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ON THE NEUMANN PROBLEM OF LINEAR HYPERBOLIC PARABOLIC COUPLED SYSTEMS

By

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Abstract. We prove the unique existence of solutions to some mixed problem of hyperbolic parabolic coupled systems with Neumann boundary condition, and we investigate how the constant in the first energy estimate depends on the coefficients of the opertors.

§1. Introduction.

Let Ω be a domain in an n-dimensional Euclidean $\boldsymbol{R}^n$, its boundary Γ being a C^∞ and compact hypersurface. Let $x=(x_1, \cdots, x_n)$ and t denote a point of $\boldsymbol{R}^n$ and a time, respectively; $\partial_t=\partial_0=\partial/\partial t$ and $\partial_j=\partial/\partial x_j$ $(j=1, \cdots, n)$. In this paper, we consider the following mixed problem;

$$
\text{(N)}\quad
\begin{cases}
\mathcal{A}_H(t)[\vec{u}]=\partial_t^2\vec{u}_H-A_H^2(t, x, \partial)\vec{u}_H-A_H^1(t, x, \partial)\partial_t\vec{u}_H \\
\qquad\qquad -A_{HP}^1(t, x, \partial)\vec{u}_P=\vec{f}_H(t, x) \qquad \text{in } [0, T]\times\Omega, \\
\mathcal{A}_P(t)[\vec{u}]=A_P^0(t, x)\partial_t\vec{u}_P-A_P^2(t, x, \partial)\vec{u}_P-A_{PH}^2(t, x, \partial)\vec{u}_H \\
\qquad\qquad -A_{PH}^1(t, x, \partial)\partial_t\vec{u}_H=\vec{f}_P(t, x) \qquad \text{in } [0, T]\times\Omega, \\
\mathcal{B}_H(t)[\vec{u}]=B_H^1(t, x, \partial)\vec{u}_H+B_{HP}(t, x)\vec{u}_P+B_H^0(t, x)\partial_t\vec{u}_H \\
\qquad\qquad =\vec{g}_H(t, x) \qquad \text{on } [0, T]\times\Gamma, \\
\mathcal{B}_P(t)[\vec{u}]=B_P^1(t, x, \partial)\vec{u}_P+B_{PH}^1(t, x, \partial)\vec{u}_H+B_{PH}^0(t, x)\partial_t\vec{u}_H \\
\qquad\qquad =\vec{g}_P(t, x) \qquad \text{on } [0, T]\times\Gamma, \\
\vec{u}_H(0, x)=\vec{u}_{H0}(x), \quad \partial_t\vec{u}_H(0, x)=\vec{u}_{H1}(x), \quad \vec{u}_P(0, x)=\vec{u}_{P0}(x) \qquad \text{in } \Omega,
\end{cases}
$$

where

$$A_H^2(t, x, \partial)\vec{v}_H=\partial_i(A_H^{ij}(t, x)\partial_j\vec{v}_H), \quad A_H^1(t, x, \partial)\vec{v}_H=A_H^{i0}(t, x)\partial_i\vec{v}_H,$$

$$A_{HP}^1(t, x, \partial)\vec{v}_P=A_{HP}^i(t, x)\partial_i\vec{v}_P, \quad A_{PH}^2(t, x, \partial)\vec{v}_H=A_{PH}^{ij}(t, x)\partial_i\partial_j\vec{v}_H,$$

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$$A_P^2(t, x, \partial)\vec{v}_P=\partial_i(A_P^{ij}(t, x)\partial_j\vec{v}_P)+A_P^i(t, x)\partial_i\vec{v}_P,$$

$$A_{PH}^1(t, x, \partial)\vec{v}_H=A_{PH}^{i0}(t, x)\partial_i\vec{v}_H,$$

$$B_H^1(t, x, \partial)\vec{v}_H=\nu_i(x)A_H^{ij}(t, x)\partial_j\vec{v}_H, \quad B_{PH}^1(t, x, \partial)\vec{v}_H=B_{PH}^i(t, x)\partial_i\vec{v}_H,$$

$$B_P^1(t, x, \partial)\vec{v}_P=\nu_i(x)A_P^{ij}(t, x)\partial_j\vec{v}_P+B_P(t, x)\vec{v}_P,$$

T is a positive constant; $\vec{u}_H$, $\vec{u}_P$, $\vec{v}_H$ and $\vec{v}_P$ are real valued functions: $\vec{u}=(\vec{u}_H, \vec{u}_P)$, $\vec{u}_H={}^t(u_{H1}, \cdots, u_{Hm_H})$, $\vec{u}_P={}^t(u_{P1}, \cdots, u_{Pm_P})$ (${}^tM$ means the transpose of M). The $\nu_i(x)$ $(i=1, \cdots, n)$ are real valued functions in $C_0^\infty(\boldsymbol{R}^n)$ such that $\nu(x)=(\nu_1(x), \cdots, \nu_n(x))$ represents the unit outer normal to Γ at $x\in\Gamma$. The summation convention is understood such as the sub and superscripts i, j take all values 1 to n. Let us introduce assumptions (A.1)-(A.4) which coefficients of the operators satisfy.

(A.1) A_H^{ij} are $m_H\times m_H$ matrices, A_{HP}^i are $m_H\times m_P$ matrices, A_P^0, A_P^{ij} and A_P^i are $m_P\times m_P$ matrices, A_{PH}^{ij} and A_{PH}^{i0} are $m_P\times m_H$ matrices, and the elements of these matrices are in $\mathscr{B}^\infty([0, T]\times\bar{\Omega})$. A_H^{i0} are $m_H\times m_H$ matrices whose elements of these are in $\mathscr{B}^\infty([-\kappa, T+\kappa]\times\bar{\Omega})$. B_{HP} is an $m_H\times m_P$ matrix, B_P is an $m_P\times m_P$ matrix, B_{PH}^0 and B_{PH}^i are $m_P\times m_H$ matrices whose elements of these are in $\mathscr{B}^\infty([0, T]\times\Gamma)$. B_H^0 is an $m_H\times m_H$ matrix whose elements of this are in $\mathscr{B}^\infty([-\kappa, T+\kappa]\times\Gamma)$. And all elements of these matrices are real-valued.

$\mathscr{B}^\infty(G)$ denotes the set of functions in $C^\infty(G)$ whose derivatives of any order are all bounded in G. For any function space S, we denote a product space $S\times\cdots\times S$ by also S.

(A.2) $\quad {}^tA_E^{ij}=A_E^{ji}$ $(E=H, P)$, ${}^tA_P^0=A_P^0$, ${}^tA_H^{i0}=A_H^{i0}$, ${}^tB_H^0=B_H^0$.

(A.3) There exist positive constants c_0, δ_0 and δ_1 such that

$$A_P^0(t, x)\geqq c_0I_{m_P};$$

$$(A_E^{ij}(t, \cdot)\partial_j\vec{u}_E, \partial_i\vec{u}_E)\geqq\delta_1\|\vec{u}_E\|_1^2-\delta_0\|\vec{u}_E\|^2$$

for any $\vec{u}_E\in H^1(\Omega)$, $t\in[0, T]$, $x\in\bar{\Omega}$ $(E=H, P)$.

$H^s(G)$ denotes the usual Sobolev space on G of order s with norm $\|\cdot\|_{s,G}$ for $s\in\boldsymbol{R}$. Put $\|\cdot\|_{s,\Omega}=\|\cdot\|_s$ and $\|\cdot\|_{0,\Omega}=\|\cdot\|$. We denote the usual inner product of $L^2(\Omega)=H^0(\Omega)$ by $(\ ,\ )$. I_m is an $m\times m$ identity matrix.

(A.4) $\quad B_H^0(t, x)-\dfrac{1}{2}\nu_i(x)A_H^{i0}(t, x)\geqq 0 \qquad$ for any $(t, x)\in[-\kappa, T+\kappa]\times\Gamma$.

