

A CHARACTERIZATION OF CLOSED s -IMAGES OF METRIC SPACES

By

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Throughout the present note, we assume that all spaces are regular topological spaces and all mappings are continuous. Let N denote the set of all natural numbers.

Recall from [7] a collection \mathcal{P} of subsets of a space X is called a k -network for X if for every compact subset K of X and every open set U of X with $K \subset U$, there is a finite subcollection \mathcal{P}' of \mathcal{P} such that $K \subset \cup \{P : P \in \mathcal{P}'\} \subset U$. A collection \mathcal{P} of subsets of a space X is called a cs -network for X if for every sequence $\{x_n : n \in N\}$ converging to a point $x \in X$ and every neighborhood U of x , there is an element $P \in \mathcal{P}$ such that $P \subset U$ and $\{x_n : n \in N\}$ is eventually in P ([4]). A space is said to be an \aleph -space if it has a σ -locally finite k -network ([6]). A mapping f from a space X to a space Y is called an s -mapping if $f^{-1}(y)$ has a countable base for each $y \in Y$.

Recently, L. Foged [2] proved an interesting characterization of Lašnev spaces: A space X is Lašnev space (i. e. X is a closed image of a metric space) if and only if X is a Fréchet space with a σ -hereditarily closure preserving k -network. On the other hand, Y. Tanaka showed that every closed s -image X of a metric space is an \aleph -space if any closed metrizable subset of X is locally compact ([9, Lemma 4.1]). (Using this result, he gave a characterization for the product space $X \times Y$ of closed s -images X and Y of metric spaces to be a k -space (see [9, Theorem 4.3]).) He asked in the same paper whether every closed s -image of a metric space is an \aleph -space. The purpose of this note is to answer the above question and simultaneously to get a characterization of Fréchet \aleph -spaces.

Our result is the following.

THEOREM. *For a regular space X , the following are equivalent.*

- (a) X is a Fréchet \aleph -space.
- (b) X is a closed s -image of a metric space.
- (c) X is a Fréchet space with a point countable, σ -closure preserving,

closed k-network.

PROOF. The implication (a)→(b) can be shown by an argument similar to that of [2, Proposition 5]. To prove the implication (b)→(c), let X be an image of a metric space Y under a closed s-mapping f . Let \mathcal{B} be a σ -locally finite base for Y . Then it is obvious that $\mathcal{P}=\{f(\bar{B}) : B\in\mathcal{B}\}$ is a point countable and σ -closure preserving family of closed sets of X . Furthermore, \mathcal{P} is a k -network for X . Indeed, let K be a compact subset of X and U an open set of X with $K\subset U$. By [5, Corollary 1.2], there is a compact subset C of Y with $C\subset f^{-1}(U)$ and $f(C)=K$. Let B_1, \dots, B_n be elements of \mathcal{B} such that $C\subset B_1\cup\dots\cup B_n\subset\bar{B}_1\cup\dots\cup\bar{B}_n\subset f^{-1}(U)$. Then $K\subset f(\bar{B}_1)\cup\dots\cup f(\bar{B}_n)\subset U$. Thus \mathcal{P} is a k -network for X and hence the implication (b)→(c) is proved. To prove the implication (c)→(a), by [1, Theorem 4], it is sufficient to show that X has a σ -discrete cs-network. Now, let $\mathcal{P}=\cup\{\mathcal{P}_n : n\in N\}$ be a point countable, σ -closure preserving and closed k -network for X , where each \mathcal{P}_n is closure preserving. Without loss of generality, we can assume that each \mathcal{P}_n is closed under finite intersections and $\mathcal{P}_n\subset\mathcal{P}_{n+1}$ for each $n\in N$. Since \mathcal{P}_n is locally countable, there is an open cover \mathcal{U}_n of X such that each member U of \mathcal{U}_n intersects at most countably many members of \mathcal{P}_n . Since X is a σ -space (see [8]), there is a σ -discrete closed refinement $\mathcal{F}_n=\cup\{\mathcal{F}_{nm} : m\in N\}$ of \mathcal{U}_n , where $\mathcal{F}_{nm}=\{F_{nm\alpha} : \alpha\in A_n\}$ is discrete in X . For each $n, m\in N$ and each $\alpha\in A_n$, we put

$$\mathcal{P}_{nm\alpha}=\{P\in\mathcal{P}_n : P\cap F_{nm\alpha}\neq\phi\}.$$

Since $\mathcal{P}_{nm\alpha}$ is countable, let $\mathcal{P}_{nm\alpha}^*=\{P_{nm\alpha}^k : k\in N\}$ be the family of all finite unions of $\mathcal{P}_{nm\alpha}$. For each $n, m\in N$ and each $\alpha\in A_n$, we put

$$W_{nm\alpha}=\cup_{i=1}^{\infty}[(\cup\{P\in\mathcal{P}_i : P\cap(\cup\{F_{nm\beta} : \beta\in A_n \text{ with } \beta\neq\alpha\})=\phi\}) - (\cup\{P\in\mathcal{P}_i : P\cap F_{nm\alpha}=\phi\})].$$

We have the following.

