TSUKUBA J. MATH.
Vol. 18 No. 1 (1994), 131—134

ON A SUFFICIENT CONDITIONS FOR MULTIVALENTLY STARLIKENESS

By

Mamoru NUNOKAWA

Let $q\in N=\{1, 2, 3, \cdots\}$ and $A(q)$ denote the class of function

$$f(z)=z^q+\sum_{n=q+1}^{\infty} a_n z^n$$

which are analytic in the open disk $E=\{z: |z|<1\}$.

A function $f(z)\in A(q)$ is called q-valently starlike with respect to the origin if and only if

$$Re\frac{zf'(z)}{f(z)}>0 \quad \text{in} \quad E.$$

There are many papers in which various sufficient conditions for multivalently starlikeness were obtained, but almost these results were got by using real part of some analytic functions.

Recently, Mocanu [3] obtained the following result by using the imaginary part of $zf''(z)/f'(z)$.

THEOREM A. *If* $f(z)\in A(1)$ *and*

$$\left|\mathrm{Im}\frac{zf''(z)}{f'(z)}\right|<\sqrt{3} \quad \textit{in } E,$$

then $f(z)$ *is univalently starlike in* E.

We need the following lemma due to [1, 2].

LEMMA 1. *Let* $w(z)$ *be analytic in* E *and suppose that* $w(0)=0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z|=r<1$ *at a point* z_0, *then we can write*

$$z_0 w'(z_0)=k w(z_0)$$

where k *is a real number and* $k\geqq 1$.

This reseach is partially supported by Grant-in-Aid for Scientific Reseach (C) (No. 04640127), Japanese Ministry of Education, Science and Culture.
Received July 6, 1992. Revised January 13, 1993.

Applying the same method as the proof of [4, Theorem 1], we can prove the following lemma:

LEMMA 2. *Let $p(z)$ be analytic in E, $p(0)=q$ and suppose that there exists a point $z_0 \in E$ such that*

$$\text{(1)} \qquad Re\ p(z)>0 \quad \text{for} \quad |z|<|z_0|$$

$Re\ p(z_0)=0$ and $p(z_0)=ia$ where a is a real number and not zero.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)}=ik$$

where

$$k \geqq \frac{1}{2}\left(\frac{q^2+a^2}{a}\right) \geqq q \qquad \text{if} \quad a>0,$$

and

$$k \leqq \frac{-1}{2}\left(\frac{q^2+a^2}{a}\right) \leqq -q \qquad \text{if} \quad a<0.$$

PROOF. Let us put

$$\text{(2)} \qquad \phi(z)=\frac{q-p(z)}{q+p(z)}.$$

Then we have that $\phi(0)=0$, $|\phi(z)|<1$ for $|z|<|z_0|$ and $|\phi(z_0)|=1$. From (1), (2) and Lemma 1, we have

$$\frac{z_0\phi'(z_0)}{\phi(z_0)}=-\frac{2z_0 p'(z_0)}{q^2-p(z_0)^2}=\frac{-2z_0 p'(z_0)}{q^2+|p(z_0)|^2} \geqq 1.$$

This shows that

$$-z_0 p'(z_0) \geqq \frac{1}{2}(q^2+|p(z_0)|^2)$$

and $z_0 p'(z_0)$ is a negative real number.

Applying the same method as the proof of [4, Theorem 1], for $a>0$, we have

$$\operatorname{Im}\frac{z_0 p'(z_0)}{p(z_0)} \geqq \frac{1}{2}\left(\frac{q^2+a^2}{a}\right) \geqq q$$

and for $a<0$, we have

$$\operatorname{Im}\frac{z_0 p'(z_0)}{p(z_0)} \leqq -\frac{1}{2}\left(\frac{q^2+a^2}{|a|}\right) \leqq -q.$$

This completes our proof.

Applying Lemma 2, we will obtain a generalized result of Theorem A.