The purpose of this paper is to prove the unique existence of solutions of (N) and to investigate how the constant in the first energy inequality depends on the coefficients of the operators. Our proof of the existence theorem is almost parallel to Shibata [6]. In § 2, we state the basic notation and main results. In § 3, we refer to the result on some elliptic boundary value problem. In § 4, assuming that

$$\text{(A.4}') \qquad B_H^0(t, x)-\frac{1}{2}\nu_i(x)A_H^{i0}(t, x)\geqq \varepsilon \qquad \text{for any } (t, x)\in[0, T]\times\Gamma,\ \varepsilon>0,$$

instead of (A.4), we prove the existence theorem. In §§ 5 and 6, reducing the problem to the case where Ω is a half space, and using the former result and the estimate of Kreis-Sakamoto type, we derive an *a priori* estimate of original problem, and then the existence theorem is obtained. The argument of § 5 is not needed when $n=1$, so that we mention the case that $n=1$ in § 7.

§ 2. Notation and main results.

First of all, we explain our notation. We always assume that functions are real-valued except for § 5. For any integers L, $M\geqq 0$, we put

$$D^L\vec{u}=(\partial_t^k\partial_x^\alpha\vec{u},\ k+|\alpha|=L), \qquad \bar{D}^L\vec{u}=(\partial_t^k\partial_x^\alpha\vec{u},\ k+|\alpha|\leqq L),$$

$$\partial^L\vec{u}=(\partial_x^\alpha\vec{u},\ |\alpha|=L), \qquad \bar{\partial}^L\vec{u}=(\partial_x^\alpha\vec{u},\ |\alpha|\leqq L).$$

If J is an interval of $\boldsymbol{R}$ and G is a domain, we put

$$X^{L,M}(J\,;G)=\bigcap_{l=0}^{L} C^l(J\,;H^{L+M-l}(G))\,;$$

$$Z^{L,M}(J\,;G)=C^L(J\,;H^{M-1}(G))\cap\bigcap_{l=0}^{L-1} C^l(J\,;H^{L+M-l}(G)).$$

Let G' be a set in $\boldsymbol{R}^k$ ($k\geqq 1$) and X, Y represent points of $\boldsymbol{R}^k$. For any integer $l\geqq 0$ and $\sigma\in(0, 1)$

$$|u|_{\infty,l,G'}=\sum_{|\alpha|\leqq l}\sup_{X\in G'}|(\partial^\alpha u)(X)| \qquad \alpha=(\alpha_1, \cdots, \alpha_k)\,;$$

$$|u|_{\infty,l+\sigma,G'}=|u|_{\infty,l,G'}+\sum_{|\alpha|=l}\sup_{\substack{X,Y\in G'\\ X\neq Y}}\frac{|(\partial^\alpha u)(X)-(\partial^\alpha u)(Y)|}{|X-Y|^\sigma}.$$

Put

$$\mathscr{B}^l(\bar{G}')=\{u\in C^l(\bar{G})\,|\,|u|_{\infty,l,G'}<\infty\}\,;$$

$$\mathscr{B}^{l+\sigma}(\bar{G}')=\{u\in C^l(\bar{G}')\,|\,|u|_{\infty,l+\sigma,G'}<\infty\}.$$

We write $|\cdot|_{\infty,l+\sigma,I}=|\cdot|_{\infty,l+\sigma,I\times\Omega}$ and $\langle\cdot\rangle_{\infty,l+\sigma,I}=|\cdot|_{\infty,l+\sigma,I\times\Gamma}$. Let us define the space of solutions $E^L(J\,;\Omega)$ by

$$E^L(J;\Omega)=\{\vec{u}_H\in X^{L,0}(J;\Omega)\,|\,\partial_t^{L-1}\bar{D}^1\vec{u}_H\in L^2(J;H^{-1/2}(\Gamma))\}$$
$$\times\{\vec{u}_P\in Z^{L-1,1}(J;\Omega)\,|\,\partial_t^{L-1}\vec{u}_P\in L^2(J;H^1(\Omega))\}.$$

$H^s(\Gamma)$ is a Hilbert space equipped with the norm $\langle\!\langle\cdot\rangle\!\rangle_s=\|\cdot\|_{s,\Gamma}$ for $s\in\boldsymbol{R}$, and put $\langle\!\langle\cdot\rangle\!\rangle=\langle\!\langle\cdot\rangle\!\rangle_0$. $\langle\,,\rangle$ denotes the usual inner product of $L^2(\Gamma)=H^0(\Gamma)$. When $n=1$, $\langle\!\langle\cdot\rangle\!\rangle_s$ stands for the absolute value $|\cdot|$ for any $s\in\boldsymbol{R}$. As the norm of $E^L(J;\Omega)$, we put

$$|||\vec{u}(t)|||_1^2=\|\bar{D}^1\vec{u}_H(t)\|^2+\|\vec{u}_P(t)\|^2$$
$$+\int_0^t\langle\!\langle\bar{D}^1\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2ds+\int_0^t\|\vec{u}_P(s)\|_1^2ds\,;$$
$$|||\vec{u}(t)|||_L^2=\|\bar{D}^L\vec{u}_H(t)\|^2+\|\partial_t^{L-1}\vec{u}_n(t)\|^2+\|\bar{D}^{L-2}\vec{u}_P(t)\|_2^2$$
$$+\int_0^t\langle\!\langle\partial_t^{L-1}\bar{D}^1\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2ds+\int_0^t\|\partial_t^{L-1}\vec{u}_P(s)\|_1^2ds\qquad\text{for } L\geqq 2.$$

For the space of right members, for $L\geqq 2$ we put

$$R^L(J;\Omega)=\{f(t,x)\in X^{L-2,0}(J;\Omega)\,|\,\partial_t^{L-1}f(t,x)\in L^2([0,T];L^2(\Omega))\}\,;$$
$$R^L(J;\Gamma)=\{g(t,x)\in X^{L-2,1/2}(J;\Gamma)\,|\,\partial_t^{L-1}g(t,x)\in L^2([0,T];H^{1/2}(\Gamma))\}.$$

Let μ be a small positive number $\in(0,1)$. For $I=[0,T]$ and $J=[-\kappa,T+\kappa]$, put

$$\mathcal{M}(l)=\sum_{E=H,P}\sum_{i,j=1}^n|A_E^{ij}|_{\infty,l,I}+|A_P^0|_{\infty,l,I}+\sum_{i,j=1}^n|A_{PH}^{ij}|_{\infty,l,I}$$
$$+\sum_{i=1}^n(|A_H^{i0}|_{\infty,l,J}+|A_{HP}^i|_{\infty,l,I}+|A_P^i|_{\infty,l,I}+|A_{PH}^{i0}|_{\infty,l,I})$$
$$+\langle B_{HP}\rangle_{\infty,l,I}+\langle B_H^0\rangle_{\infty,l,J}+\langle B_P\rangle_{\infty,l,I}+\sum_{i=0}^n|B_{PH}^i|_{\infty,l,I\times\Gamma}\,;$$
$$\mathcal{M}(1+\mu)=\sum_{i,j=1}(|A_H^{ij}|_{\infty,1+\mu,I}+|A_P^{ij}|_{\infty,1,I}+|A_{PH}^{ij}|_{\infty,1,I})+|A_P^0|_{\infty,1,I}$$
$$+\sum_{i=1}^n(|A_H^{i0}|_{\infty,1+\mu,J}+|A_{HP}^i|_{\infty,1,I}+|A_{PH}^{i0}|_{\infty,1,I}+|A_P^i|_{\infty,1,I})$$
$$+\langle B_{HP}\rangle_{\infty,1,I}+\langle B_H^0\rangle_{\infty,1,J}+\langle B_P\rangle_{\infty,1,I}+\sum_{i=0}^n\langle B_{PH}^i\rangle_{\infty,1,I}\,;$$
$$\mathcal{M}(2)=\sum_{E=H,P}\sum_{i,j=1}^n|A_E^{ij}|_{\infty,2,I}+|A_P^0|_{\infty,1,I}$$
$$+\sum_{i=1}^n(|A_H^{i0}|_{\infty,1+\mu,J}+|A_{HP}^i|_{\infty,1,I}+|A_P^i|_{\infty,1,I}+|A_{PH}^{i0}|_{\infty,1,I})$$
$$+\langle B_{HP}\rangle_{\infty,2,I}+\langle B_H^0\rangle_{\infty,2,J}+\langle B_P\rangle_{\infty,2,I}+\sum_{i=0}^n\langle B_{PH}^i\rangle_{\infty,2,I}\,.$$

