TSUKUBA J. MATH.
Vol. 17 No. 2 (1993), 497–512

COMPLETE SPACE-LIKE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF AN INDEFINITE SPACE FORM

By

Qing-ming CHENG* and Soon Meen CHOI

1. Introduction.

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite Riemannian manifold of index p and of constant curvature c, which is called *an indefinite space form of index p*. According to $c>0$, $c=0$ or $c<0$ it is denoted by $S_p^{n+p}(c)$, $\boldsymbol{R}_p^{n+p}$ or $H_p^{n+p}(c)$. A submanifold M of an indefinite space form $M_p^{n+p}(c)$ is said to be *space-like* if the induced metric on M from that of the ambient space is positive definite. It is pointed out by some physicians that space-like hypersurfaces with constant mean curvature of arbitrary spacetimes get interested in relativity theory and an entire space-like hypersurface with constant mean curvature of an indefinite space form are studied by many authors (for examples: [1], [2], [3], [4], [7], [12] and so on).

Now, for a complete space-like submanifold M with parallel mean curvature vector of $S_p^{n+p}(c)$, it is also seen by the first author [5] that M is totally umbilic if $n=2$ and $h^2\leqq 4c$ or if $n>2$ and $h^2<4(n-1)c$, where H denotes the mean curvature, i.e., the norm of the mean curvature vector and $h=nH$. On the other hand, the first author and Nakagawa [6] investigated the total umbilicness of such hypersurfaces from the different point of view. They proved that the squared norm S of the second fundamental form of M is bounded from above by $S_+(1)$ and if $\sup S<S_-(1)$ and $H^2\leqq c$, then M is totally umbilic, where

$$S_{\pm}(p)=-pnc+\frac{nh^2\pm(n-2)\sqrt{h^4-4(n-1)ch^2}}{2(n-1)}.$$

In this paper, we research the similar problem to the above property for the complete space-like submanifolds with parallel mean curvature vector of an indefinite space form. That is, we prove the following

---

* The project is supported by National Natural Science Foundation of China.
Received November 5, 1992.

THEOREM 1. *Let M be an n-dimensional complete space-like submanifold with parallel mean curvature vector of an indefinite space form $M_p^{n+p}(c)$. If the one of the following conditions is satisfied:*

(1) $c \leqq 0$,

(2) $c>0$ *and* $n^2H^2 \geqq 4(n-1)c$,

then

$$S \leqq S_+(p)+K(p), \tag{1.1}$$

where $K(p)$ is a constant defined by

$$K(p)=(p-1)H\{nH+\sqrt{n(n-1)\{S_+(1)-nH^2\}}\}.$$

THEOREM 2. *The hyperbolic cylinder $H^1(c_1)\times \boldsymbol{R}^{n-1}$ in $\boldsymbol{R}_1^{n+1}$ is the only complete connected space-like n-dimensional submanifolds with parallel mean curvature vector of $\boldsymbol{R}_p^{n+p}$ satisfying $S=S_+(p)+K(p)$.*

THEOREM 3. *The hyperbolic cylinder $H^1(c_1)\times H^{n-1}(c_2)$ of $H_1^{n+1}(c)$ and the maximal submanifolds $H^{n_1}(c_1)\times\cdots\times H^{n_{p+1}}(c_{p+1})$ of $H_p^{n+p}(c)$ are the only complete connected space-like n-dimensional submanifolds with parallel mean curvature vector satisfying $S=S_+(p)+K(p)$, where $c_r=(n/n_r)c$ and $\sum_{r=1}^{p+1} n_r=n$ in the latter case.*

2. Standard models.

This section is concerned with some standard models of complete space-like submanifolds with parallel mean curvature vector of an indefinite space form $M_p^{n+p}(c)$, $c \leqq 0$. In particular, we only consider non-totally umbilic cases. Moreover, the squared norms of the second fundamental forms of such standard models are calculated. Without loss of generality, an $(n+p)$-dimensional indefinite Euclidean space $\boldsymbol{R}_p^{n+p}$ of index $p(\geqq 1)$ can be first regarded as a product manifold of

$$\boldsymbol{R}_1^{n_1+1}\times\cdots\times\boldsymbol{R}_1^{n_p+1}\times\boldsymbol{R}^m,$$

where $\sum_{r=1}^{p} n_r+m=n$. With respect to the standard orthonormal basis of $\boldsymbol{R}_p^{n+p}$ a class of space-like submanifolds

$$H^{n_1}(c_1)\times\cdots\times H^{n_p}(c_p)\times\boldsymbol{R}^m$$

of $\boldsymbol{R}_p^{n+p}$ is defined as the Pythagorean product

$$H^{n_1}(c_1)\times\cdots\times H^{n_p}(c_p)\times\boldsymbol{R}^m$$
$$=\left\{(x_1,\cdots,x_{p+1})\in\boldsymbol{R}_p^{n+p}=\boldsymbol{R}_1^{n_1+1}\times\cdots\times\boldsymbol{R}_1^{n_p+1}\times\boldsymbol{R}^m:\ |x_r|^2=-\frac{1}{c_r}>0\right\},$$

where $r=1, \cdots, p$ and $|\ |$ denotes the norm defined by the product on the Minkowski space R_1^{k+1} which is given by $\langle x, x\rangle=-(x_0)^2+\sum_{j=1}^k(x_j)^2$. The mean curvature vector h of M is given by

$$(2.1)\qquad h=-\frac{1}{n}\sum_{r=1}^{p} n_r c_r x_r$$

at $(x_1, \cdots, x_{p+1})\in M$, which is parallel in the normal bundle of M. The number $S_+(1)$ and the squared norm S of the second fundamental form are given by

$$(2.2)\qquad S_+(1)=n^2H^2=-\sum_{r=1}^{p} n_r^2 c_r, \qquad S=-\sum_{r=1}^{p} n_r c_r .$$

Then we get

$$S_+(p)+K(p)=pn^2H^2=-p\sum_{r=1}^{p} n_r^2 c_r \geqq S,$$

where the equality holds if and only if $p=1$ and $n_1=1$.

