $\alpha\in\lambda+1, n\in\omega$. Then $(x,\alpha)\in\{x\}\times C_\alpha\subset G_n\subset H$, and hence $(x,\alpha)\in U_\beta\times C_\beta$ for some $\beta\in\lambda+1$. By $\alpha\in C_\beta$, we have $\beta\leq\alpha$, and consequently $x\in U_\beta\subset U_\alpha$. This prove that $V_{\alpha,n}\subset U_\alpha$ for $\alpha\in\lambda+1$ and $n\in\omega$, and so $\mathcal{V}$ is an indexed refinement of $\mathcal{U}$. Therefore X is a $P(\lambda+1)$-space.

The proof of Theorem 3.5 is completed.

Let μ be an infinite cardinal, and let λ be an arbitrary ordinal with $\lambda\leq\mu$. Then $W(\mu+1)$ contains $W(\lambda+1)$ as a closed subspace. Consequently, if $X\times W(\mu+1)$ is normal, then $X\times W(\lambda+1)$ is normal. Therefore the following corollary is a direct consequence of Theorems 1.2 and 3.5 (together with Lemma 2.1).

COROLLARY 3.6 (K. Kunen). *Let μ be an infinite cardinal. If the product $X\times$*

$W(\mu+1)$ of a space X with $W(\mu+1)$ is normal, then X is μ-paracompact and normal.

The converse of Corollary 3.6 is also true by Morita [9, Theorem 2.2].

4. Application (II). Let X be a normal space and Y be a compact Hausdorff space. As for the normality of the product $X\times Y$, the following result is well-known:

(A) *If X is $w(Y)$-paracompact, then $X\times Y$ is normal* (Morita [9]).

Here $w(Y)$ is the weight of Y. Recently, in [6], the author introduced a cardinal function v such that $v(Y)\leq w(Y)$, and he obtained a result which covers the above result (A); namely,

(B) *if X is $v(Y)$-paracompact and $w(Y)$-collectionwise normal, then $X\times Y$ is normal.*

While, as a necessary condition, there is the following remarkable result:

(C) *If $X\times Y$ is normal, then X is $w(Y)$-collectionwise normal* (Rudin [12], or Morita and Hoshina [10]).

In this section we shall give another necessary condition for the normality of $X\times Y$.

Let λ be an infinite cardinal. According to Arhangel'skiĭ [1], a well-ordered set $\{y_\alpha|\alpha\in\lambda\}$ consisting of points of a space Y is said to be a *free sequence* of length λ in Y, if $\overline{\{y_\beta|\beta<\alpha\}}\cap\overline{\{y_\beta|\alpha\leq\beta\}}=\phi$ for each $\alpha\in\lambda$. Let λ and μ be two infinite cardinals such that $\lambda\leq\mu$. If Y contains a free sequence of length μ, then Y contains that of length λ; indeed, the subset $\{y_\alpha|\alpha\in\lambda\}$ of a free sequence $\{y_\alpha|\alpha\in\mu\}$ of length μ in Y is obviously a free sequence of length λ in Y.

THEOREM 4.1. *Let μ be an infinite cardinal, and let Y be a compact Hausdorff space in which there exists a free sequence of length μ. If the product $X\times Y$ of a space X with Y is normal, then X is μ-paracompact.*

PROOF. Let λ be an arbitrarily fixed infinite cardinal with $\lambda\leq\mu$. Let $\{y_\alpha|\alpha\in\lambda\}$ be a free sequence of length λ in Y; Y contains a free sequence of length λ, since it contains a free sequence of length μ. For each $\alpha\in\lambda+1$, we define

$$C_\alpha=\bigcap_{\beta<\alpha}\overline{\{y_\gamma|\beta\leq\gamma\}}\,,\qquad D_\alpha=Y-\overline{\{y_\beta|\beta<\alpha\}}\,.$$

Then the collection $\{C_\alpha|\alpha\in\lambda+1\}$ (resp. $\{D_\alpha|\alpha\in\lambda+1\}$) is a decreasing sequence of closed (resp. open) sets of Y. Moreover, $C_\beta=\bigcap_{\alpha<\beta}C_\alpha$ for each limit ordinal $\beta\in\lambda+1$.