Second of all, we shall explain the compatibility condition, which $\vec{u}_{H0}$, $\vec{u}_{H1}$, $\vec{u}_{P0}$, $\vec{f}_E$ and $\vec{g}_E$ $(E=H, P)$ should satisfy in order that solutions to (N) exist. For a moment, we assume that solutions $\vec{u}=(\vec{u}_H, \vec{u}_P)$ to (N) exist and that

(2.1) $$\vec{u}=(\vec{u}_H, \vec{u}_P)\in E^L([0, T]; \Omega) \quad \text{for } L\geqq 2.$$

Put

(2.2) $$\vec{u}_{Hk}=\partial_t^k\vec{u}_H(0) \quad (0\leqq k\leqq L), \qquad \vec{u}_{Pk}=\partial_t^k\vec{u}_P(0) \quad (0\leqq k\leqq L-1),$$

which are represented in terms of initial data, right members $\vec{f}_H$, $\vec{f}_P$ and their derivatives. For example

$$\vec{u}_{H2}=A_H^2(0, x, \partial)\vec{u}_{H0}+A_H^1(0, x, \partial)\vec{u}_{H1}+A_{HP}^1(0, x, \partial)\vec{u}_{P0}+\vec{f}_H(0, x);$$

$$\vec{u}_{P1}=A_P^0(0, x)^{-1}\{A_P^2(0, x, \partial)\vec{u}_{P0}+A_{PH}^2(0, x, \partial)\vec{u}_{H0}+A_{PH}^1(0, x, \partial)\vec{u}_{H1}+\vec{f}_P(0, x)\},$$

and so on. It follows from (2.1) that

(2.3) $$\begin{aligned}&\vec{u}_{Hk}\in H^{L-k}(\Omega) \quad 0\leqq k\leqq L,\\ &\vec{u}_{Pk}\in H^{L-k}(\Omega) \quad 0\leqq k\leqq L-2, \qquad \vec{u}_{PL-1}\in L^2(\Omega).\end{aligned}$$

In view of the trace theorem to the boundary, the boundary condition in (N) requires that

(2.4) $$\begin{aligned}&\partial_t^k(B_H^1(t, x, \partial)\vec{u}_H+B_{HP}(t, x)\vec{u}_P+B_H^0(t, x)\partial_t\vec{u}_H)|_{t=0}=\partial_t^k\vec{g}_H(0) \quad \text{on } \Gamma;\\ &\partial_t^k(B_P^1(t, x, \partial)\vec{u}_P+B_{PH}^1(t, x, \partial)\vec{u}_H+B_{PH}^0(t, x)\partial_t\vec{u}_H)|_{t=0}=\partial_t^k\vec{g}_P(0) \quad \text{on } \Gamma,\end{aligned}$$

for $0\leqq k\leqq L-2$. Such conditions are also represented in terms of initial data, right members $\vec{f}_E$, $\vec{g}_E$ $(E=H, P)$ and their derivatives. When (2.4) holds, we say that $\vec{u}_{H0}$, $\vec{u}_{H1}$, $\vec{u}_{P0}$, $\vec{f}_E$ and $\vec{g}_E$ $(E=H, P)$ satisfy the compatibility conditions of order $L-2$.

Now, we shall state our main results.

THEOREM 2.1. *Let $T>0$. Assume that* (A.1)-(A.4) *are valid.*

(1) *If $\vec{u}_{H0}\in H^2(\Omega)$, $\vec{u}_{H1}\in H^1(\Omega)$, $\vec{u}_{P0}\in H^2(\Omega)$, $\vec{f}_E\in R^2([0, T]; \Omega)$ and $\vec{g}_E\in R^2([0, T]; \Gamma)$, and if* (2.3) *and* (2.4) *are satisfied for $L=2$, then there exists a unique solution $\vec{u}=(\vec{u}_H, \vec{u}_P)\in E^2([0, T]; \Omega)$ to* (N).

(2) *Assume that $n\geqq 2$ and let μ be a small positive number $\in(0, 1)$. Then, there exists a constant $C=C(\mathcal{M}(1+\mu), T)>0$ such that for any $t\in[0, T]$ and $\vec{u}\in(\vec{u}_H, \vec{u}_P)\in X^{2,0}\times Z^{1,1}([0, T]; \Omega)$ the following estimates hold:*

(2.5) $$\begin{aligned}|||\vec{u}(t)|||_1^2\leqq C\Big\{&\|\bar{D}^1\vec{u}_H(0)\|^2+\|\vec{u}_P(0)\|^2\\ &+\sum_{E=H,P}\int_0^t(\|\mathcal{A}_E(s)[\vec{u}(s)]\|^2+\langle\!\langle\mathcal{B}_E(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2)ds\Big\};\end{aligned}$$

$$(2.6)\qquad \|\partial_t\vec{u}_H(t)\|^2+\|\vec{u}_H(t)\|^2_{\mathcal{J}(t)}+\|\vec{u}_P(t)\|^2_{\mathcal{J}(t)}$$

$$\leqq e^{ct}\Big[\|\partial_t\vec{u}_H(0)\|^2+\|\vec{u}_H(0)\|^2_{\mathcal{J}(0)}+\|\vec{u}_P(0)\|^2_{\mathcal{J}(0)}$$

$$+C\int_0^t\{\sum_{E=H,P}(\|\mathcal{A}_E(s)[\vec{u}(s)]\|^2+\langle\!\langle\mathcal{B}_E(s)[\vec{u}(s)]\rangle\!\rangle^2_{1/2})$$

$$+\|\vec{u}_P(s)\|_1^2+\langle\!\langle\bar{D}^1\vec{u}_H(s)\rangle\!\rangle^2_{-1/2}\}\,ds\Big],$$

where

$$(2.7)\qquad \|\vec{u}_H(t)\|^2_{\mathcal{J}(s)}=(A_H^{ij}(s)\partial_j\vec{u}_P(t),\ \partial_i\vec{u}_H(t))+\delta_0\|\vec{u}_H(t)\|^2;$$

$$\|\vec{u}_P(t)\|^2_{\mathcal{J}(s)}=(A_P^0(s)\vec{u}_P(t),\ \vec{u}_P(t)).$$

(3) *When* $n=1$, *the estimate* (2.5) *and* (2.6) *are valid with* $\mu=0$.

THEOREM 2.2. *Let* $T>0$ *and* L *be an integer* $\geqq 3$. *Assume that* (A.1)-(A.4) *are valid. If* $\vec{u}_{H0}\in H^L(\Omega)$, $\vec{u}_{H1}\in H^{L-1}(\Omega)$, $\vec{u}_{P0}\in H^L(\Omega)$, $\vec{f}_E\in R^L([0,T];\Omega)$ *and* $\vec{g}_E\in R^L([0,T];\Gamma)$, *and if* (2.3) *and* (2.4) *are satisfied, then there exists a unique solution* $\vec{u}=(\vec{u}_H,\vec{u}_P)\in E^L([0,T];\Omega)$ *to* (N).

REMARK. (1) We do not need the estimate (2.6) in this paper. But to prove the existence theorem of nonlinear problem, (2.6) is needed. (cf. [4], [5]).

(2) By a suitable extension of the coefficients of the operators with respect to t, we know that the assumptions (A.1)-(A.4) are equivalent to the following assumptions (a.1)-(a.5).

(a.1) A_H^{ij} and A_H^{i0} are $m_H\times m_H$ matrices, A_{HP}^i are $m_H\times m_P$ matrices, A_P^0, A_P^{ij} and A_P^i are $m_P\times m_P$ matrices, A_{PH}^{i0} and A_{PH}^{ij} are $m_P\times m_H$ matrices, and the elements of these matrices in $\mathcal{B}^\infty(\boldsymbol{R}\times\bar{\Omega})$. B_{HP} and B_H^0 are $m_H\times m_H$ matrices, B_P is an $m_P\times m_P$ matrix, B_{PH}^0 and B_{PH}^i are $m_P\times m_H$ matrices, and the elements of these matrices are in $\mathcal{B}^\infty(\boldsymbol{R}\times\Gamma)$;

$$(a.2)\qquad {}^tA_E^{ij}=A_E^{ji}\ (E=H,P),\quad {}^tA_H^{i0}=A_H^{i0},\quad {}^tA_P^0=A_P^0,\quad {}^tB_H^0=B_H^0;$$

(a.3) there exist c_0, δ_0 and δ_1 such that

$$A_P^0(t,x)\geqq c_0I_{m_P};$$

$$(A_E^{ij}(t,\cdot)\partial_j\vec{u}_E,\ \partial_i\vec{u}_E)\geqq\delta_1\|\vec{u}_E\|_1^2-\delta_0\|\vec{u}_E\|^2$$

for any $\vec{u}_E\in H^1(\Omega)$ and $t\in\boldsymbol{R}$, $x\in\bar{\Omega}$, $(E=H,P)$;

$$(a.4)\qquad B_H^0(t,x)-\frac{1}{2}\nu_i(x)A_H^{i0}(t,x)\geqq 0\qquad \text{for any } (t,x)\in\boldsymbol{R}\times\Gamma;$$

(a.5) if we write $A_E^{ij}=(A_{Ea}^{ijb})$, then there exists $T_0>0$ such that $A_{Ea}^{ijb}=\delta_{ij}\delta_{ab}$ $(E=H,P)$ and other functions vanish for $|t|>T_0$.