Next we consider an n-dimensional space-like submanifold of $H_p^{n+p}(c)$, $p\geqq 1$. Without loss of generality, an $(n+p+1)$-dimensional indefinite Euclidean space R_{p+1}^{n+p+1} of index $(p+1)$ can be first regarded as a product manifold of

$$R_1^{n_1+1}\times \cdots \times R_1^{n_{p+1}+1},$$

where $\sum_{r=1}^{p+1} n_r=n$. With respect to the standard orthonormal basis of R_{p+1}^{n+p+1} a class of space-like submanifolds

$$H^{n_1}(c_1)\times \cdots \times H^{n_{p+1}}(c_{p+1})$$

of R_{p+1}^{n+p+1} is defined as the Pythagorean product

$$H^{n_1}(c_1)\times\cdots\times H^{n_{p+1}}(c_{p+1})$$
$$=\left\{(x_1, \cdots, x_{p+1})\in R_{p+1}^{n+p+1}=R_1^{n_1+1}\times\cdots\times R_1^{n_{p+1}+1} :\ |x_r|^2=-\frac{1}{c_r}>0\right\},$$

where $r=1, \cdots, p+1$. The mean curvature vector h of M is given by

$$(2.3)\qquad h=-\frac{1}{n}\sum_{r=1}^{p+1}(n_r c_r x_r)+cx$$

at $x=(x_1, \cdots, x_{p+1})\in M$, which is parallel in the normal bundle of M. The norm H of the mean curvature vector h and the squared norm S of the second fundamental form are given by

$$(2.4)\qquad h^2=n^2H^2=n^2c-\sum_{r=1}^{p+1} n_r^2 c_r, \qquad S=\sum_{r=1}^{p+1} n_r(c-c_r)=nc-\sum_{r=1}^{p+1} n_r c_r .$$

When M is maximal, it satisfies $n_r c_r=nc$ for any index r by (2.3), which yields $S=-pnc$. Then we get $S_+(p)+K(p)-S=0$, because of $S_+(p)=-pnc$ and $K(p)=0$.

Suppose that $H\neq 0$. By a theorem of Ki, Kim and Nakagawa [9], if $p=1$, then we have $S_{+}(1)-S=0$. On the other hand, we have $S_{+}(1)>h^{2}-nc$, because of $c<0$. So it is seen that if $p\geqq 2$, then we obtain

$$S_{+}(p)+K(p)-S>h^{2}-pnc+(p-1)h^{2}-S=ph^{2}-pnc-S\geqq 0$$

by (2.4). In order to prove the last inequality, the following lemma is prepared. The proof of this lemma is the only calculus and hence it is omitted.

LEMMA 2.1. *Let* $a_{1}, \cdots, a_{p+1}$ *be numbers not less than* 1 *satisfying* $\sum a_{r}=n$ *and* $b_{1}, \cdots, b_{p+1}$ *be negative numbers satisfying* $\sum(1/b_{r})=(1/b)$. *Then we have*

$$\sum\{a_{r}-p(a_{r})^{2}\}b_{r}\geqq n(p+1-pn)b.$$

3. Preliminaries.

Throughout this paper all manifolds are assumed to be smooth, connected without boundary. We discuss in smooth category. Let $M_{p}^{n+p}(c)$ be an $(n+p)$-dimensional indefinite Riemannian manifold of constant curvature c whose index is p, which is called *an indefinite space form of constant curvature c and with index p*. Let M be an n-dimensional submanifold of an $(n+p)$-dimensional indefinite space form $M_{p}^{n+p}(c)$ of index p. The submanifold M is said to be *space-like* if the induced metric on M from that of the ambient space is positive definite. We choose a local field of orthonormal frames $e_{1}, \cdots, e_{n+p}$ adapted to the indefinite Riemannian metric of $M_{p}^{n+p}(c)$ and the dual coframes $\omega_{1}, \cdots, \omega_{n+p}$ in such a way that, restricted to the submanifold M, $e_{1}, \cdots, e_{n}$ are tangent to M. Then connection forms $\{\omega_{AB}\}$ of $M_{p}^{n+p}(c)$ are characterized by the structure equations

$$\begin{cases} d\omega_{A}+\sum\varepsilon_{B}\omega_{AB}\wedge\omega_{B}=0, \qquad \omega_{AB}+\omega_{BA}=0, \\ d\omega_{AB}+\sum\varepsilon_{C}\omega_{AC}\wedge\omega_{CB}=\Omega_{AB}, \\ \Omega_{AB}=-\dfrac{1}{2}\sum\varepsilon_{C}\varepsilon_{D}R'_{ABCD}\omega_{C}\wedge\omega_{D}, \end{cases} \tag{3.1}$$

$$R'_{ABCD}=c\varepsilon_{A}\varepsilon_{B}(\delta_{AD}\delta_{BD}-\delta_{AC}\delta_{BD}), \tag{3.2}$$

where $\varepsilon_{A}=1$ for an index $A\leqq n$, $\varepsilon_{A}=-1$ for an index $A\geqq n+1$, and Ω_{AB} (resp. R'_{ABCD}) denotes the indefinite Riemannian curvature form (resp. the components of the indefinite Riemannian curvature tensor R$'$) of $M_{p}^{n+p}(c)$. Therefore the components of the Ricci curvature tensor *Ric$'$* and the scalar curvature r' of $M_{p}^{n+p}(c)$ are given as

$$R'_{AB}=c(n+p-1)\varepsilon_{A}\delta_{AB}, \qquad r'=(n+p)(n+p-1)c.$$

In the sequel, the following convention on the range of indices is used, unless otherwised stated:

$$1\leqq A, B, \cdots\leqq n+p;\quad 1\leqq i, j, \cdots\leqq n;\quad n+1\leqq\alpha, \beta, \cdots\leqq n+p.$$

We agree that the repeated indices under a summation sign without indication are summed over the respective range. The canonical forms $\{\omega_A\}$ and the connection forms $\{\omega_{AB}\}$ restricted to M are also denoted by the same symbols. We then have

$$\omega_\alpha=0 \quad \text{for} \quad \alpha=n+1, \cdots, n+p. \tag{3.3}$$

We see that $e_1, \cdots, e_n$ is a local field of orthonormal frames adapted to the induced Riemannian metric on M and $\omega_1, \cdots, \omega_n$ is a local field of its dual coframes on M. It follows from (3.1), (3.3) and Cartan's lemma that we have

$$\omega_{\alpha i}=\sum h_{ij}^\alpha\omega_j, \qquad h_{ij}^\alpha=h_{ji}^\alpha. \tag{3.4}$$

