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FACTORIZATION THEOREM FOR PERFECT MAPS BETWEEN METRIZABLE SPACES

By

Yoshie TAKEUCHI

1. Introduction. We assume that all spaces are normal and all maps are continuous. We write $A\in\mathrm{ANR}$ for a space A if A is an ANR for the class of all compact metrizable spaces.

Given spaces X and A we write $\dim X\leqq A$ if for any closed subset F of X any map $f: F\rightarrow A$ can be extended to X. For a map $\xi: X\rightarrow X_0$ we write $\dim\xi\leqq A$ if $\dim\xi^{-1}(x_0)\leqq A$ for any $x_0\in X_0$. It is kown that a space X satisfies the relation $\dim X\leqq S^n$ for the n-sphere S^n if and only if X satisfies the inequality $\dim X\leqq n$ in the sense of the covering dimension.

Our purpose in this paper is to prove the following theorem;

THEOREM. *Let $A\in\mathrm{ANR}$, let ξ be a closed map of a space X into a paracompact space X_0, ζ be a perfect map of a metrizable space Z into a metrizable space Z_0, and let $f: X\rightarrow Z$ and $f_0: X_0\rightarrow Z_0$ be maps such that $\zeta f=f_0\xi$ and $\dim\xi\leqq A$. Then there are metrizable spaces Y and Y_0, a perfect map $\eta: Y\rightarrow Y_0$, and maps $g: X\rightarrow Y$, $g_0: X_0\rightarrow Y_0$, $h: Y\rightarrow Z$ and $h_0: Y_0\rightarrow Z_0$ such that $\eta g=g_0\xi$, $\zeta h=h_0\eta$, $hg=f$, $h_0g_0=f_0$, $\dim\eta\leqq A$, $w(Y_0)\leqq\max(w(X_0), w(Z_0))$, and $\dim Y_0\leqq\dim X_0$.*

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{h} & Z \\ \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\zeta} \\ X_0 & \xrightarrow[g_0]{} & Y_0 & \xrightarrow[h_0]{} & Z_0 \end{array}$$

For a map $\zeta: Z\rightarrow Z_0$ we write $w(\zeta)\leqq\tau$ if there is an embedding $i: Z\rightarrow Z_0\times I^{\tau}$ such that $\zeta=\mathrm{pr}\, i$, where I^{τ} is the Tikhonov cube of weight τ and $\mathrm{pr}: Z_0\times I^{\tau}\rightarrow Z_0$ is the projection.

In [9] Pasynkov proved a similar theorem to the above theorem, in which he added the property that $w(\eta)\leqq\tau$, if $w(\xi)\leqq\tau$, in the case that X, X_0, Z, Z_0 are compact (which are not assumed to be metrizable).

However, in [7] Pasynkov stated that, if f is a perfect map between

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metrizable spaces, the relation $w(f)\leqq\omega$ holds. Therefore, in the above theorem we need not to add the property that $w(\eta)\leqq\tau$ if $w(\xi)\leqq\tau$.

2. Proof of Theorem. The above theorem is an easy consequence of Lemmas 2 and 3 (cf. [9]). We need Lemma 1 to prove Lemma 2. The idea of the proof of Theorem is essentially due to Pasynkov.

LEMMA 1 ([9, (5.2)]). *Let $Y\in$ ANR. Then for any metric ρ in Y there is an $\varepsilon>0$ with the following properties; if f is a map of a compact space X into Y and g is a map of a closed set F in X into Y such that* $d(g, f|_F)=\max\{\rho(g(x), f(x));\ x\in F\}<\varepsilon$, *then g can be extended to X.*

LEMMA 2. *Under the condition of Theorem there are metrizable spaces Y and Y_0, a perfect map $\eta: Y\to Y_0$, and maps $g: X\to Y$, $g_0: X_0\to Y_0$, $h: Y\to Z$ and $h_0: Y_0\to Z_0$ such that $\eta g=g_0\xi$, $\zeta h=h_0\eta$, $hg=f$, $h_0g_0=f_0$,* $w(Y_0)\leqq\max(w(X_0), w(Z_0))$, $\dim Y_0\leqq\dim X_0$ *and for any $y_0\in Y_0$, any compact $F\subset\eta^{-1}(y_0)$ and any map $\chi: h(F)\to A$, the map $\chi h|_F$ can be extended to $\eta^{-1}(y_0)$.*

