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EXISTENCE OF ALL THE ASYMPTOTIC λ-TH MEANS FOR CERTAIN ARITHMETICAL CONVOLUTIONS

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Abstract. Let E designate either of the classical error terms for the summatory functions of the arithmetical functions $\phi(n)/n$ and $\sigma(n)/n$ (ϕ is Euler's function and σ the divisor function).

By following an idea of Codecà's [3] and by refining some of his estimates we prove that $|E|$ has asymptotic λ-th order means for all positive real numbers λ. We also prove that E has asymptotic k-th order means for all positive integers k, and that this mean is zero whenever k is odd.

The results obtained can be applied to functions other than E as well, such as the functions P and Q of Hardy and Littlewood [8], or the divisor functions $G_{-1,k}$ [9].

1. Introduction.

We consider

$$H(x)=\sum_{n\leq x}\frac{\phi(n)}{n}-\frac{6}{\pi^{2}}x\,, \tag{1.1}$$

$$F(x)=\sum_{n\leq x}\frac{\sigma(n)}{n}-\frac{\pi^{2}}{6}x+\frac{1}{2}\log x+\frac{\gamma}{2}+1\,, \tag{1.2}$$

$$Q(x)=\sum_{n\leq x}\frac{1}{n}\sin(x/n)\,, \tag{1.3}$$

and

$$P(x)=\sum_{n\leq x}\frac{1}{n}\cos(x/n)\,, \tag{1.4}$$

where ϕ denotes Euler's function, $\sigma(n)$ the sum of the positive divisors of n, and γ Euler's constant. These functions are unbounded; more precisely we

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know that [13, 5]

$$H(x)=\Omega(\log\log\log x) \tag{1.5}$$

and

$$H(x)=\Omega_{\pm}(\log\log\log\log x), \tag{1.6}$$

that [12, 2]

$$F(x)=\Omega_{-}(\log\log x) \tag{1.7}$$

and

$$\limsup_{x\to\infty} F(x)=+\infty, \tag{1.8}$$

and that [8, 4]

$$P(x)=\Omega_{+}(\log\log x) \tag{1.9}$$

and

$$Q(x)=\Omega_{\pm}((\log\log x)^{1/2}). \tag{1.10}$$

However, H [13], F [14] and Q [15] are known to have an asymptotic first mean; F [16] and H [1] even have square means. By λ-th mean we mean

DEFINITION. For a real function E defined on $[1, +\infty)$ and a real positive number λ we call—as long as the involved limit exists—

$$M(E, \lambda)=\lim_{x\to\infty}\frac{1}{x}\int_1^x E^{\lambda}(t)dt \tag{1.11}$$

the *asymptotic λ-th mean* of E.

In a recent article [3], P. Codecà obtains for any positive real number λ

$$\int_1^x |E(t)|^{\lambda}dt=O_{\lambda}(x), \tag{1.12}$$

if E is one of the functions defined in (1.1) through (1.4). In this paper we prove that in fact, for the same E,

$$M(E, k) \text{ exists for all positive integers } k, \tag{1.13}$$

and that

$$M(|E|, \lambda) \text{ exists for all positive real numbers } \lambda \tag{1.14}$$

(Theorems 1 and 2).

We conclude this introduction by noting that quantitative estimates of the constants $M(|E|, \lambda)$ for large λ are worth seeking for: in the case where $E=H$ for instance, they might provide precious information on the behaviour of the distribution function

$$D_H(s)=\lim_{x\to\infty}\frac{1}{x}|\{n\leqq x, H(n)\geqq s\}| \tag{1.15}$$

[10], which by a result of Erdös and Shapiro's [6] exists and is continuous. D_H in turn has a close relationship with the function $X_H(x)$ that counts the number of changes in sign of H in the interval $(1, x)$ [11].

Since $M(|E|, \lambda)=M(E, \lambda)$ for $\lambda=2k$ with k a positive integer, this case seems easier to handle; as yet we can only estimate the related $M(E, 2k+1)$ if $E=H$, F or Q, for all nonnegative integers k (Theorem 3).