The second fundamental form α and the mean curvature vector $\boldsymbol{h}$ of M are defined by

$$\alpha=-\sum h_{ij}^\alpha\omega_i\omega_j e_\alpha, \qquad \boldsymbol{h}=-\frac{1}{n}\sum(\sum_i h_{ii}^\alpha)e_\alpha.$$

The mean curvature H is defined by

$$H=|\boldsymbol{h}|=\frac{1}{n}\sqrt{\sum(\sum_i h_{ii}^\alpha)^2}. \tag{3.5}$$

Let $S=\sum(h_{ij}^\alpha)^2$ denote the squared norm of the second fundamental form α of M. The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations

$$\begin{cases} d\omega_i+\sum\omega_{ij}\wedge\omega_j=0, \qquad \omega_{ij}+\omega_{ji}=0, \\ d\omega_{ij}+\sum\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}, \\ \Omega_{ij}=-\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l, \end{cases} \tag{3.6}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R) of M. Therefore, from (3.1) and (3.6), the Gauss equation is given by

$$R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})-\sum(h_{il}^\alpha h_{jk}^\alpha-h_{ik}^\alpha h_{jl}^\alpha). \tag{3.7}$$

The components of the Ricci curvature Ric and the scalar curvature r are given by

$$R_{jk}=(n-1)c\delta_{jk}-\sum h_{ii}^\alpha h_{jk}^\alpha+\sum h_{ji}^\alpha h_{ik}^\alpha, \tag{3.8}$$

$$r=n(n-1)c-n^2H^2+\sum(h_{ij}^\alpha)^2. \tag{3.9}$$

We also have

$$d\omega_{\alpha\beta}-\sum\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}=-\frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j, \tag{3.10}$$

where

$$R_{\alpha\beta ij}=-\sum(h^\alpha_{il}h^\beta_{jl}-h^\alpha_{jl}h^\beta_{il}).$$

The Codazzi equation and the Ricci formula for the second fundamental form are given by

$$h^\alpha_{ijk}-h^\alpha_{ikj}=0, \tag{3.11}$$

$$h^\alpha_{ijkl}-h^\alpha_{ijlk}=-\sum h^\alpha_{im}R_{mjkl}-\sum h^\alpha_{mj}R_{mikl}+\sum h^\beta_{ij}R_{\beta\alpha kl}, \tag{3.12}$$

where h^α_{ijk} and h^α_{ijkl} denote the components of the covariant differentials $\nabla\alpha$ and $\nabla^2\alpha$ of the second fundamental form, respectively. The Laplacian Δh^α_{ij} of the components h^α_{ij} of the second fundamental form α is given by

$$\Delta h^\alpha_{ij}=\sum h^\alpha_{ijkk}.$$

From (3.12) we get

$$\Delta h^\alpha_{ij}=\sum_k h^\alpha_{kkij}-\sum h^\alpha_{km}R_{mijk}-\sum h^\alpha_{mi}R_{mkjk}+\sum h^\beta_{ki}R_{\beta\alpha jk}. \tag{3.13}$$

The following generalized maximum principle due to Omori [11] and Yau [15] will play an important role in this paper.

THEOREM 3.1. *Let M be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from above on M, then for any $\varepsilon>0$, there exists a point p in M such that*

$$F(p)+\varepsilon>\sup F, \qquad |\operatorname{grad} F|(p)<\varepsilon, \qquad \Delta F(p)<\varepsilon.$$

The following lemma is already known.

LEMMA 3.2. *Let $a_1, \cdots, a_n$ be real numbers satisfying $\sum a_i=0$ and $\sum a_i^2=k^2$ for $k>0$. Then we have*

$$|\sum a_i^3|\leqq(n-2)\sqrt{\frac{1}{n(n-1)}}k^3,$$

where the equality holds if and only if $n-1$ of them are equal with each other.

4. Pseudo-umbilic submanifolds.

Let M be an n-dimensional space-like submanifold with parallel mean curvature vector $\boldsymbol{h}$ of an indefinite space form $M^{n+p}_p(c)$. Because the mean curvature vector is parallel, the mean curvature is constant. Suppose that

$H\neq 0$. We choose e_{n+1} in such a way that its direction coincides with that of the mean curvature vector. Then it is easily seen that we have

$$\omega_{\alpha n+1}=0, \qquad H=\text{constant}, \tag{4.1}$$

$$H^{\alpha}H^{n+1}=H^{n+1}H^{\alpha}, \tag{4.2}$$

$$trH^{n+1}=nH, \qquad trH^{\alpha}=0 \tag{4.3}$$

for any $\alpha\neq n+1$, where H^{α} denotes an $n\times n$ symmetric matrix (h_{ij}^{α}).

A submanifold M is said to be *pseudo-umbilic*, if it is umbilic with respect to the direction of the mean curvature vector $\boldsymbol{h}$, that is,

$$h_{ij}^{n+1}=H\delta_{ij}. \tag{4.4}$$

We denote by μ an $n\times n$ symmetric matrix with $\mu_{ij}=h_{ij}^{n+1}-H\delta_{ij}$. Then we have

$$tr\mu=0, \qquad |\mu|^{2}=tr(\mu)^{2}=\sum(\mu_{ij})^{2}=tr(H^{n+1})^{2}-nH^{2}. \tag{4.5}$$

So the pseudo-umbilic submanifolds are characterized by the property $\mu=0$. A non-negative function τ is defined by $\tau^{2}=\sum_{\beta\neq n+1}(h_{ij}^{\beta})^{2}$. We then have

$$S=|\mu|^{2}+\tau^{2}+nH^{2}. \tag{4.6}$$

Hence it is seen that $|\mu|^{2}$ as well as τ^{2} are independent of the choice of the frame fields and they are functions defined globally on M.

PROPOSITION 4.1. *Let M be n-dimensional complete space-like submanifold with parallel mean curvature vector of an indefinite space form $S_{p}^{n+p}(c)$. If it satisfies*

$$n^{2}c\geqq n^{2}H^{2}\geqq 4(n-1)c, \qquad S\leqq S_{-}(1),$$

then M is pseudo-umbilic, where H denotes the mean curvature, i.e., the norm of the mean curvature vector.