PROOF. Since $w(\zeta)\leqq\omega$, there is a embedding $i: Z\to Z_0\times I^\omega$ such that $\zeta=\mathrm{pr}\,i$, where I^ω is the Hilbert cube and $\mathrm{pr}: Z_0\times I^\omega\to Z_0$ is the projection. We denote by p the projection of $Z_0\times I^\omega$ onto I^ω. We choose a countable base $\{O_n: n=1, 2, \cdots\}$ for I^ω that is closed under finite unions. We fix a metric ρ on A and choose $\varepsilon>0$ in accordance with Lemma 1. For any n we fix a countable dense set C_n in $\mathrm{C}(\bar{O}_n, A)$, which is the space of maps from $\bar{O}_n$ to A with the metric of uniform convergence.

We fix n and $\varphi\in C_n$. For each $x_0\in X_0$ we consider the set $\Phi(x_0)=\xi^{-1}(x_0)\cap f^{-1}i^{-1}p^{-1}(\bar{O}_n)$. Since $\dim\xi\leqq A$ and $A\in$ ANR, the map $\varphi pif: \Phi(x_0)\to A$ can be extended to $\xi^{-1}(x_0)$ and then to a neighbourhood $V(\xi^{-1}(x_0))$ as a map $\Psi x_0: V(\xi^{-1}(x_0))\to A$.

Every point x of $\xi^{-1}(x_0)$ has a neighbourhood $O_x\subset V(\xi^{-1}(x_0))$ such that

$$\operatorname{diam}\varphi(pif(O_x)\cap\bar{O}_n)<\varepsilon/4 \quad\text{and}$$
$$\operatorname{diam}\Psi x_0(O_x)<\varepsilon/4.$$

Since ξ is closed, there is a neighbourhood $V(x_0)$ of x_0 such that $\xi^{-1}(V(x_0))\subset\cup\{O_x: x\in\xi^{-1}(x_0)\}$ and $\Phi(x_0')\subset\cup\{O_x: x\in\Phi(x_0)\}$ for any $x_0'\in V(x_0)$. Hence, for any $x_0'\in V(x_0)$ and every $x'\in\Phi(x_0')$ we can find a point $x\in\Phi(x_0)$ such that $x'\in O_x$, and hence,

$$(1)\qquad \rho(\Psi x_0(x'), \varphi pif(x'))\leqq\rho(\Psi x_0(x'), \Psi x_0(x))+\rho(\varphi pif(x), \varphi pif(x'))$$
$$<\varepsilon/4+\varepsilon/4=\varepsilon/2.$$

By paracompactness of X_0 there is a σ-discrete cozero cover $\omega(n, \varphi)=\bigcup_{j=1}^{\infty}\{U_{j(\lambda)}: j(\lambda)\in\Gamma_j\}$ of X_0 such that $\omega(n, \varphi)$ refines $\{V(x_0): x_0\in X_0\}$. For any j and each $j(\lambda)\in\Gamma_j$ we take $x_{j(\lambda)}\in X_0$ such that $U_{j(\lambda)}\subset V(X_{j(\lambda)})$. For each j we denote by $H_j(n, \varphi)$ the Hedgehog space (see [3]) constructed by $\{[0, 1]_{j(\lambda)}=[0, 1]: j(\lambda)\in\Gamma_j\}$. There is a function $g_{0j}(n, \varphi): X_0\to H_j(n, \varphi)$ such that $U_{j(\lambda)}=g_{0j}(n, \varphi)^{-1}(0, 1]_{j(\lambda)}$ for any $j(\lambda)\in\Gamma_j$. We denote by $P_j(n, \varphi)$ the partial product (see [6]) with base $H_j(n, \varphi)$ and fiber A with respect to the open set $\cup\{(0, 1]_{j(\lambda)}: j(\lambda)\in\Gamma_j\}$; we denote by $\eta_j(n, \varphi)$ its projection onto $H_j(n, \varphi)$ and by $\pi_{j(\lambda)}$ the projection of $(0, 1]_{j(\lambda)}\times A$ onto A. There is a map $g_j(n, \varphi): X\to P_j(n, \varphi)$ such that $g_{0j}(n, \varphi)\xi=\eta_j(n, \varphi)g_j(n, \varphi)$ and for any $j(\lambda)\in\Gamma_j$ $\pi_{j(\lambda)}g_j(n, \varphi)=\Psi x_{j(\lambda)}$ in $U_{j(\lambda)}$.