2. Notation and statement of the results.

We denote by α a real bounded sequence that satisfies, for some real constant K,

$$\sum_{n\leq x}\alpha(n)=Kx+o(x), \tag{2.1}$$

and by f a real periodic function with period T, of bounded variation, such that

$$\int_0^T f(t)dt=0.$$

If the real function g, defined on $[1, +\infty)$, satisfies

$$g(x)=\sum_{n\leq x}\frac{\alpha(n)}{n}f(x/n)+o(1), \tag{2.3}$$

then we shall say that $g\in C(\alpha, f)$.

For the functions defined in (1.1) and (1.2), for instance, elementary calculation—with, in the case of H, an application of the prime number theorem—shows that

$$H\in C(-\mu, \psi) \tag{2.4}$$

and

$$F\in C(-1, \psi), \tag{2.5}$$

where μ is Moebius' function, $\psi(y)=\{y\}-1/2$ (with $\{y\}$ the fractional part of y), and 1 denotes the arithmetic function with constant value one. As for the functions of (1.3) and (1.4), we have by definition

$$Q\in C(1, \sin) \tag{2.6}$$

and

$$P\in C(1, \cos). \tag{2.7}$$

Much better information on such a function can be obtained if the corresponding sum (2.3) can be truncated. We shall say that $g\in C(\alpha, f)$ belongs to $C_z(\alpha, f)$ if, for K as in (2.1), we have

$$g(x)=\sum_{n\leq z}\frac{\alpha(n)}{n}f(x/n)+K\int_1^\infty\frac{f(u)}{u}du+o(1) \tag{2.8}$$

for some increasing and unbounded function $z=z(x)=o(x)$ $(x\to\infty)$. In the sequel these conditions on z will be assumed; if in addition z satisfies $z(x)=o(x^\varepsilon)$ for all positive ε, we shall say that z is *slowly varying*. Also, we shall refer to the constant on the right side of (2.8) by $K(g)$.

For instance we have

THEOREM 1. *There is a slowly varying function z such that*

$$H\in C_z(-\mu, \psi) \qquad (K(H)=0) \tag{2.9}$$

$$F\in C_z(-1, \psi) \qquad \left(K(F)=-\frac{1}{2}\log 2\pi+1\right) \tag{2.10}$$

$$Q\in C_z(1, \sin) \qquad \left(K(Q)=\int_1^\infty \frac{\sin u}{u}\,du\right), \tag{2.11}$$

and

$$P\in C_z(1, \cos) \qquad \left(K(P)=\int_1^\infty \frac{\cos u}{u}\,du\right). \tag{2.12}$$

Assertion (1.13) is thus a consequence of the following theorem easily deducible by induction from Codecà's Theorem 1 [3].

THEOREM A. *If $g\in C_z(\alpha, f)$ for some α, f and slowly varying z, then*

$$M(g, k) \ \text{ exists for all positive integers } k. \tag{2.13}$$

In order to obtain assertion (1.14) we need more, namely

THEOREM 2. *If g satisfies the hypotheses of Theorem A, then*

$$M(|g|, \lambda) \ \text{ exists for all positive real numbers } \lambda. \tag{2.14}$$

In the proof of Theorem 2, we shall use another result of Codecà's [3, (5.5) and Theorem 2].

THEOREM B. *If g satisfies the hypotheses of Theorem A and if*

$$g_y(x)=\sum_{n\leq y}\frac{\alpha(n)}{n}f(x/n), \tag{2.15}$$

then

$$\lim_{N\to\infty}\left(\limsup_{x\to\infty}\frac{1}{x}\int_1^x |g_z(t)-g_N(t)|^\lambda dt\right)=0, \tag{2.16}$$

and (as a consequence) g_z is a B^λ almost periodic function.

Note that we also have (this will be used later)

$$\int_1^x |g_N(t)|^\lambda dt=O_\lambda(x), \tag{2.17}$$

where the implied constant does not depend on N.

Also note that the last assertion of Theorem B implies, with Theorem 1, that the functions H, F, Q and P are B^λ almost periodic for all positive real numbers λ.