PROOF. In order to prove this property it suffices to show $\mu=0$. From (3.13), the Gauss equation (3.7) and (3.10), we have

$$\begin{aligned}\Delta h_{ij}^{n+1}=nch_{ij}^{n+1}-ncH\delta_{ij}+\sum h_{km}^{n+1}h_{mk}^{\beta}h_{ij}^{\beta}-2\sum h_{ik}^{\beta}h_{km}^{n+1}h_{mj}^{\beta}\\ +\sum h_{im}^{n+1}h_{mk}^{\beta}h_{kj}^{\beta}-nH\sum h_{im}^{n+1}h_{mj}^{n+1}+\sum h_{ik}^{\beta}h_{km}^{\beta}h_{mj}^{n+1}.\end{aligned} \tag{4.7}$$

Accordingly we obtain from (4.2)

$$\begin{aligned}\frac{1}{2}\Delta|\mu|^{2}=\sum(h_{ijk}^{n+1})^{2}+nc\sum(h_{ij}^{n+1})^{2}-n^{2}cH^{2}\\ +\sum h_{km}^{n+1}h_{mk}^{\beta}h_{ij}^{\beta}h_{ij}^{n+1}-2\sum h_{ik}^{\beta}h_{km}^{n+1}h_{mj}^{\beta}h_{ij}^{n+1}+\sum h_{im}^{n+1}h_{mk}^{\beta}h_{kj}^{\beta}h_{ij}^{n+1}\end{aligned}$$

$$-nH\sum h_{im}^{n+1}h_{mj}^{n+1}h_{ij}^{n+1}+\sum h_{ik}^{\beta}h_{km}^{\beta}h_{mj}^{n+1}h_{ij}^{n+1}$$

and hence we see

$$\begin{aligned}\frac{1}{2}\Delta|\mu|^2&=\sum(h_{ijk}^{n+1})^2+nc\sum(h_{ij}^{n+1})^2\\ &\quad -n^2cH^2-nH\,\mathrm{tr}(H^{n+1})^3-\sum_{\beta\neq n+1}\mathrm{tr}(H^{n+1}H^{\beta}-H^{\beta}H^{n+1})^2\\ &\quad +\{\mathrm{tr}(H^{n+1})^2\}^2+\sum_{\beta\neq n+1}\{\mathrm{tr}(H^{n+1}H^{\beta})\}^2.\end{aligned}\tag{4.8}$$

On the other hand, because of

$$\mathrm{tr}(H^{n+1})^3=\mathrm{tr}\,\mu^3+3H\{\mathrm{tr}(H^{n+1})^2-nH^2\}+nH^3,$$

we get

$$\begin{aligned}\frac{1}{2}\Delta|\mu|^2&\geqq(|\mu|^2+nH^2)^2-nH\{\mathrm{tr}\,\mu^3+3H|\mu|^2+nH^3\}+nc|\mu|^2\\ &=|\mu|^2(|\mu|^2+nc-nH^2)-nH\,\mathrm{tr}\,\mu^3.\end{aligned}\tag{4.9}$$

Because of $\mathrm{tr}\,\mu=0$, we can apply Lemma 3.2 to the eigenvalues of μ and obtain

$$|\mathrm{tr}\,\mu^3|\leqq\frac{n-2}{\sqrt{n(n-1)}}|\mu|^3.\tag{4.10}$$

Hence we obtain

$$\frac{1}{2}\Delta|\mu|^2\geqq|\mu|^2\Big(|\mu|^2-nH\frac{n-2}{\sqrt{n(n-1)}}|\mu|+nc-nH^2\Big),\tag{4.11}$$

where we have used (4.9) and (4.10). From (3.8) we know that the Ricci curvature of M is bounded from below. Putting $F=-1/\sqrt{|\mu|^2+a}$ for any positive number a. Since M is complete and space-like, we can apply the Generalized Maximum Principle (Theorem 3.1) to the function F. For any given positive number $\varepsilon>0$, there exists a point p at which F satisfies

$$\sup F<F(p)+\varepsilon,\qquad |\mathrm{grad}\,F|(p)<\varepsilon,\qquad \Delta F(p)<\varepsilon.\tag{4.12}$$

Consequently the following relationship

$$\frac{1}{2}F(p)^4\Delta|\mu|^2(p)<3\varepsilon^2-F(p)\varepsilon\tag{4.13}$$

can be derived by the simple and direct calculations. For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m\to0$ $(m\to\infty)$ and $\varepsilon_m>0$, there exists a point sequence $\{p_m\}$ such that $\{F(p_m)\}$ converges to $F_0=\sup F$ by (4.12). On the other hand, it follows from (4.13) that we have

$$\frac{1}{2}F(p_m)^4\Delta|\mu|^2(p_m)<3\varepsilon_m{}^2-F(p_m)\varepsilon_m.\tag{4.14}$$

The right hand side of (4.14) converges to 0 because F is bounded. Accordingly, for any positive number $\varepsilon>0$ ($\varepsilon<2$) there exists a sufficiently large integer m for which we have

$$F(p_m)^4\Delta|\mu|^2(p_m)<\varepsilon.$$

Hence we get

$$(2-\varepsilon)|\mu|^4(p_m)-2nH\frac{n-2}{\sqrt{n(n-1)}}|\mu|^3(p_m)$$
$$+2(nc-nH^2-\varepsilon a)|\mu|^2(p_m)-\varepsilon a^2<0.$$

Thus the sequence $\{|\mu|^2(p_m)\}$ is bounded and the definition of F gives rise to

$$\lim_{m\to\infty}|\mu|^2(p_m)=\sup|\mu|^2. \tag{4.15}$$

Therefore the supremum of F satisfies $F_0=\sup F<0$. According to (4.14) we have

$$\lim_{m\to\infty}\sup\Delta|\mu|^2(p_m)\leqq 0. \tag{4.16}$$

Thus (4.11) and (4.16) yield

$$0\geqq\sup|\mu|^2\Big(\sup|\mu|^2-nH\frac{n-2}{\sqrt{n(n-1)}}\sup|\mu|+nc-nH^2\Big). \tag{4.17}$$

Taking account of (4.5) we have

$$\sup\sum(h_{ij}^{n+1})^2=nH^2 \quad\text{or}\quad S_-(1)\leqq\sup\sum(h_{ij}^{n+1})^2\leqq S_+(1), \tag{4.18}$$

from which combining with the assumption of Proposition 4.1 it follows that we have

$$\sup\sum(h_{ij}^{n+1})^2=nH^2.$$

This means that $\mu=0$ because of (4.5) and therefore M is pseudo-umbilic. □

The inequality (4.17) holds on the space-like submanifold M of $M_p^{n+p}(c)$. Accordingly, in this case we have

$$\sup\sum(h_{ij}^{n+1})^2=nH^2 \quad\text{or}\quad \sup\sum(h_{ij}^{n+1})^2\leqq S_+(1). \tag{4.19}$$

REMARK. When $p=1$, the hypersurface M becomes totally umbilic under the assumption of Proposition 4.1, which means that this property is a generalization of the theorem due to the first author and Nakagawa [6].