- (1) $F_{nm\alpha}\subset W_{nm\alpha}$ for each $\alpha\in A_n$ and $n\in N$.
- (2) $W_{nm\alpha}\cap W_{nm\beta}=\phi$ for each $\alpha, \beta\in A_n$ with $\alpha\neq\beta$.

For each $n, m, k, r\in N$ and each $\alpha\in A_n$, we put

$$Q_{nm\alpha}^r=\cup\{P\in\mathcal{P}_r : P\subset W_{nm\alpha}\},$$

and

$$Q_{nm}^{kr}=\{P_{nm\alpha}^k\cap Q_{nm\alpha}^r : \alpha\in A_n\}.$$

Finally, we put

$$Q=\cup\{Q_{nm}^{kr} : (n, m, k, r)\in N\times N\times N\times N\}.$$

By (2), it follows that Q_{nm}^{kr} is discrete in X . To show that Q is a cs-network for X , let $\{x_n : n \in N\}$ be a sequence in X which converges to a point $x \in X$ and U a neighborhood of x . Since \mathcal{P} is a k-network for X , there are a number $n \in N$ and a finite subcollection \mathcal{P}_n' of \mathcal{P}_n such that $\{x_n : n \in N\}$ is eventually in $\cup \{P : P \in \mathcal{P}_n'\}$, $\cup \{P : P \in \mathcal{P}_n'\} \subset U$ and $x \in \cap \{P : P \in \mathcal{P}_n'\}$. Since \mathcal{F}_n is a cover of X , there are a number $m \in N$ and an element $\alpha \in A_n$ such that $x \in F_{nm\alpha}$. Then $\cup \{P : P \in \mathcal{P}_n'\} \in \mathcal{P}_{nm\alpha}^*$. Let us put $\cup \{P : P \in \mathcal{P}_n'\} = P_{nm\alpha}^k$ for some $k \in N$. On the other hand, since every sequence converging to a point of $F_{nm\alpha}$ is eventually in $W_{nm\alpha}$, it follows that there are a number $r \in N$ with $r \geq n$ and a finite subcollection \mathcal{P}_r' of \mathcal{P}_r such that $\{x_n : n \in N\}$ is eventually in $\cup \{P : P \in \mathcal{P}_r'\}$ and $\cup \{P : P \in \mathcal{P}_r'\} \subset W_{nm\alpha}$ by [1, Lemma 3]. Therefore, $Q = P_{nm\alpha}^k \cap Q_{nm\alpha}^r$ ($\in Q_{nm}^{kr} \subset Q$) contains a tail of $\{x_n : n \in N\}$ and Q is contained in U . Hence Q is a cs-network for X . This completes the proof.

REMARK 1. (i) By the theorem, every closed s-image of a metric space is an \aleph -space. This is an affirmative answer to the Tanaka's question stated before.

(ii) The proof of the implication (c) \rightarrow (a) in the theorem showed that every regular space with a point countable, σ -closure preserving and closed k-network is an \aleph -space. This is an affirmative answer to a question in [11] whether every regular space with a point countable, σ -hereditarily closure preserving closed k-network is an \aleph -space.

REMARK 2. In the statement (c) of the theorem, the assumption of the "closedness" of the k-network can not be dropped. Indeed, let X be the discrete sum $\bigoplus \{I_\alpha : \alpha < \omega_1\}$ of the copies I_α , $\alpha < \omega_1$, of the unit closed interval $I = [0, 1]$. Let A be the subset of X consisting of all zero's. Let $Y = X/A$ be the quotient space. It is well known that Y has no point countable closed k-network (cf. [10] or [3]). On the other hand, L. Foged [2] pointed out that every Lašnev space has a σ -hereditarily closure preserving and point countable k-network. Hence Y has a point countable, σ -closure preserving k-network.

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