§ 3. On an elliptic boundary value problem.

Our purpose in this section is to solve the following elliptic boundary value problem in Ω with parameter $t\in[0, T]$:

$$-A_H^2(t, \cdot, \partial)\vec{u}_H - A_{HP}^1(t, \cdot, \partial)\vec{u}_P + \lambda_H\vec{u}_H = \vec{f}_H(t) \quad \text{in } \Omega, \tag{3.1a}$$

$$-A_P^2(t, \cdot, \partial)\vec{u}_P - A_{PH}^2(t, \cdot, \partial)\vec{u}_H + \lambda_P\vec{u}_P = \vec{f}_P(t) \quad \text{in } \Omega, \tag{3.1b}$$

$$B_H^1(t, \cdot, \partial)\vec{u}_H + B_{HP}(t, \cdot)\vec{u}_P = \vec{g}_H(t) \quad \text{on } \Gamma, \tag{3.1c}$$

$$B_P^1(t, \cdot, \partial)\vec{u}_P + B_{PH}^1(t, \cdot, \partial)\vec{u}_H = \vec{g}_P(t) \quad \text{on } \Gamma. \tag{3.1d}$$

where λ_H and λ_P are constants determined below, and $\vec{f}_H, \vec{f}_P \in C^0([0, T]; L^2(\Omega))$ and $\vec{g}_H, \vec{g}_P \in C^0([0, T]; H^{1/2}(\Gamma))$. First we consider the following problem:

$$\begin{cases} -A_H^2(t, \cdot, \partial)\vec{u}_H + \lambda_H\vec{u}_H = \vec{f} & \text{in } \Omega, \\ B_H^1(t, \cdot, \partial)\vec{u}_H = \vec{g} & \text{on } \Gamma. \end{cases} \tag{3.2}$$

Let us define the bilinear form $D_H(t, \cdot, \cdot)$ associated with (3.2) by

$$D_H(t, \vec{w}_H, \vec{v}_H) = (A_H^{ij}\partial_j\vec{w}_H, \partial_i\vec{v}_H) + \lambda_H(\vec{w}_H, \vec{v}_H) \quad \text{for } \vec{w}_H, \vec{v}_H \in H^1(\Omega). \tag{3.3}$$

By Schwarz's inequality and (A.3), we have

$$|D_H(t, \vec{w}_H, \vec{v}_H)| \leqq C(\lambda_H, \mathcal{M}(0))\|\vec{w}_H\|_1\|\vec{v}_H\|_1; \tag{3.4}$$

$$D_H(t, \vec{w}_H, \vec{w}_H) \geqq \delta_1\|\vec{w}_H\|_1^2 \quad \text{as } \lambda_H \geqq \delta_0. \tag{3.5}$$

By the Lax and Milgram theorem, we know that for any $\vec{f}\in L^2(\Omega)$ and $\vec{g}\in H^{1/2}(\Gamma)$, there exists a unique solution $\vec{w}_H\in H^1(\Omega)$ of variational equation:

$$D_H(t, \vec{w}_H, \vec{v}_H) = (\vec{f}, \vec{v}_H) + \langle\vec{g}, \vec{v}_H\rangle \quad \text{for any } \vec{v}_H\in H^1(\Omega). \tag{3.6}$$

To prove $\vec{w}_H\in H^2(\Omega)$, straightening the boundary locally, we study the case that $\Omega=\boldsymbol{R}_+^n$, where $\boldsymbol{R}_+^n=\{x\in\boldsymbol{R}^n \mid x_n>0\}$ (cf. [6], § 3). For any $h\neq 0$ such that $|h|$ is small, put

$$[\vec{w}_H]_h = \{\vec{w}_H(y+h\vec{e}_p) - \vec{w}_H(y)\}/h \quad \text{where } \vec{e}_p = (0, \cdots, \overset{\overset{p}{\vee}}{1}, \cdots, 0) \tag{3.7}$$

for $p=1, \cdots, n-1$. By (3.5),

$$D_H(t, [\vec{w}_H]_h, \vec{v}_H) \leqq \{\|\vec{f}\| + \langle\!\langle\vec{g}\rangle\!\rangle_{1/2} + C(\mathcal{M}(1))\|\partial^1\vec{w}_H\|\}\,\|\partial^1\vec{v}_H\|_{\boldsymbol{R}_+^n}. \tag{3.8}$$

Here, we have used that $\|[\vec{w}_H]_h\|_{\boldsymbol{R}_+^n} \leqq \|\partial^1\vec{w}_H\|_{\boldsymbol{R}_+^n}$. Putting $\vec{v}_H=[\vec{w}_H]_h$ in (3.8), by (3.5) we have

$$\|\partial^1[\vec{w}_H]_h\|_{\boldsymbol{R}_+^n} \leqq C(\mathcal{M}(1))(\|\vec{f}\| + \langle\!\langle\vec{g}\rangle\!\rangle_{1/2} + \|\partial^1\vec{w}_H\|), \tag{3.9}$$

which implies that $\partial_p\vec{w}_H\in H^1(\boldsymbol{R}_+^n)$, $p=1, \cdots, n-1$. Noting that A_H^{nn} is non-singular and $\vec{w}_H$ satisfies (3.2) in the distribution sense, we see $\partial_n^2 \vec{w}_H \in L^2(\boldsymbol{R}_+^n)$ and we have

$$(3.10) \qquad \|\partial^2 \vec{w}_H\| \leqq C(\mathcal{M}(1), \delta_1)(\|\vec{f}\| + \langle\!\langle \vec{g} \rangle\!\rangle_{1/2} + \|\vec{w}_H\|_1).$$

From (3.5) and (3.6), it follows that

$$(3.11) \qquad \|\vec{w}_H\|_1 \leqq C(\mathcal{M}(0))(\|\vec{f}\| + \langle\!\langle \vec{g} \rangle\!\rangle_{1/2}).$$

By (3.10) and (3.11), we have

$$(3.12) \qquad \|\vec{w}_H\|_2 \leqq C(\mathcal{M}(1))(\|\vec{f}\| + \langle\!\langle \vec{g} \rangle\!\rangle_{1/2}).$$

Moreover, by integration by parts we see that $\vec{w}_H$ is a strong solution to (3.2). In particular, if we substitute $\vec{f}_H(t)$ for $\vec{f}$ and $\vec{g}_H(t)$ for $\vec{g}$, there exists a unique solution $\vec{u}_H^0 \in H^2(\Omega)$ to (3.2) satisfying the estimate:

$$(3.13) \qquad \|\vec{u}_H^0\|_2 \leqq C(\mathcal{M}(1))(\|\vec{f}_H(t)\| + \langle\!\langle \vec{g}_H(t) \rangle\!\rangle_{1/2}).$$