5. Proof of Theorem 1.

In this section the squared norm S of the second fundamental form of M is estimated from above. Let M be an n-dimensional space-like submanifold with parallel mean curvature vector $\boldsymbol{h}$ of an indefinite space form $M_p^{n+p}(c)$.

PROOF OF THEOREM 1. Because the mean curvature vector is parallel, the mean curvature is constant. If $H=0$, then from Theorem 1.1 due to Ishihara [8], we know that M is totally geodesic if $c\geqq 0$ and $S\leqq -npc$ if $c<0$. Hence Theorem 1 is true. Next we may suppose $H\neq 0$. We choose e_{n+1} in such a way that its direction coincides with that of the mean curvature vector. Then we get (4.1), (4.2) and (4.3). From (3.13), the Gauss equation (3.7) and (3.10) we get

$$\begin{aligned}\frac{1}{2}\Delta\tau^2 &= \sum_{\alpha\neq n+1}(h_{ijk}^{\alpha})^2+\sum_{\alpha\neq n+1}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}\\ &=\sum_{\alpha\neq n+1}(h_{ijk}^{\alpha})^2+nc\tau^2+\sum_{\alpha\neq n+1}h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta}h_{ij}^{\alpha}\\ &\quad -2\sum_{\alpha\neq n+1}h_{ik}^{\beta}h_{km}^{\alpha}h_{mj}^{\beta}h_{ij}^{\alpha}+\sum_{\alpha\neq n+1}h_{im}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta}h_{ij}^{\alpha}\\ &\quad -nH\sum_{\alpha\neq n+1}h_{im}^{\alpha}h_{mj}^{n+1}h_{ij}^{\alpha}+\sum_{\alpha\neq n+1}h_{ik}^{\beta}h_{km}^{\beta}h_{mj}^{\alpha}h_{ij}^{\alpha},\end{aligned}$$

and hence we get

$$\begin{aligned}\frac{1}{2}\Delta\tau^2 &= \sum_{\alpha\neq n+1}(h_{ijk}^{\alpha})^2+nc\tau^2+\sum_{\alpha,\beta\neq n+1}h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta}h_{ij}^{\alpha}\\ &\quad -2\sum_{\alpha,\beta\neq n+1}h_{ik}^{\beta}h_{km}^{\alpha}h_{mj}^{\beta}h_{ij}^{\alpha}+\sum_{\alpha,\beta\neq n+1}h_{im}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta}h_{ij}^{\alpha}\\ &\quad +\sum_{\alpha,\beta\neq n+1}h_{ik}^{\beta}h_{km}^{\beta}h_{mj}^{\alpha}h_{ij}^{\alpha}+\sum_{\alpha\neq n+1}h_{km}^{\alpha}h_{mk}^{n+1}h_{ij}^{n+1}h_{ij}^{\alpha}\\ &\quad -2\sum_{\alpha\neq n+1}h_{ik}^{n+1}h_{km}^{\alpha}h_{mj}^{n+1}h_{ij}^{\alpha}+\sum_{\alpha\neq n+1}h_{im}^{\alpha}h_{mk}^{n+1}h_{kj}^{n+1}h_{ij}^{\alpha}\\ &\quad -nH\sum_{\alpha\neq n+1}h_{im}^{\alpha}h_{mj}^{n+1}h_{ij}^{\alpha}+\sum_{\alpha\neq n+1}h_{ik}^{n+1}h_{km}^{n+1}h_{mj}^{\alpha}h_{ij}^{\alpha}.\end{aligned}\tag{5.1}$$

We put $S_{\alpha\beta}=\sum h_{ij}^{\alpha}h_{ij}^{\beta}$ for any α, $\beta\neq n+1$. Then $(S_{\alpha\beta})$ is a $(p-1)\times(p-1)$ symmetric matrix. It can assumed to be diagonal for a suitable choice of $e_{n+2}, \cdots, e_{n+p}$. Set $S_{\alpha}=S_{\alpha\alpha}$. We then have $\tau^2=\sum S_{\alpha}$. In general, for a matrix $A=(a_{ij})$, we define $N(A)=\mathrm{tr}(A^tA)$. Hence we get

$$\begin{aligned}&\sum_{\alpha,\beta\neq n+1}h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta}h_{ij}^{\alpha}-2\sum_{\alpha,\beta\neq n+1}h_{ik}^{\beta}h_{km}^{\alpha}h_{mj}^{\beta}h_{ij}^{\alpha}\\ &\quad +\sum_{\alpha,\beta\neq n+1}h_{im}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta}h_{ij}^{\alpha}+\sum_{\alpha,\beta\neq n+1}h_{jm}^{\alpha}h_{mk}^{\beta}h_{ki}^{\beta}h_{ij}^{\alpha}\\ &=\sum_{\alpha\neq n+1}(S_{\alpha})^2+\sum_{\alpha,\beta\neq n+1}N(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha}).\end{aligned}$$

Obviously, we see

$$\sum_{\alpha,\beta\neq n+1}N(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})\geqq 0.\tag{5.2}$$

Suppose $p\geqq 2$. Let

$$(p-1)\sigma_1=\tau^2=\sum S_\alpha,$$

$$(p-1)(p-2)\sigma_2=2\sum_{\alpha<\beta,\,\alpha,\,\beta\neq n+1} S_\alpha S_\beta.$$