We perform these construction for all n and all $\varphi\in C_n$. We now set

$$Y'=Z\times\Pi\{P_j(n, \varphi): j=1, 2, \cdots, \varphi\in C_n, n=1, 2, \cdots\},$$

$$Y_0'=Z_0\times\Pi\{H_j(n, \varphi): j=1, 2, \cdots, \varphi\in C_n, n=1, 2, \cdots\}.$$

Clearly Y' and Y_0' are metrizable. We denote by h (resp. h_0) the projection of Y' onto Z (resp. Y_0' onto Z_0) and for any n, each $\varphi\in C_n$ and each j we denote by $g_j^{\varphi}(n, \varphi)$ (resp. $g_{0j}^{\omega}(n, \varphi)$ the projection of Y' onto $P_j(n, \varphi)$ (resp. Y_0' onto $H_j(n, \varphi)$). We set

$$\eta=\Pi\{\xi, \eta_j(n, \varphi): j=1, 2, \cdots, \varphi\in C_n, n=1, 2, \cdots\},$$

$$g=\Delta\{f, g_j(n, \varphi): j=1, 2, \cdots, \varphi\in C_n, n=1, 2, \cdots\} \quad \text{and}$$

$$g_0=\Delta\{f_0, g_{0j}(n, \varphi): j=1, 2, \cdots, \varphi\in C_n, n=1, 2, \cdots\}.$$

Clearly η is perfect and for any n, any $\varphi\in C_n$ and each j

(2) $$\eta_j(n, \varphi)g_j^{\varphi}(n, \varphi)=g_{0j}^{\omega}(n, \varphi)\eta,$$

$$g_j^{\omega}(n, \varphi)g=g_j(n, \varphi), \quad g_{0j}^{\omega}(n, \varphi)g_0=g_{0j}(n, \varphi);$$

(3) $$hg=f, \quad h_0g_0=f_0, \quad \eta g=g_0\xi, \quad \zeta h=h_0\eta,$$

We set $Y_0=g_0(X_0)$ and $Y=\overline{g(X)}\cap\eta^{-1}(Y_0)$. If we now regard η, h, g_j^{ω}, (n, φ) and h_0, $g_{0j}^{\omega}(n, \varphi)$ as the restrictions of these maps to Y and Y_0, respectivery, then (2), (3) remain valid, and η is perfect.

We fix a point $y_0\in Y_0$, a compact set $F\subset\eta^{-1}(y_0)$ and a map $\chi: h(F)\to$ A. We shall prove that χh can be extended to $\eta^{-1}(y_0)$. Since $h(F)\subset\zeta^{-1}(h_0(y_0))$, there is a map $\varphi': pih(F)\to A$ such that $\chi=\varphi' pi$, and hence $\chi h=\varphi' pih$.

Since $A\in$ANR, we may assume that φ' is defined on some $\bar{O}_n$ with $O_n\supset pih(F)$. Since C_n is dense in $C(\bar{O}_n, A)$, by [9, Lemma 5.1] there is a map $\varphi\in C_n$ homotopic to $\varphi pih: F\to A$. Since $\omega(n, \varphi)$ is a cover of X_0, there is j and

$j(\lambda)\in\Gamma_j$ such that $t_0=g^{\varphi}_{0j}(n, \varphi)(y_0)\in(0, 1]_{j(\lambda)}$. For any $y\in F$ $pih(y)\in O_n$, $g^{\varphi}_j(n, \varphi)(y)\in\{t_0\}\times A\subset(0, 1]_{j(\lambda)}\times A$ and $g(X)$ is dense in Y, hence there is $y'\in g(X)$ such that $pih(y')\in O_n$, $g^{\varphi}_j(n, \varphi)(y')\in(0, 1]_{j(\lambda)}\times A$,

$$\rho(\pi_{j(\lambda)}g^{\varphi}_j(n, \varphi)(y), \pi_{j(\lambda)}g^{\varphi}_j(n, \varphi)(y'))<\varepsilon/4 \quad\text{and}$$

$$\rho(\varphi pih(y), \varphi pih(y'))<\varepsilon/4.$$

We take a point $x'\in X$ such that $g(x')=y'$, then $pif(x')=pih(y')\in O_n$, and since $g_{0j}(n, \varphi)\xi(x')=\eta_j(n, \varphi)g^{\varphi}_j(n, \varphi)(y')\in(0, 1]_{j(\lambda)}$, we have $x'\in\xi^{-1}g_{0j}(n, \varphi)^{-1}(0, 1]_{j(\lambda)}=\xi^{-1}U_{j(\lambda)}$. We set $x'_0=\xi(x')$ then $x'_0\in U_{j(\lambda)}\subset V(x_{j(\lambda)})$ and $X'\in\varphi(x'_0)$. From (1), we have