The following theorem determines the value of $M(E, 2k+1)$ as mentioned before:

THEOREM 3. *If g satisfies the hypotheses of Theorem A, and if*

$$f(t)=-f(-t) \tag{2.18}$$

except possibly on a set of measure zero, then

$$M(g-K(g), 2k+1)=0 \qquad (k=0, 1, 2 \cdots) \tag{2.19}$$

Other applications. The functions (recall (2.5))

$$G_{a,k}(x)=\sum_{n\leq\sqrt{x}} n^a \psi_k(x/n),$$

where $\psi_k(y)=B_k(\{y\})$ is the k-th Bernoulli polynomial "modulo 1", are closely related to various divisor problems (see for instance [9]). Theorem 2 is applicable to $G_{-1,k}$ for all k, and Theorem 3 for all odd k. (We shall omit the proof of this for $k>1$, very similar to that for $k=1$: Walfisz'argument [17, Chapter III] can be easily generalised if one uses the Fourier expansion for ψ_k instead of that for $\psi=\psi_1$.)

3. Proof of Theorem 1.

Most of the material needed in the proof essentially exists in the literature [3, 7, 9, 17], and rather than repeat lengthy arguments, we choose, to save space, to refer systematically to it.

a) *Proof of* (2.9): *H.* First we have

$$\sum_{x\exp(-\sqrt{\log x})<n\leq x} \frac{\mu(n)}{n}\psi(x/n)=o(1) \tag{3.1}$$

instead of Codecà's weaker [3, Lemma 5], where he shows that the left side of (3.1) is $O(1)$; the same argument, with a stronger version [17, p. 146] of the prime number theorem than the one he uses shows that in fact it is $o(1)$.

Next we have, for some slowly varying function $z=z(x)$,

$$\sum_{z<n\leq x\exp(-\sqrt{\log x})} \frac{\mu(n)}{n}\psi(x/n)=o(1). \tag{3.2}$$

This is essentially Hilfssätze 4 and 5 of [17, pp 141-144]: one may replace BQv^{-2} on the right side of (22) by $BQv^{-8/3}$, thus improving the conclusion of Hilfssatz 4; by using this better estimate to improve (31), one eventually obtains (3.2) instead of Hilfssatz 5. Note that although this argument of Walfisz' uses the assumption that x is an integer, this is a superfluous hypothesis, since

$$\sum_{y<n\leq x}\frac{\mu(n)}{n}\psi(x/n)=\sum_{y<n\leq x}\frac{\mu(n)}{n}\psi([x]/n)+O(y^{-1}). \tag{3.3}$$

Assertion (2.9) now follows from (2.4), (3.1) and (3.2).

b) *Proof of* (2.10): *F*. First we have

$$\sum_{\sqrt{x}<n\leq x}\frac{1}{n}\psi(x/n)=\frac{1}{2}\log(2\pi)-1+o(1): \tag{3.4}$$

this is a special case of [9, Theorem 2]. Then, for some slowly varying z,

$$\sum_{z<n\sqrt{x}}\frac{1}{n}\psi(x/n)=o(1): \tag{3.5}$$

this can be easily derived from the proof of Satz 1 in [17, p. 94-95] by being less generous in estimate (28) p. 95.

c) *Proof of* (2.11) *and* (2.12): *P and Q*. By [7, p. 9] we have

$$\sum_{\exp(\log x/\log\log x)<n\leq\sqrt{x}}\frac{1}{n}\exp(ix/n)=o(1). \tag{3.6}$$

Next, an application of the Euler-Mac Laurin sum formula yields, for $1>\varepsilon>x^{-1/2}$,

$$\sum_{\varepsilon x<n\leq x}\frac{1}{n}\exp(ix/n)=\int_1^\infty\frac{e^{iu}}{u}du+O(\varepsilon^{-2}x^{-1}+\varepsilon). \tag{3.7}$$

Finally, for $\varepsilon>x^{-1/2}$ we have

$$\sum_{\sqrt{x}<n\leq\varepsilon x}\frac{1}{n}\exp(ix/n)=O(\varepsilon^{1/4}), \tag{3.8}$$

which can easily be obtained from the unnumbered estimate [7, p. 8]

$$\sum_{a\leq n\leq b\leq 2a}\frac{1}{n}\exp(ix/n)=O((a/x)^{1/4})\qquad(a>\sqrt{x}>6). \tag{3.9}$$

(2.11) and (2.12) now follow from (3.6), (3.7), (3.8) if $\varepsilon=\varepsilon(x):=x^{-1/3}$.