MAIN THEOREM. *Let $f(z)\in A(q)$ and suppose that*

$$1+\frac{zf''(z)}{f'(z)}\neq ik \quad \text{in} \quad E, \tag{3}$$

where k is a real number and $|k|\geqq\sqrt{3}q$.

Then $f(z)$ is q-valently starlike in E.

PROOF. Let us put

$$p(z)=\frac{zf'(z)}{f(z)}$$

where $p(0)=q$. From the assumption (3), we easily have

$$p(z)\neq 0 \quad \text{in} \quad E.$$

In fact, if $p(z)$ has a zero of order n at $z=\alpha\in E$, then we can put

$$p(z)=(z-\alpha)^n p_1(z), \qquad (n\in N)$$

where $p_1(z)$ is analytic in E and $p_1(\alpha)\neq 0$.

Then we have

$$\begin{aligned} 1+\frac{zf''(z)}{f'(z)} &= \frac{zp'(z)}{p(z)}+p(z) \\ &= \frac{nz}{z-\alpha}+\frac{zp_1'(z)}{p_1(z)}+(z-\alpha)^n p_1(z). \end{aligned} \tag{4}$$

But, the imaginary part of (4) can take any infinite values when z approaches α.

This contradicts (3). Hence we have

$$p(z)\neq 0 \quad \text{in} \quad E.$$

Therefore, if there exists a point $z_0\in E$ such that $Re\, p(z)>0$ for $|z|<|z_0|$,

$$Re\, p(z_0)=0 \quad \text{and} \quad p(z_0)=ia,$$

then we have

$$p(z_0)\neq 0 \quad \text{and} \quad a\neq 0.$$

From Lemma 2 and (4), for $a>0$, we have

$$\begin{aligned} 1+\frac{z_0 f''(z_0)}{f'(z_0)} &= \frac{z_0 p'(z_0)}{p(z_0)}+p(z_0) \\ &= i\left(\mathrm{Im}\frac{z_0 p'(z_0)}{p(z_0)}+\mathrm{Im}\, p(z_0)\right) \end{aligned}$$

and

$$\mathrm{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}+p(z_0)\right)\geqq\frac{1}{2}\left(\frac{q^2+3a^2}{a}\right)\geqq\sqrt{3}\,q.$$

For $a<0$, we have

$$1+\frac{z_0 f''(z_0)}{f'(z_0)}=i\left(\mathrm{Im}\,\frac{z_0 p'(z_0)}{p(z_0)}+\mathrm{Im}\,p(z_0)\right)$$

and so

$$\mathrm{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}+p(z_0)\right)\leqq-\frac{1}{2}\left(\frac{q^2+3a^2}{|a|}\right)\leqq-\sqrt{3}\,q.$$

These contradict (3). Hence we have

$$Re\,p(z)>0 \quad \text{in} \quad E.$$

This shows that $f(z)$ is q-valently starlike in E.

This completes our proof.

From Main theorem, we easily have the following result.

COROLLARY. *Let $f(z)\in A(q)$ and suppose that there exists a real number R for which*

$$\left|\frac{zf''(z)}{f'(z)}-R\right|<\sqrt{(R+1)^2+3q^2} \qquad in \quad E.$$

Then $f(z)$ is q-valently starlike in E.

Acknowledgement. The author sincerly appreciates the helpful comments made by the referee.

References

[1] Jack, I.S., Functions starlike and convex of order α, J. London Math. Soc. **3** (1971), 469-474.

[2] Miller, S.S. and Mocanu, P.T., Second order differential inequalities in the complex plane, J. Math. Anal. Appl. **65** (1978), 289-305.

[3] Mocanu, P.T., Some integral operators and starlike functions, Rev. Roumaine Math. Pures Appl. **31** (1986), 231-235.

[4] Nunokawa, M., On properties of Non-Carathéodory functions, Proc. Japan Acad. **68**, No. 6 (1992), 152-153.

Department of Mathematics
University of Gunma
Aramaki, Maebashi 371
Japan
e-mail, nunokawa@la. gunma-u.ac.jp