And if we substitute $A_{HP}^1(t, \cdot, \partial)\vec{u}_P$ for $\vec{f}$ and $-B_{HP}(t, \cdot)\vec{u}_P$ for $\vec{g}$, there exists a unique solution $\vec{u}_H(\vec{u}_P) \in H^2(\Omega)$ to (3.2) satisfying the estimate:

$$(3.14) \qquad \|\vec{u}_H(\vec{u}_P)\|_2 \leqq C(\mathcal{M}(1))\|\vec{u}_P\|_1,$$

which implies that $\vec{u}_H(\vec{u}_P)$ is a bounded linear operator of $\vec{u}_P \in H(\Omega)$ to $\vec{u}_H(\vec{u}_P) \in H^2(\Omega)$. Put $\vec{u}_H = \vec{u}_H^0 + \vec{u}_H(\vec{u}_P)$. $\vec{u}_H$ is a solution to (3.1a) and (3.1c) for given $\vec{u}_P \in H^1(\Omega)$. Noting the above facts, we consider the following problem:

$$(3.15) \qquad \begin{cases} -A_P^2(t, \cdot, \partial)\vec{u}_P - A_{PH}^2(t, \cdot, \partial)\vec{u}_H(\vec{u}_P) + \lambda_P \vec{u}_P & \\ \qquad = \vec{f}_P(t) + A_{PH}^2(t, \cdot, \partial)\vec{u}_H^0 & \text{in } \Omega, \\ B_P^1(t, \cdot, \partial)\vec{u}_P + B_{PH}^1(t, \cdot, \partial)\vec{u}_H(\vec{u}_P) = \vec{g}_P(t) - B_{PH}^1(t, \cdot, \partial)\vec{u}_H^0 & \text{on } \Gamma. \end{cases}$$

The associated bilinear form is

$$(3.16) \qquad D_P(t, \vec{u}_P, \vec{v}_P) = (A_P^{ij}\partial_j \vec{u}_P, \partial_i \vec{v}_P) - (A_{PH}^{ij}\partial_i\partial_j \vec{u}_H(\vec{u}_P), \vec{v}_P)$$
$$-(A_P^i \partial_i \vec{u}_P, \vec{v}_P) + \langle B_{PH}^i \partial_i \vec{u}_H(\vec{u}_P), \vec{v}_P\rangle + \langle B_P \vec{u}_P, \vec{v}_P\rangle.$$

From Schwartz's inequality and (3.14) it follows that

$$(3.17) \qquad |D_P(t, \vec{u}_P, \vec{v}_P)| \leqq C(\mathcal{M}(1))\|\vec{u}_P\|_1 \|\vec{v}_P\|_1 \qquad \text{for any } \vec{u}_P, \vec{v}_P \in H^1(\Omega);$$

$$(3.18) \qquad D_P(t, \vec{u}_P, \vec{u}_P) \geqq \frac{\delta_1}{2}\|\vec{u}_P\|_1^2 \qquad \text{for any } \vec{u}_P \in H^1(\Omega),$$

provided that λ_P is sufficiently large.

The Lax-Milgram theorem yields that there exists a unique $\vec{u}_P \in H^1(\Omega)$ such that

$$(3.19)\qquad D_P(t,\ \vec{u}_P,\ \vec{v}_P)=(\vec{f}_P(t),\ \vec{v}_P)+(A^{ij}_{PH}(t)\partial_i\partial_j\vec{u}^0_H,\ \vec{v}_P)$$
$$+\langle\vec{g}_P(t),\ \vec{v}_P\rangle-\langle B^i_{PH}(t)\partial_i\vec{u}^0_H,\ \vec{v}_P\rangle\,.$$

Furthermore, employing the same argument as above, we see $\vec{u}_P\in H^2(\Omega)$ and obtain

$$(3.20)\qquad \|\vec{u}_P\|_2\leqq C(\mathcal{M}(1))(\|\vec{f}_P(t)\|+\langle\!\langle\vec{g}_P(t)\rangle\!\rangle_{1/2}+\|\vec{u}^0_H\|_2)\,.$$

Combining (3.13), (3.14) and (3.20), we have

$$(3.12)\qquad \|\vec{u}_H\|_2+\|\vec{u}_P\|_2\leqq C(\mathcal{M}(1))\sum_{E=H,P}(\|\vec{f}_E(t)\|+\langle\!\langle\vec{g}_E(t)\rangle\!\rangle_{1/2})\,.$$

$\vec{u}=(\vec{u}_H,\ \vec{u}_P)$ is a unique solution to (3.1a)-(3.1d). Moreover, $\vec{u}=(\vec{u}_H,\ \vec{u}_P)$ depends on time t, so that we write $\vec{u}=\vec{u}(t,\ x)=\vec{u}(t)$. By (3.21), we have

$$\|\vec{u}_H(t)-\vec{u}_H(t')\|_2+\|\vec{u}_P(t)-\vec{u}_P(t')\|_2$$
$$\leqq C(\mathcal{M}(2))\{\sum_{E=H,P}(\|\vec{f}_E(t)-\vec{f}_E(t')\|+\langle\!\langle\vec{g}_E(t)-\vec{g}_E(t')\rangle\!\rangle_{1/2})$$
$$+|t-t'|(\|\vec{u}_H(t')\|_2+\|\vec{u}_P(t')\|_2)\}\,.$$

Therefore, we see that $\vec{u}_H$ and $\vec{u}_P\in C^0([0,\ T]\,;H^2(\Omega))$. In the similar manner, we can get the higher regularity of the solutions. Namely, we have following theorem:

THEOREM 3.1. *Assume that* (A.1)-(A.4) *are valid*

(1) *For any* $\vec{f}_E\in C^0([0,\ T]\,;L^2(\Omega))$ *and* $\vec{g}_E\in C^0([0,\ T]\,;H^{1/2}(\Gamma))$ $(E=H,\ P)$, *there exist constants* λ_H *and* λ_P *depending only on* $\mathcal{M}(0)$ *such that* (3.1) *admits a unique solution* $\vec{u}=(\vec{u}_H,\ \vec{u}_P)\in C^0([0,\ T]\,;H^2(\Omega))$ *satisfying* (3.21).

(2) *Let* L *and* K *be integers* $\geqq 0$. *If* $\vec{f}_E\in C^K([0,\ T]\,;H^L(\Omega))$ *and* $\vec{g}_E\in C^K([0,\ T]\,;H^{L+1/2}(\Gamma))$ $(E=H,\ P)$, *then* $\vec{u}=(\vec{u}_H,\ \vec{u}_P)\in C^K([0,\ T]\,;H^{L+2}(\Omega))$.

§4. An existence theorem under the assumption (A.4′).

In this section, assuming that

$$(\mathrm{A.4'})\qquad B^0_H(t,\ x)-\frac{1}{2}\nu_i(x)A^{i0}_H(t,\ x)\geqq\varepsilon\qquad\text{for any }(t,\ x)\in[0,\ T]\times\Gamma,\ \varepsilon>0\,,$$

in stead of (A.4), we shall prove the existence theorem of (N). At first, we calculate the energy estimate.

LEMMA 4.1. *Assume that* (A.1)-(A.4) *hold. For any* $\vec{u}_H\in X^{2,0}([0,\ T]\,;\Omega)$ *and* $\vec{u}_P\in Z^{1,1}([0,\ T]\,;\Omega)$, *the identity*

$$(4.1)\quad \frac{1}{2}\frac{d}{dt}\{\|\partial_t\vec{u}_H(t)\|^2+\|\vec{u}_H(t)\|^2_{\mathcal{J}(t)}+\|\vec{u}_P(t)\|^2_{\mathcal{J}(t)}\}$$

$$+\left\langle\left(B_H^0(t)-\frac{1}{2}\nu_i A_H^{i0}(t)\right)\partial_t\vec{u}_H(t),\ \partial_t\vec{u}_H(t)\right\rangle+(A_P^{ij}(t)\partial_j\vec{u}_P(t),\ \partial_i\vec{u}_P(t))$$

$$+\langle B_{HP}(t)\vec{u}_P(t),\ \partial_t\vec{u}_H(t)\rangle+\langle B_{PH}^0(t)\partial_t\vec{u}_H(t)+B_{PH}^i(t)\partial_i\vec{u}_H(t),\ \vec{u}_P(t)\rangle$$

$$\cong(\mathcal{A}_H(t)[\vec{u}(t)],\ \partial_t\vec{u}_H(t))+(\mathcal{A}_P(t)[\vec{u}(t)],\ \vec{u}_P(t))$$

$$+\langle\mathcal{B}_H(t)[\vec{u}(t)],\ \partial_t\vec{u}_H(t)\rangle+\langle\mathcal{B}_P(t)[\vec{u}(t)],\ \vec{u}_P(t)\rangle$$

holds for $t\in[0, T]$, *where* $A\cong B$ *means that*

$$(4.2)\quad |A-B|\leqq C(\mathcal{M}(1))(\|\bar{D}^1\vec{u}_H(t)\|^2+\|\vec{u}_P(t)\|^2+\|\vec{u}_P(t)\|\,\|\bar{\partial}^1\vec{u}_P(t)\| +\|\bar{D}^1\vec{u}_H(t)\|\,\|\bar{\partial}^1\vec{u}_P(t)\|).$$