Then we get

$$\sum(S_\alpha)^2=(p-1)(\sigma_1)^2+(p-1)(p-2)\{(\sigma_1)^2-\sigma_2\},$$

$$(p-1)^2(p-2)\{(\sigma_1)^2-\sigma_2\}=\sum_{\alpha<\beta,\,\alpha,\,\beta\neq n+1}(S_\alpha-S_\beta)^2.$$

Hence we obtain

$$\begin{aligned}&\sum_{\alpha,\,\beta\neq n+1} h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta h_{ij}^\alpha-2\sum_{\alpha,\,\beta\neq n+1} h_{ik}^\beta h_{km}^\alpha h_{mj}^\beta h_{ij}^\alpha\\ &+\sum_{\alpha,\,\beta\neq n+1} h_{im}^\alpha h_{mk}^\beta h_{kj}^\beta h_{ij}^\alpha+\sum_{\alpha,\,\beta\neq n+1} h_{ik}^\beta h_{km}^\beta h_{mj}^\alpha h_{ij}^\alpha\\ &\geqq(p-1)(\sigma_1)^2=\frac{1}{p-1}\tau^4.\end{aligned}\tag{5.3}$$

Then the equations (5.1), (5.2) and (5.3) imply

$$\begin{aligned}\frac{1}{2}\Delta\tau^2\geqq nc\tau^2&+\frac{1}{p-1}\tau^4+\sum_{\alpha\neq n+1} h_{km}^\alpha h_{mk}^{n+1} h_{ij}^{n+1} h_{ij}^\alpha-2\sum_{\alpha\neq n+1} h_{ik}^{n+1} h_{km}^\alpha h_{mj}^{n+1} h_{ij}^\alpha\\ &+\sum_{\alpha\neq n+1} h_{im}^\alpha h_{mk}^{n+1} h_{kj}^{n+1} h_{ij}^\alpha-nH\sum_{\alpha\neq n+1} h_{im}^\alpha h_{mj}^{n+1} h_{ij}^\alpha+\sum_{\alpha\neq n+1} h_{ik}^{n+1} h_{km}^{n+1} h_{mj}^\alpha h_{ij}^\alpha.\end{aligned}$$

For a fixed index α, since $H^\alpha H^{n+1}=H^{n+1}H^\alpha$, we can choose $\{e_1, \cdots, e_n\}$ such that

$$h_{ij}^\alpha=\lambda_i^\alpha\delta_{ij},\qquad h_{ij}^{n+1}=\lambda_i\delta_{ij}.$$

Then we get

$$\begin{aligned}&\sum h_{km}^\alpha h_{mk}^{n+1} h_{ij}^{n+1} h_{ij}^\alpha-2\sum h_{ik}^{n+1} h_{km}^\alpha h_{mj}^{n+1} h_{ij}^\alpha+\sum h_{im}^\alpha h_{mk}^{n+1} h_{kj}^{n+1} h_{ij}^\alpha\\ &\quad-nH\sum h_{im}^\alpha h_{mj}^{n+1} h_{ij}^\alpha+\sum h_{ik}^{n+1} h_{km}^{n+1} h_{mj}^\alpha h_{ij}^\alpha\\ &\quad=(\sum\lambda_i\lambda_i^\alpha)^2-nH\sum\lambda_i(\lambda_i^\alpha)^2.\end{aligned}$$

We notice here that eigenvalues λ_i's are bounded by (4.19). In order to estimate the last term on the above equation, the following property is prepared.

LEMMA 5.1. *Let $a_1, \cdots, a_n$ be real numbers satisfying $\sum a_i=0$ and let $b_1, \cdots, b_n$ be also real numbers. Then we have*

$$\sum a_i(b_i)^2\leqq\sqrt{\frac{n-1}{n}}\sqrt{\sum(a_i)^2}\sum(b_i)^2,$$

where the equality holds if and only if the $n-1$ of a_i's are equal with each other and the corresponding $n-1$ of b_i's are equal to 0.

PROOF. We consider the function $f=\sum a_i(b_i)^2$ with constraint $\sum a_i=0$, $\sum(a_i)^2=a$ and $\sum(b_i)^2=b$. Then there exists a critical point of f on $\boldsymbol{R}^{2n}$ at which we have

$$(b_i)^2+\mu_1+2\mu_2 a_i=0, \qquad 2a_ib_i+2\mu_3 b_i=0. \tag{5.4}$$

From (5.4) we get

$$\mu_1=-\frac{1}{n}b,$$

and the critical value of f is equal to $-\mu_3 b=-2\mu_2 a$, and therefore we have

$$a_i=-\mu_3, \quad \text{or} \quad b_i=0. \tag{5.5}$$

If $a_i=-\mu_3$ for any index i, then we get $f=0$, because of $\sum a_i=0$. If $a_i=-\mu_3$, $1\leqq i\leqq m$ and $b_j=0$, $m+1\leqq j\leqq n$, then we have from (5.4)

$$2\mu_2 a_j=\frac{1}{n}b, \qquad j=m+1, \cdots, n.$$

If $\mu_2=0$, then $f=0$. Without loss of generality, we may suppose $\mu_2\neq 0$. Thus we see

$$a_j=\frac{b}{2n\mu_2}, \qquad j=m+1, \cdots, n,$$

which yields

$$m\mu_3=(n-m)\frac{b}{2n\mu_2}. \tag{5.6}$$

From (5.4) and (5.5) it follows that we obtain

$$\mu_2=\pm\frac{1}{2}\sqrt{\frac{n-m}{nm}}\frac{b}{\sqrt{a}},$$

which means that we obtain

$$|f|\leqq\sqrt{\frac{n-1}{n}}\sqrt{\sum(a_i)^2}\sum(b_i)^2.$$

If the equality holds, then we have $m=1$ and $a_j=\pm\sqrt{a/n(n-1)}$, $b_j=0$, $2\leqq j\leqq n$. The converse is obvious. □

According to Lemma 5.1 we have

$$\begin{aligned}(\sum_i\lambda_i\lambda_i^\alpha)^2-nH\sum_i\lambda_i(\lambda_i^\alpha)^2 &\geqq -nH\sum_i(\lambda_i-H)(\lambda_i^\alpha)^2-nH^2\sum_i(\lambda_i^\alpha)^2\\ &=-nH(\sum_i\mu_i(\lambda_i^\alpha)^2+H\sum_i(\lambda_i^\alpha)^2)\\ &\geqq -nH\Big(\sqrt{\frac{n-1}{n}}\sqrt{\sum(h_{ij}^{n+1})^2-nH^2}+H\Big)\operatorname{tr}(H^\alpha)^2.\end{aligned}$$