$$\begin{aligned}&\rho(\pi_{j(\lambda)}g^{\varphi}_j(n, \varphi)(y'), \varphi pih(y'))\\ &=\rho(\pi_{j(\lambda)}g_j(n, \varphi)(x'), \varphi pif(x'))\\ &=\rho(\Phi x_{j(\lambda)}(x'), \varphi pif(x'))<\varepsilon/2.\end{aligned}$$

Hence, we see that

$$\begin{aligned}&\rho(\pi_{j(\lambda)}g^{\varphi}_j(n, \varphi)(y), \varphi pih(y))\\ &\leqq\rho(\pi_{j(\lambda)}g^{\varphi}_j(n, \varphi)(y), \pi_{j(\lambda)}g^{\varphi}_j(n, \varphi)(y'))\\ &\quad+\rho(\pi_{j(\lambda)}g^{\varphi}_j(n, \varphi)(y'), \varphi pih(y'))\\ &\quad+\rho(\varphi pih(y'), \varphi pih(y))\\ &<\varepsilon/4+\varepsilon/2+\varepsilon/4=\varepsilon.\end{aligned}$$

The map $\pi_{j(\lambda)}g^{\varphi}_j(n, \varphi)$ is defined on $\eta^{-1}(y_0)$. By Lemma 1. φpih can be extended to $\eta^{-1}(y_0)$, and by Homotopy extension theorem (see e.g. [4]) χh can be also extended to $\eta^{-1}(y_0)$.

The fact that $w(Y_0)\leqq\max(w(X_0), w(Z_0))$ is evident.

We claim that we may assume that $\dim Y_0\leqq\dim X_0$. By [8, Theorem 2.] there is a metrizable space Y'_0 and maps $g'_0: X_0\to Y'_0$ and $h''_0: Y'_0\to Y_0$ such that $w(Y'_0)\leqq w(Y_0)$. $\dim Y'_0\leqq\dim X_0$ and $g_0=h''_0g'_0$. We denote by Y' the fan product of Y'_0 and Y with respect to h''_0 and η (see [1. Supplement to Ch. 1, §2]); by η' and h'' we denote that projections of Y' into Y'_0 and Y, respectively, and by g' a map of X into Y' such that $\eta'g'=g'_0\xi$ and $h''g'=g$. If we replace Y, Y_0, g, g_0, h, h_0 and η with Y', Y'_0, g', g'_0, hh'', $h_0h''_0$ and η', respectively, then these spaces and maps are what is required (cf. [9]).

Lemma 2 has been proved.

LEMMA 3 ([9, Lemma 5.3]). *Suppose that* $A\in$ ANR *and* $\{T_n, h_{n+1,n}\}$ $(n=0, 1, \cdots)$

is an inverse sequence of compact spaces such that for any n, any compact $F\subset T_{n+1}$, *and any map* $\chi: h_{n+1,n}(F)\rightarrow A$, *the map* $\chi h_{n+1,n}|_F$ *has an extension to* T_{n+1}. *Then* $\dim T\leqq A$ *for the limit* T *of the sequence in question.*

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References

[1] Aleksandrov, P. S. and Pasynkov, B. A., Introduction to dimension theory (in Russian). Nauka Moscow, 1973.
[2] Borusk, K., Theory of retracts. PWN, 1967.
[3] Engelking, R., General topology. PWN, 1977.
[4] ———, Dimension theory. PWN, 1978.
[5] Hu, S. T., Theory of retracts, Wayne State Unive. Press, 1965.
[6] Pasynkov, B. A., Partial topological products. Soviet Math. Dokl. 5 (1964), 167-170.
[7] ———, On the dimension and geometry of mappings. Soviet Math. Dokl. 16 (1975), No. 2, 384-388.
[8] ———, Factorization theorem in dimension theory. Russian Math. Surveys 36: 3 (1981), 175-209.
[9] ———, A theorem on ω-maps for maps. Russian. Math. Surveys 39: 5 (1984), 125-153.

Yoshie TAKEUCHI
Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki, 305