PROOF. If we calculate $(\mathcal{A}_H(t)[\vec{u}(t)]+\delta_0\vec{u}_H,\ \partial_t\vec{u}_H(t))$ and $(\mathcal{A}_P(t)[\vec{u}(t)],\ \vec{u}_P(t))$ and combine the resulting formulas, we have (4.1). ■

LEMMA 4.2. *Assume that* (A.1)-(A.2) *hold. Let* $B(t, x)=B(t)$ *is an* $m_P\times m_H$ *matrix of functions in* $\mathcal{B}^\infty([0, T]\times\Gamma)$. *Let* $\vec{u}_H\in X^{2,0}([0, T]; \Omega)$ *and* $\vec{u}_P\in Z^{1,1}([0, T]; \Omega)$. *Then the following estimates are valid*:

$$(4.3)\quad |\langle B(t)\partial_i\vec{u}_H(t),\ \vec{u}_P(t)\rangle|$$

$$\leqq C(\mathcal{M}(0),\ \sigma)(\|\vec{u}_H(t)\|_1^2+\|\vec{u}_P(t)\|^2+\langle\!\langle\mathcal{B}_H(t)[\vec{u}(t)]\rangle\!\rangle^2_{-1/2})$$

$$+\sigma\|\vec{u}_P(t)\|_1^2+\sigma\langle\!\langle\partial_t\vec{u}_H(t)\rangle\!\rangle^2 \quad \textit{for any } \sigma>0;$$

$$(4.4)\quad \langle\!\langle\partial_i\vec{u}_H(t)\rangle\!\rangle_{-1/2}\leqq C(\mathcal{M}(0))(\|\vec{u}_H(t)\|_1+\langle\!\langle\mathcal{B}_H(t)[\vec{u}(t)]\rangle\!\rangle_{-1/2} +\langle\!\langle\vec{u}_P(t)\rangle\!\rangle_{-1/2}+\langle\!\langle\partial_t\vec{u}_H(t)\rangle\!\rangle_{-1/2}).$$

PROOF. Put $\vec{g}_H(t)=\mathcal{B}_H(t)[\vec{u}(t)]$. Using the local coordinates, we can straighten the bouundary locally, so that it is sufficient to prove the lemma in the case that $\Omega=\boldsymbol{R}_+^n$. Since A_H^{nn} is invertible, we write on Γ:

$$(4.5)\quad \partial_n\vec{u}_H(t,\ x',\ 0)=A_H^{nn}(t,\ x',\ 0)^{-1}\{-\vec{g}_H(t,\ x',\ 0)+B_{HP}(t,\ x')\vec{u}_P(t,\ x',\ 0) +B_H^0(t,\ x')\partial_t\vec{u}_H(t,\ x',\ 0)-\sum_{j=1}^{n-1}A_H^{nj}(t,\ x',\ 0)\partial_j\vec{u}_H(t,\ x',\ 0)\}.$$

By (4.5) we can prove (4.3) and (4.4) easily. ■

LEMMA 4.3. *Assume that* (A.1)-(A.4′) *are valid. For any* $T>0$, *the following estimates hold*: (E.1) *there exist* $C_i=C_i(\mathcal{M}(1),\ \varepsilon)$ $i=1,\ 2$ *such that*

$$|\!|\!|\vec{u}(t)|\!|\!|_1^2\leqq C_1e^{C_2t}\{\|\bar{D}^1\vec{u}_H(0)\|^2+\|\vec{u}_P(0)\|^2$$

$$+\sum_{E=H,P}\int_0^t(\|\mathcal{A}_E(s)[\vec{u}(s)]\|^2+\langle\!\langle\mathcal{B}_E(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2)ds\}$$

for any $\vec{u}_H\in X^{2,0}([0, T_1]; \Omega)$; $\vec{u}_P\in Z^{1,1}([0, T_1]; \Omega)$ *and for any* $T_1\in(0, T]$.

(E.2) *there exist* $C_i=C_i(\mathcal{M}(1), \varepsilon)$ $i=3, 4$ *such that*

$$\|\partial_t\vec{u}_H(t)\|^2+\|\vec{u}_H(t)\|_{\mathcal{G}(t_0)}^2+\|\vec{u}_P(t)\|_{\mathcal{G}(t_0)}^2$$
$$\leqq e^{C_3(t-t_1)}\{\|\partial_t\vec{u}_H(t_1)\|^2+\|\vec{u}_H(t_1)\|_{\mathcal{G}(t_0)}^2+\|\vec{u}_P(t_1)\|_{\mathcal{G}(t_0)}^2$$
$$+C_4\sum_{E=H,P}\int_{t_1}^t(\|\mathcal{A}_E(t_0)[\vec{u}(s)]\|^2+\langle\!\langle\mathcal{B}_E(t_0)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2)ds\}$$

for any $t\in[t_1, t_2]$, $\vec{u}_H\in X^{2,0}([t_1, t_2]; \Omega)$, $\vec{u}_P\in Z^{1,1}([t_1, t_2]; \Omega)$ *and* $t_0, t_1, t_2\in[0, T]$ $(t_1<t_2)$.

PROOF. By Lemmas 4.1 and 4.2 and (A.4′), we have

$$\frac{d}{ds}\{\|\partial_s\vec{u}_H(s)\|^2+\|\vec{u}_H(s)\|_{\mathcal{G}(s)}^2+\|\vec{u}_P(s)\|_{\mathcal{G}(s)}^2\}\tag{4.6}$$
$$+\varepsilon\langle\!\langle\partial_s\vec{u}_H(s)\rangle\!\rangle^2+\delta_1\|\vec{u}_P(s)\|_1^2$$
$$\leqq C(\mathcal{M}(1), \delta_1, \varepsilon)\{\sum_{E=H,P}(\|\mathcal{A}_E(s)[\vec{u}(s)]\|^2+\langle\!\langle\mathcal{B}_E(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2)$$
$$+\|\vec{u}_H(s)\|_1^2+\|\partial_s\vec{u}_H(s)\|^2+\|\vec{u}_P(s)\|^2\};$$

$$\frac{d}{ds}\{\|\partial_s\vec{u}_H(s)\|^2+\|\vec{u}_H(s)\|_{\mathcal{G}(t_0)}^2+\|\vec{u}_P(s)\|_{\mathcal{G}(T_0)}^2\}\tag{4.7}$$
$$\leqq C(\mathcal{M}(1), \varepsilon)\{\sum_{E=H,P}(\|\mathcal{A}_E(t_0)[\vec{u}(s)]\|^2+\langle\!\langle\mathcal{B}_E(t_0)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2)$$
$$+\|\partial_s\vec{u}_H(s)\|^2+\|\vec{u}_H(s)\|_{\mathcal{G}(t_0)}^2+\|\vec{u}_P(s)\|_{\mathcal{G}(t_0)}^2\}.$$

Combining (4.4) and (4.6) and integrating the resulting formula on $[0, t]$, we obtain (E.1) by Gronwall's inequality. And also (E.2) can be obtained from (4.7). ■

Now, we shall prove

$$\begin{aligned}&\mathcal{A}_H(t_0)[\vec{u}(t, x)]=\vec{f}_H(t, x), \quad \mathcal{A}_P(t_0)[\vec{u}(t, x)]=\vec{f}_P(t, x) && \text{in } [t_1, t_2]\times\Omega,\\ &\mathcal{B}_H(t_0)[\vec{u}(t, x)]=0, \quad \mathcal{B}_P(t_0)[\vec{u}(t, x)]=0 && \text{on } [t_1, t_2]\times\Gamma,\\ &\vec{u}_H(t_1, x)=\vec{u}_{H0}(x),\ \partial_t\vec{u}_H(t_1, x)=\vec{u}_{H1}(x),\ \vec{u}_P(t_1, x)=\vec{u}_{P0}(x) && \text{in } \Omega.\end{aligned}\tag{4.8}$$