The right hand side of the inequality above does not depend on the choice of frame fields. Therefore we have

$$\sum_{\alpha\neq n+1} h^{\alpha}_{km}h^{n+1}_{mk}h^{n+1}_{ij}h^{\alpha}_{ij}-2\sum_{\alpha\neq n+1} h^{n+1}_{ik}h^{\alpha}_{km}h^{n+1}_{mj}h^{\alpha}_{ij}+\sum_{\alpha\neq n+1} h^{\alpha}_{im}h^{n+1}_{mk}h^{n+1}_{kj}h^{\alpha}_{ij}$$

$$-nH\sum_{\alpha\neq n+1} h^{\alpha}_{im}h^{n+1}_{mj}h^{\alpha}_{ij}+\sum_{\alpha\neq n+1} h^{\alpha}_{jm}h^{n+1}_{mk}h^{n+1}_{ki}h^{\alpha}_{ij}$$

$$\geqq -nH\left(\sqrt{\frac{n-1}{n}}\sqrt{\sum(h^{n+1}_{ij})^2-nH^2}+H\right)\tau^2 .$$

Thus we have

$$\frac{1}{2}\Delta\tau^2\geqq\left\{nc-nH\left(\sqrt{\frac{n-1}{n}}\sqrt{\sum(h^{n+1}_{ij})^2-nH^2}+H\right)\right\}\tau^2+\frac{1}{p-1}\tau^4 .$$

Making use of the same proof as in the proof of $|\mu|^2$ above, we have

$$0\geqq\left\{nc-nH\left(\sqrt{\frac{n-1}{n}}\sqrt{\sum(h^{n+1}_{ij})^2-nH^2}+H\right)\right\}\tau^2+\frac{1}{p-1}\tau^4 .$$

Thus from (4.19) we get

$$\sup\tau^2\leqq(p-1)\left\{nH\left(\sqrt{\frac{n-1}{n}}\sqrt{S_+(1)-nH^2}+H\right)-nc\right\}. \tag{5.7}$$

The equality (4.6), the inequalities (4.19) and (5.7) yield

$$S\leqq S_+(p)+(p-1)H\{nH+\sqrt{n(n-1)\{S_+(1)-nH^2\}}\}.$$

Hence we complete the proof of Theorem 1. □

REMARK. When M is maximal (i.e., $H=0$), Theorem 1 implies $S\leqq -npc$. Ishihara [8] obtained this relation for complete maximal space-like submanifolds. When $p=1$, Theorem 1 becomes $S\leqq S_+(1)$. This result is obtained by the first author and Nakagawa [6]. Hence Theorem 1 generalizes the results above.

6. Proof of Theorems 2 and 3.

Let M be an n-dimensional complete space-like submanifold with parallel mean curvature vector of $M^{n+p}_p(c)$, $c\leqq 0$. We assume $S=S_+(p)+K(p)$. Then the equalities of all inequalities in the previous sections have to hold. Consequently, from (4.8) and (5.7) it is seen that

$$h^{\alpha}_{ijk}=0 \tag{6.1}$$

for any i, j, k and α. Also from (4.2) and (5.7) it follows that

$$H^{\alpha}H^{\beta}=H^{\beta}H^{\alpha}$$

for any α and β. The equations imply that all of H^{α} are simultaneously diagonalizable and the normal connection in the normal bundle of M is flat. Hence we can choose a suitable basis $\{e_i\}$ such that

(6.2) $$h_{ij}^{\alpha}=\lambda_i^{\alpha}\delta_{ij}$$

for any i, j and α. The submanifold M is said to be *isoparametric* [13] if the normal connection is flat and the characteristic polynomial of the shape operator A_ξ has constant coefficients over the domain of any local parallel normal field ξ.

LEMMA 6.1. *M is isoparametric.*

PROOF. Since the normal connection is flat, it is seen that there exist locally p mutually orthogonal unit normal vector fields which are parallel in the normal bundle. So we can choose a suitable parallel basis $\{e_\alpha\}$ and then we have $\omega_{\alpha\beta}=0$. Hence, since we have

(6.3) $$\sum h_{ijk}^{\alpha}\omega_k=dh_{ij}^{\alpha}-\sum h_{kj}^{\alpha}\omega_{ki}-\sum h_{ik}^{\alpha}\omega_{kj}+\sum h_{ij}^{\beta}\omega_{\beta\alpha},$$

setting $i=j$ in the above equation and using (6.1) we get $dh_{ii}^{\alpha}=0$. Hence h_{ii}^{α} is constant and M is isoparametric. □

LEMMA 6.2. *M is of non-positive curvature.*

PROOF. Suppose that there exist indices i, j and α such that $h_{ii}^{\alpha}\neq h_{jj}^{\alpha}$. From the equation (6.3) we get

$$\sum h_{kj}^{\alpha}\omega_{ki}+\sum h_{ik}^{\alpha}\omega_{kj}=(h_{ii}^{\alpha}-h_{jj}^{\alpha})\omega_{ij}=0,$$

from which it follows that $\omega_{ij}=0$. Accordingly, we have

$$\sum\omega_{ik}\wedge\omega_{kj}=0.$$

In fact, for any fixed indices i and α we denote by $[i]$ the set consisting of indices k such that $h_{ii}^{\alpha}=h_{kk}^{\alpha}$. Then we have $[i]\neq[j]$ by the supposition and hence we get

$$\sum_k\omega_{ik}\wedge\omega_{kj}=\sum_{k\in[i]}\omega_{ik}\wedge\omega_{kj}+\sum_{k\in[j]}\omega_{ik}\wedge\omega_{kj}+\sum_{k\notin[i]\cup[j]}\omega_{ik}\wedge\omega_{kj},$$

each term of which vanishes identically. By the structure equation

$$d\omega_{ij}+\sum\omega_{ik}\wedge\omega_{kj}=-\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l,$$

we obtain

$$R_{ijji}=c-\sum_\beta\lambda_i^{\beta}\lambda_j^{\beta}=0.$$

Next, suppose that $h_{ii}^{\alpha}=h_{jj}^{\alpha}$ for any distinct indices i and j and for any index α. Then the Gauss equation implies

$$R_{ijji}=c-\sum_{\alpha}(h_{ii}^{\alpha})^2=c-\sum_{\alpha}(\lambda_i^{\alpha})^2\leqq 0,$$

because of $c\leqq 0$.