4. Proof of Theorem 2.

Let $\bar{g}_N(t):=g_N(t)+K(g)$. If ν and ε are positive real numbers, then it follows from Theorem B that for some $N_0=N_0(\nu,\varepsilon)$, whenever $N\geqq N_0$ and x is sufficiently large, we have

$$\int_1^x|g(t)-\bar{g}_N(t)|^\nu dt\leqq\varepsilon x. \tag{4.1}$$

This implies that

$$\int_1^x|g(t)|^\lambda dt=\int_1^x|\bar{g}_N(t)|^\lambda dt+xR_{\lambda,N}(x), \tag{4.2}$$

where $\lim_{N\to\infty}\limsup_{x\to\infty}|R_{\lambda,N}(x)|=0$. Indeed if k is the (positive) integer such that $k-1<\lambda\leqq k$, $\varepsilon>0$, and N is an integer large enough to satisfy (4.1) for $\nu=2\mu$, where $\mu:=\lambda/k$, then by Schwarz inequality,

$$\int_1^x ||g(t)|^\lambda-|\bar{g}_N(t)|^\lambda|dt\leqq\left(\int_1^x(|g(t)|^\mu-|\bar{g}_N(t)|^\mu)^2dt\right)^{1/2}$$

$$\times\left(\int_1^x\left(\sum_{n=0}^{k-1}|g(t)|^{\mu n}|\bar{g}_N(t)|^{\mu(k-1-n)}\right)^2dt\right)^{1/2}$$

$$=:\sqrt{\alpha\beta},\qquad \text{say.} \tag{4.3}$$

Since $\mu\leqq 1$, we have $||g(t)|^\mu-|\bar{g}_N(t)|^\mu|\leqq|g(t)-\bar{g}_N(t)|^\mu$, whence by (4.1)

$$\alpha\leqq\varepsilon x. \tag{4.4}$$

And $\beta\leqq k^2\left(\int_1^x|g(t)|^{2\mu(k-1)}dt+\int_1^x|\bar{g}_N(t)|^{2\mu(k-1)}dt\right)$, whence by Theorem A and a direct consequence of (2.17),

$$\beta=O(x). \tag{4.5}$$

In view of (4.3) and (4.4), this concludes the proof of (4.2).

We proceed to prove Theorem 2. Since g_N is a periodic function, so is $|\bar{g}_N|^\lambda$. Hence

$$\int_1^x|\bar{g}_N(t)|^\lambda dt\sim K_N x\qquad(x\to\infty), \tag{4.6}$$

where by (2.17) the sequence $\{K_N\}_{N=1}^\infty$ is bounded, and has thus a subsequence $\{K_{N_i}\}_{i=1}^\infty$ that converges to some constant C_λ. By (4.2) we must then have

$$\int_1^x|g(t)|^\lambda dt\sim C_\lambda x\qquad(x\to\infty) \tag{4.7}$$

(and in fact the whole sequence $\{K_N\}$ converges to C_λ).

5. Proof of Theorem 3.

We have [3, (4.1) and (4.2)]

$$\int_1^x g_z^m(t)dt=\sum_{1\leqq n_j\leqq z}\alpha(n_1)\cdots\alpha(n_m)\int_{\lambda/N}^{x/N}f(N_1u)\cdots f(N_mu)du \tag{5.1}$$

where $N:=n_1\cdots n_m$, $N_j:=N/n_j$ $(j=1,\cdots,m)$, and $\lambda:=\max(w(n_1),\cdots,w(n_m))$, w denoting the inverse of z. For $j=1,\cdots,m$, the function $f_j(u):=f(N_ju)$ is periodic of period T/N_j, and so is thus $G(u):=f_1(u)\cdots f_m(u)$, with period $P:=T/(N_1,\cdots,N_m)$. Now by (2.18) $f_j(u)=-f_j(-u)$ $(j=1,\cdots,m)$, and thus, if m is odd, $G(u)=-G(-u)$, except possibly on a set of measure zero. Hence, for all real numbers a,

$$\int_{a}^{a+P} G(u)du=0 \tag{5.2}$$

From (5.1) and (5.2) we obtain (2.19), since G and α are bounded, and since z is slowly varying.

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