Here and hereafter, t_0, t_1 and t_2 are always any fixed times on $[0, T]$ such that $t_1<t_2$. Let $H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ be Hilbert space with norm

$$|||\mathcal{V}|||^2=\|\vec{u}_{H0}\|_1^2+\|\vec{u}_{H1}\|^2+\|\vec{u}_P\|^2\tag{4.9}$$

for $\mathcal{U}=(\vec{u}_{H0}, \vec{u}_{H1}, \vec{u}_P)\in H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$. Put

(4.10a) $$(\mathcal{U}, \mathcal{V})_{\mathcal{H}(t)}=(A_H^{ij}(t)\partial_j\vec{u}_{H0}, \partial_i\vec{v}_{H0})+\delta_0(\vec{u}_{H0}, \vec{v}_{H0})$$
$$+(\vec{u}_{H1}, \vec{v}_{H1})+(A_P^0(t)\vec{u}_P, \vec{v}_P);$$

(4.10b) $$\|\mathcal{U}\|^2_{\mathcal{H}(t)}=(\mathcal{U}, \mathcal{U})_{\mathcal{H}(t)}=\|\vec{u}_{H0}\|^2_{\mathcal{J}(t)}+\|\vec{u}_{H1}\|^2+\|\vec{u}_P\|^2_{\mathcal{J}(t)},$$

where $\mathcal{U}=(\vec{u}_{H0}, \vec{u}_{H1}, \vec{u}_P)$, $\mathcal{V}=(\vec{v}_{H0}, \vec{v}_{H1}, \vec{v}_P)\in H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$. By (A.2) and (A.3), we know that $(\ ,\ )_{\mathcal{H}(t)}$ is a bilinear form and

(4.11) $$\min(1, \delta_1, c_0)|||\mathcal{U}|||^2\leqq\|\mathcal{U}\|^2_{\mathcal{H}(t)}\leqq C(\mathcal{M}(0))|||\mathcal{U}|||^2$$

for any $\mathcal{U}\in H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$, $t\in[0, T]$. Let $\mathcal{H}(t)$ denote the Hilbert space $H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ equipped with inner product $(\ ,\ )_{\mathcal{H}(t)}$. Put

(4.12a) $$\mathcal{A}(t)\mathcal{U}=\begin{bmatrix} \vec{u}_{H1} \\ A_H^2(t,\cdot,\partial)\vec{u}_{H0}+A_H^1(t,\cdot,\partial)\vec{u}_{H1}+A_{HP}^1(t,\cdot,\partial)\vec{u}_P \\ (A_P^0)^{-1}\{A_P^2(t,\cdot,\partial)\vec{u}_P+A_{PH}^2(t,\cdot,\partial)\vec{u}_{H0}+A_{PH}^1(t,\cdot,\partial)\vec{u}_{H1}\} \end{bmatrix},$$

(4.12b) $$\mathcal{B}(t)\mathcal{U}=\begin{bmatrix} B_H^1(t,\cdot,\partial)\vec{u}_{H0}+B_{HP}(t,\cdot)\vec{u}_P+B_H^0(t,\cdot)\vec{u}_{H1} \\ B_P^1(t,\cdot,\partial)\vec{u}_P+B_{PH}^1(t,\cdot,\partial)\vec{u}_{H0}+B_{PH}^0(t,\cdot)\vec{u}_{H1} \end{bmatrix}$$

for $\mathcal{U}=H^2(\Omega)\times H^1(\Omega)\times H^2(\Omega)$;

(4.12c) $$\mathcal{D}(t)=\{\mathcal{U}\in H^2(\Omega)\times H^1(\Omega)\times H^2(\Omega)\mid\mathcal{B}(t)\mathcal{U}=0 \quad\text{on } \Gamma.\}.$$

LEMMA 4.4. *Assume that* (A.1)-(A.4′) *are valid. Then there exists a* $C=C(\mathcal{M}(1))$ *such that*

(4.13) $$\|(\lambda I-\mathcal{A}(t)\mathcal{U}\|_{\mathcal{H}(t)}\geqq(\lambda-C)\|\mathcal{U}\|_{\mathcal{H}(t)}$$

for any $\lambda>C$ *and* $\mathcal{U}\in\mathcal{D}(t)$. *Here, I is the identity operator.*

PROOF. Since

$$\|(\lambda I-\mathcal{A}(t))\mathcal{U}\|^2_{\mathcal{H}(t)}=\lambda^2\|\mathcal{U}\|^2_{\mathcal{H}(t)}+\|\mathcal{A}(t)\mathcal{U}\|^2_{\mathcal{H}(t)}-2\lambda(\mathcal{A}(t)\mathcal{U}, \mathcal{U}),$$

if we have

(14.4) $$2(\mathcal{A}(t)\mathcal{U}, \mathcal{U})_{\mathcal{H}(t)}\leqq C\|\mathcal{U}\|^2_{\mathcal{H}(t)} \qquad\text{for any } \mathcal{U}\in\mathcal{D}(t),$$

where $C=C(\mathcal{M}(1))$, we can get (4.13) immediately. Since $\mathcal{B}(t)\mathcal{U}=0$ on Γ, by integration by parts we have

$$(\mathcal{A}(t)\mathcal{U}, \mathcal{U})_{\mathcal{H}(t)}\leqq-\Big\langle(B_H^0(t)-\frac{1}{2}\nu_iA_H^{i0}(t))\vec{u}_{H1}, \vec{u}_{H1}\Big\rangle-(A_P^{ij}(t)\partial_j\vec{u}_P, \partial_i\vec{u}_P)$$
$$+C(\sigma, \mathcal{M}(1))\|\mathcal{U}\|_{\mathcal{H}(t)}+\sigma\|\vec{u}_P\|_1+\sigma\langle\!\langle\vec{u}_{H1}\rangle\!\rangle$$

for any $\sigma>0$. Here we have used the same idea as in Lemma 4.2 to estimate the boundary terms. Therefore, by (A.3) and (A.4′), we have (4.14), which completes the proof. ■

LEMMA 4.5. *Assume that* (A.1)-(A.4′) *are valid. Then, there exists a* $C=C(\mathcal{M}(1), \varepsilon)$ *such that for any* $\lambda>C$, $\lambda I-\mathcal{A}(t)$ *is a bijective map of* $\mathcal{D}(t)$ *onto* $\mathcal{H}(t)$. *If we denote its inverse by* $(\lambda I-\mathcal{A}(t))^{-1}$, *then*

$$\|(\lambda I-\mathcal{A}(t))^{-1}\mathcal{U}\|_{\mathcal{H}(t)}\leqq(\lambda-C)^{-1}\|\mathcal{U}\|_{\mathcal{H}(t)} \tag{4.15}$$

for any $\lambda>C$, $\mathcal{U}\in\mathcal{D}(t)$.

PROOF. In view of Lemma 4.4, it is sufficient to prove the bijectiveness. Namely, for given $\mathcal{V}=(\vec{v}_{H0}, \vec{v}_{H1}, \vec{v}_P)\in\mathcal{H}(t)$ we shall prove the unique existence of $\mathcal{U}=(\vec{u}_{H0}, \vec{u}_{H1}, \vec{u}_P)\in\mathcal{D}(t)$ such that $(\lambda I-\mathcal{A}(t))\mathcal{U}=\mathcal{V}$. If we use the relation of the first components: $\lambda\vec{u}_{H0}-\vec{u}_{H1}=\vec{v}_{H0}$, we rewrite the relation of the second, the third components and the condition that $\mathcal{U}\in\mathcal{D}(t)$ as follows:

$$\begin{aligned}
&-A_H^2(t, \cdot, \partial)\vec{u}_{H0}-\lambda A_H^1(t, \cdot, \partial)\vec{u}_{H0}-A_{HP}^1(t, \cdot, \partial)\vec{u}_P+\lambda^2\vec{u}_{H0}=\vec{f}_H(t) && \text{in } \Omega,\\
&-A_P^2(t, \cdot, \partial)\vec{u}_P+\lambda A_P^0(t, \cdot)\vec{u}_P-A_{PH}^2(t, \cdot, \partial)\vec{u}_{H0}-\lambda A_{PH}^1(t, \cdot, \partial)\vec{u}_{H0}=\vec{f}_P(t) && \text{in } \Omega,\\
&B_H^1(t, \cdot, \partial)\vec{u}_{H0}+B_{HP}(t, \cdot)\vec{u}_P+\lambda B_H^0(t, \cdot)\vec{u}_{H0}=\vec{g}_H(t) && \text{on } \Gamma,\\
&B_P^1(t, \cdot, \partial)\vec{u}_P+B_{PH}^1(t, \cdot, \partial)\vec{u}_{H0}+\lambda B_{PH}^0(t, \cdot)\vec{u}_{H0}=\vec{g}_P(t) && \text{on } \Gamma,
\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
&\vec{f}_H(t)=-A_H^1(t, \cdot, \partial)\vec{v}_{H0}+\lambda\vec{v}_{H0}+\vec{v}_{H1}\in L^2(\Omega),\\
&\vec{f}^d(t)=A_P^0(t)\vec{v}_P-A_{PH}^1(t, \cdot, \partial)\vec{v}_{H0}\in L^2(\Omega),\\
&\vec{g}_H(t)=B_H^0(t)\vec{v}_{H0}\in H^1(\Omega), \qquad \vec{g}_P(t)=B_{PH}^0(t)\vec{v}_{H0}\in H^1(\Omega).
\end{aligned}$$