Thus M is of non-positive curvature. □

PROOF OF THEOREM 2. By a theorem due to Koike [10] and Lemmas 6.1 and 6.2 it is seen that M is locally congruent to the product submanifold $H^{n_1}(c_1)\times\cdots\times H^{n_q}(c_q)\times \boldsymbol{R}^m$ of $\boldsymbol{R}_q^{n+q}$, where $\sum_{r=1}^q n_r+m=n$ and $1\leqq q\leqq p$. Then M can be naturally regarded as the space-like submanifold of $\boldsymbol{R}_p^{n+p}$ whose mean curvature vector is given by (2.1). It is also parallel in the normal bundle of M in $\boldsymbol{R}_p^{n+p}$. The constant $S_+(1)$ and the squared norm S of the second fundamental form are given by (2.2). Therefore it is seen that we have

$$S_+(p)+K(p)=-p\sum_{r=1}^{q} n_r{}^2 c_r=S,$$

which implies $p=q=1$ and $n_1=1$. Accordingly the hyperbolic cylinder $H^1(c_1)\times \boldsymbol{R}^{n-1}$ of $\boldsymbol{R}_1^{n+1}$ is the complete connected space-like hypersurface with constant mean curvature whose squard norm S attaining the maximal value. □

PROOF OF THEOREM 3. When $p=1$ it is seen by a theorem due to Ki, Kim and Nakagawa [9] that the hyperbolic cylinder $H^1(c_1)\times H^{n-1}(c_2)$ is the complete spacelike hypersurface with constant mean curvature of $H_1^{n+1}(c)$ satisfying the given condition.

Suppose next that $p\geqq 2$. By means of Koike's theorem and Lemmas 6.1 and 6.2 again, M is locally congruent to the product submanifold $H^{n_1}(c_1)\times\cdots\times H^{n_{q+1}}(c_{q+1})$ in $H_q^{n+q}(c')$, where $\sum_{r=1}^{q+1} n_r=n$, $\sum_{r=1}^{q+1}(1/c_r)=(1/c')\geqq(1/c)$ and $H_q^{n+q}(c')$ is a totally umbilic submanifold of $H_p^{n+p}(c)$. The mean curvature vector of M in $H_q^{n+q}(c')$ is denoted by $\boldsymbol{h}'$, which is parallel in the normal bundle of M in $H_q^{n+q}(c')$. Then the mean curvature vector $\boldsymbol{h}$ of M of $H_p^{n+p}(c)$ is given by $\boldsymbol{h}=\boldsymbol{h}'+\boldsymbol{h}''$, where h'' is the mean curvature vector of $H_q^{n+q}(c')$ in $H_p^{n+p}(c)$. Consequently the mean curvature vector $\boldsymbol{h}$ is parallel in the normal bundle $N(M)$ and the mean curvature H and the squared norm S of M in $H_p^{n+p}(c)$ are given by

$$h^2=n^2H^2=n^2c-\sum_{r=1}^{q+1} n_r{}^2 c_r,$$

$$S=nc-\sum_{r=1}^{q+1} n_r c_r.$$

We have $S_+(1)\geqq h^2-nc$, because of $c<0$. So it is seen by Lemma 2.1 that we obtain

(6.4) $$S_+(p)+K(p)-S\geqq h^2-pnc+(p-1)h^2-S=ph^2-pnc-S\geqq 0,$$

where the equality holds if and only if $H=0$. Accordingly, if we have $S=S_+(p)+K(p)$, then H must vanish identically. This implies that Theorem 3 is proved by a theorem due to Ishihara [8]. □

References

[1] K. Abe, N. Koike and S. Yamaguchi, Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form, Yokohama Math. J. **35** (1987), 123-136.
[2] R. Aiyama and Q. M. Cheng, Complete space-like hypersurfaces with constant mean in a Lorentz space form of dimension 4, Kodai Math. J. **15** (1992), 375-386.
[3] K. Akutagawa, On space-like hypersurfaces with constant mean curvature in the de Sitter space, Math. Z. **196** (1987), 13-19.
[4] E. Calabi, Examples of Bernstein problems for some nonlinear equations, Proc. Pure Appl. Math. **15** (1970), 223-230.
[5] Q. M. Cheng, Complete space-like submanifolds in de Sitter space with parallel mean curvature vector, Math. Z. **206** (1991), 333-339.
[6] Q. M. Cheng and H. Nakagawa, Totally umbilical hypersurfaces, Hiroshima J. Math. **20** (1990), 1-10.
[7] S. Y. Cheng and S. T. Yau, Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces, Ann. of Math. **104** (1976), 407-419.
[8] T. Ishihara, Maximal spacelike submanifolds of a pseudo Riemannian space of constant mean curvature, Michigan Math. J. **35** (1988), 345-352.
[9] U.-H. Ki, H.-J. Kim and H. Nakagawa, On space-like hypersurfaces with constant mean curvature of a Lorentz space form, Tokyo J. of Math. **14** (1991), 205-216.
[10] N. Koike, Proper isoparametric semi-Riemannian submanifolds in a semi-Riemannian space form, Tsukuba J. Math. **13** (1989), 131-146.
[11] H. Omori, Isometric immersions of Riemannian manifolds, J. Math. Soc. Japan **19** (1967), 205-214.
[12] J. Ramanathan, Complete spacelike hypersurfaces of constant mean curvature in a de Sitter space, Indiana Univ. Math. J. **36** (1987), 349-359.
[13] C. L. Terng, Isoparametric submanifolds and their coxeter groups, J. Differential Geometry **21** (1985), 79-107.
[14] A. E. Treibergs, Entire hypersurfaces of constant mean curvature in Minkowski 3-space, Invent. Math. **66** (1982), 39-56.
[15] S. T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure and Appl. Math. **28** (1975), 201-208.

Institute of Mathematics, Fudan University, Shanghai 200433, P. R. China
Institute of Mathematics, University of Tsukuba, 305 Ibaraki, Japan