Now, to prove that X has the property $P(\lambda)$, let $\mathcal{U}=\{U_\alpha|\alpha\in\lambda\}$ be a λ-increasing open cover of X. Put

$$H=\bigcup_{\alpha\in\lambda}(U_\alpha\times D_\alpha)\,,$$

then H is an open set of $X\times Y$. For each $\alpha\in\lambda$, we have

$$C_\lambda \subset C_{\alpha+1} = \overline{\{y_\beta | \alpha \leq \beta\}} \subset Y - \overline{\{y_\beta | \beta < \alpha\}} = D_\alpha .$$

Hence $X \times C_\lambda \subset H$, because $\{U_\alpha | \alpha \in \lambda\}$ is a cover of X. From the assumption of the normality of $X \times Y$, we have a sequence $\{G_n | n \in \omega\}$ of open sets of $X \times Y$ such that

$$X \times C_\lambda \subset G_0 \subset \bar{G}_0 \subset G_1 \subset \cdots \subset G_n \subset \bar{G}_n \subset G_{n+1} \subset \cdots \subset H .$$

Define

$$V_{\alpha,n} = \{x \in X | \{x\} \times C_\alpha \subset G_n\} \quad \text{for} \quad \alpha \in \lambda+1, n \in \omega ,$$

then, by Lemma 3.1, the collection $\mathcal{V}^* = \{V_{\alpha,n} | \alpha \in \lambda+1, n \in \omega\}$ is a $(\lambda+1, \bar{\omega})$-increasing open cover of X. Since λ is a limit ordinal, the subcollection $\mathcal{V} = \{V_{\alpha,n} | \alpha \in \lambda, n \in \omega\}$ of $\mathcal{V}^*$ is a cover of X, so that $\mathcal{V}$ is a $(\lambda, \bar{\omega})$-increasing open cover of X. It remains to prove that $\mathcal{V}$ is an indexed refinement of $\mathcal{U}$. To do this, let $x \in V_{\alpha,n}$ for $\alpha \in \lambda$ and $n \in \omega$. Then $\{x\} \times C_\alpha \subset G_n \subset H$. From the definition of C_α, we have $y_\alpha \in C_\alpha$, and so $(x, y_\alpha) \in H$. Hence $(x, y_\alpha) \in U_\beta \times D_\beta$ for some $\beta \in \lambda$. From the definition of D_β and the fact $y_\alpha \in D_\beta$, we obtain $\beta \leq \alpha$. Hence $x \in U_\beta \subset U_\alpha$. This prove that $\mathcal{V}$ is an indexed refinement of $\mathcal{U}$. Hence X is a $P(\lambda)$-space.

Thus it is proved that X has the property $P(\lambda)$ for each infinite (regular) cardinal $\lambda \leq \mu$. Therefore, by Theorem 1.2, X is μ-paracompact.

REMARK. Corollary 3.6 is also a corollary to Theorem 4.1, since the space $W(\mu+1)$ contains a free sequence of length μ.

LEMMA 4.2 (Arhangel'skiĭ [1]). *For a non-discreate compact Hausdorff space Y,*

$$t(Y) = \sup\{\lambda | \textit{there is a free sequence of length } \lambda \textit{ in } Y\} .$$

Let μ be an infinite cardinal. A space X is said to be *μ^--paracompact*, if each open cover with cardinality $< \mu$ has a locally finite open refinement. Obviously, μ-paracompactness (in the usual sence) is equivalent to $(\mu^+)^-$-paracompactness, where μ^+ denotes the cardinal successor of μ; that is, μ^+ is the smallest cadinal greater than μ. It is also obvious that X is μ^--paracompact if and only if X is λ-paracompact for each infinite cardinal $\lambda < \mu$.

LEMMA 4.3. *If an infinite cardinal μ is singular (=non-regular), then*

$$\mu\textit{-paracompact} \Longleftrightarrow \mu^-\textit{-paracompact} .$$

This follows from Lemma 2.10.

Let Y be a non-discrete compact Hausdorff space. Temporarily, we say that Y has the *property* (*), if it satisfies either one of the following two conditions:

$(*)_1$ There exists $\max\{\lambda | \text{there is a free sequence of length } \lambda \text{ in } Y\}$.

$(*)_2$ $t(Y)$ is singular.

LEMMA 4.4. *Let Y be a non-discrete compact Hausdorff space. If $t(Y)$ is not weakly inaccessible, then Y has the property* (*).

PROOF. Each non-discrete compact Hausdorff space contains at least a free sequence of length ω. Hence, in case $t(Y)=\omega$, by Lemma 4.2, Y satisfies the condition $(*)_1$ so that it has the property (*). If $t(Y)$ has the cardinal predecessor μ, i.e., $t(Y)=\mu^+$, then it is obvious from Lemma 4.2 that Y satisfies $(*)_1$ and hence Y has the property (*). Of course, in case that $t(Y)$ is singular, Y has the property (*). Therefore, in case that $t(Y)$ is not weakly inaccessible, Y has always the property (*).

THEOREM 4.5. *Let Y be a non-discrete compact Hausdorff space. If the product $X\times Y$ of a space X with Y is normal, then X is $t(Y)^-$-paracompact. If, furthermore, Y has the property* (*), *then X is $t(Y)$-paracompact.*

PROOF. By Lemma 4.2, for each infinite cardinal $\mu<t(Y)$, there exists a free sequence of length μ in Y. Therefore, by Theorem 4.1, X is μ-paracompact, and hence X is $t(Y)^-$-paracompact. Consequently, if $t(Y)$ is singular, X is $t(Y)$-paracompact by Lemma 4.3. On the other hand, in case that Y satisfies the condition $(*)_1$, $t(Y)=\max\{\lambda|$there is a free sequence of length λ in $Y\}$ by Lemma 4.2. Hence Y contains a free sequence of length $t(Y)$, and so X is $t(Y)$-paracompact by Theorem 4.1. In either case, X is $t(Y)$-paracompact when Y has the property (*).

COROLLARY 4.6. *Let Y be a non-discrete compact Hausdorff space such that $t(Y)$ is not weakly inaccessible. If $X\times Y$ is normal, then X is $t(Y)$-paracompact.*

As is well-known, the space $W(\mu^+)$ is not μ^+-paracompact; indeed, the open cover $\{O(\alpha)|\alpha\in\mu^+\}$, where $O(\alpha)=\{\beta|\beta\leq\alpha\}$, of $W(\mu^+)$ has no locally finite open refinements. Therefore we have

COROLLARY 4.7 (Nogura [11]). *Let μ be an infinite cardinal, and let Y be a non-discrete compact Hausdorff space. If $W(\mu^+)\times Y$ is normal, then $t(Y)\leq\mu$.*

The converse of Corollary 4.7 holds ([11]).

THEOREM 4.8. *Let X be a normal space and let Y be a non-discrete compact Hausdorff space with the property* (*).

(a) *In case $t(Y)=w(Y)$, the product $X\times Y$ is normal if and only if X is $t(Y)$-paracompact.*

(b) *In case $t(Y)=v(Y)$, $X\times Y$ is normal if and only if X is $t(Y)$-paracompact and $w(Y)$-collectionwise normal.*

PROOF. (a) (resp. (b)) follows from Theorem 4.5 together with (A) (resp. (B) and (C)) above-mentioned.

COROLLARY 4.9. *Let X be a normal space and let Y be a non-discrete dyadic compact Hausdorff space with the property (*). Then the product $X\times Y$ is normal if and only if X is $t(Y)$-paracompact.*

PROOF. By [2], $t(Y)=w(Y)$ for each non-discrete dyadic compact Hausdorff space Y. Therefore the corollary follows immediately from Theorem 4.8.

The following example shows that Theorem 4.8 or the latter part of Theorem 4.5 is not necessarily true if we omit the property (*) from Y.

EXAMPLE 4.10. Assume that μ be a weakly inaccessible cardinal. Let X be the space $W(\mu)$, and let Y be the one-point compactification of the topological sum of disjoint spaces Y_λ's where λ runs over all infinite cardinals less than μ and Y_λ is the space (homeomorphic to) $W(\lambda+1)$ for each λ. Then we have the following facts:

(a) X is a collectionwise normal space which is μ^--paracompact (more strongly, μ^--compact) but not μ-paracompact.

(b) Y is a compact Hausdorff space with $t(Y)=w(Y)=\mu$.

(c) $X\times Y$ is normal.

From the assumption that μ is wekly inaccessible, we have

(i) $cf(\mu)=\mu>\omega$,

(ii) $\mu=\sup\{\lambda|\lambda$ is an infinite cardinal less than $\mu\}$.

(i) and (ii) are respectively essential for (a) and (b); (c) is a special case of the following proposition:

PROPOSITION 4.11. *Let ν be an infinite cardinal. Let X be a countably paracompact ν-collectionwise normal space and Y be the one-point compactification of the topological sum of disjoint compact Hausdorff spaces $Y_\alpha, \alpha\in\nu$. If $X\times Y_\alpha$ is normal for each $\alpha\in\nu$, then $X\times Y$ is normal.*

The proof of Proposition 4.11 is analogous to that of [6, Proposition 3.4].

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