If we prove that there exists a constant C such that for any $\lambda>C$ the problem (4.16) admits a unique solution $(\vec{u}_{H0}, \vec{u}_P)\in H^2(\Omega)\times H^2(\Omega)$, then $\mathcal{U}=(\vec{u}_{H0}, \lambda\vec{u}_{H0}-\vec{v}_{H0}, \vec{u}_P)$ is a required vector. At first, for given $\vec{u}_P\in H^1(\Omega)$ we consider the following problem:

$$\left\{\begin{aligned}
&-A_H^2(t, \cdot, \partial)\vec{u}_{H0}-\lambda A_H^1(t, \cdot, \partial)\vec{u}_{H0}+\lambda^2\vec{u}_{H0}\\
&\qquad=\vec{f}_H(t)+A_{HP}^1(t, \cdot, \partial)\vec{u}_P && \text{in } \Omega,\\
&B_H^1(t, \cdot, \partial)\vec{u}_{H0}+\lambda B_H^0(t, \cdot)\vec{u}_{H0}=\vec{g}_H(t)-B_{HP}(t, \cdot)\vec{u}_P && \text{on } \Gamma.
\end{aligned}\right. \tag{4.17a}$$

The associated bilinear form is

$$\begin{aligned}
D_H(t, \vec{u}_H, \vec{v}_H)=&(A_H^{ij}(t)\partial_j\vec{u}_H, \partial_i\vec{v}_H)-\lambda(A_H^{i0}(t)\partial_i\vec{u}_H, \vec{v}_H)\\
&+\lambda^2(\vec{u}_H, \vec{v}_H)+\lambda\langle B_H^0(t)\vec{u}_H, \vec{v}_H\rangle \quad \text{for } \vec{u}_H, \vec{v}_H\in H^1(\Omega).
\end{aligned} \tag{4.18}$$

By Schwaltz's inequality, (A.3) and (A.4′), we have

(4.19) $$|D_H(t, \vec{u}_H, \vec{v}_H)| \leqq C(\lambda, \mathcal{M}(0)) \|\vec{u}_H\|_1 \|\vec{v}_H\|_1 ;$$

(4.20) $$D_H(t, \vec{u}_H, \vec{u}_H) \geqq \delta_1 \|\vec{u}_H\|_1^2 - \delta_0 \|\vec{u}_H\|^2 + \lambda\varepsilon \langle\!\langle \vec{u}_H \rangle\!\rangle^2 + \lambda^2 \|\vec{u}_H\|^2 - \lambda C(\mathcal{M}(1)) \|\vec{u}_H\|^2$$
$$\geqq \delta_1 \|\vec{u}_H\|_1^2 + \frac{\lambda^2}{2} \|\vec{u}_H\|^2 + \lambda\varepsilon \langle\!\langle \vec{u}_H \rangle\!\rangle^2 \geqq \delta_1 \|\vec{u}_H\|_1^2$$

for any large λ. Therefore, employing the same arguments as in the proof of Theorem 3.1, we see that there exist a unique solution $\vec{u}_H^0 \in H^2(\Omega)$ and a unique solution $\vec{u}_H(\vec{u}_P) \in H^2(\Omega)$ such that

(4.17a.1) $$\begin{cases} -A_H^2(t, \cdot, \partial)\vec{u}_H^0 - \lambda A_H^1(t, \cdot, \partial)\vec{u}_H^0 + \lambda^2 \vec{u}_H^0 = \vec{f}_H(t) & \text{in } \Omega, \\ B_H^1(t, \cdot, \partial)\vec{u}_H^0 + \lambda B_H^0(t)\vec{u}_H^0 = \vec{g}_H(t) & \text{on } \Gamma, \end{cases}$$

(4.17a.2) $$\begin{cases} -A_H^2(t, \cdot, \partial)\vec{u}_H(\vec{u}_P) - \lambda A_H^1(t, \cdot, \partial)\vec{u}_H(\vec{u}_P) + \lambda^2 \vec{u}_H(\vec{u}_P) = A_{HP}^1(t, \cdot, \partial)\vec{u}_P & \text{in } \Omega, \\ B_H^1(t, \cdot, \partial)\vec{u}_H(\vec{u}_P) + \lambda B_H^0(t)\vec{u}_H(\vec{u}_P) = -B_{HP}(t)\vec{u}_P & \text{on } \Gamma. \end{cases}$$

In particular, $\vec{u}_H(\vec{u}_P)$ is a bounded linear operator of $\vec{u}_P \in H^1(\Omega)$ to $\vec{u}_H(\vec{u}_P) \in H^2(\Omega)$, and $\vec{u}_H = \vec{u}_H^0 + \vec{u}_H(\vec{u}_P)$ satisfies (4.17a). By (4.20) and (4.17a.2) we see that there exists $\lambda_H(\mathcal{M}(1))$ such that

(4.21) $$\delta_1 \|\vec{u}_H(\vec{u}_P)\|_1^2 + \frac{1}{4}\lambda^2 \|\vec{u}_H(\vec{u}_P)\|^2 + \frac{\lambda\varepsilon}{2} \langle\!\langle \vec{u}_H(\vec{u}_P) \rangle\!\rangle^2$$
$$\geqq \frac{C}{\lambda^2} \|\vec{u}_P\|_1^2 + \frac{C}{\lambda\varepsilon} \langle\!\langle \vec{u}_P \rangle\!\rangle^2 \qquad \text{for any } \lambda > \lambda_H(\mathcal{M}(1)),$$

where $C = C(\mathcal{M}(0))$. Now, we consider the following:

(4.17b) $$\begin{cases} -A_P^2(t, \cdot, \partial)\vec{u}_P + \lambda A_P^0(t)\vec{u}_P - A_{PH}^2(t, \cdot, \partial)\vec{u}_H(\vec{u}_P) \\ \qquad -\lambda A_{PH}^1(t, \cdot, \partial)\vec{u}_H(\vec{u}_P) = \vec{f}_P(t) + A_{PH}^2(t, \cdot, \partial)\vec{u}_H^0 + \lambda A_{PH}^1(t, \cdot, \partial)\vec{u}_H^0 & \text{in } \Omega, \\ B_P^1(t, \cdot, \partial)\vec{u}_P + B_{PH}^1(t, \cdot, \partial)\vec{u}_H(\vec{u}_P) + \lambda B_{PH}^0(t, \cdot)\vec{u}_H(\vec{u}_P) \\ \qquad = \vec{g}_P(t) - \lambda B_{PH}^0(t, \cdot)\vec{u}_H^0 - B_{PH}^1(t, \cdot, \partial)\vec{u}_H^0 & \text{on } \Gamma. \end{cases}$$

The associated bilinear form is

(4.22) $$D_P(t, \vec{u}_P, \vec{v}_P) = (A_P^{ij}(t)\partial_j \vec{u}_P, \partial_i \vec{v}_P) - (A_P^i(t)\partial_i \vec{u}_P, \vec{v}_P) + \lambda(A_P^0(t)\vec{u}_P, \vec{v}_P)$$
$$+ (A_{PH}^{ij}(t)\partial_j \vec{u}_H(\vec{u}_P), \partial_i \vec{v}_P) + ((\partial_i A_{PH}^{ij}(t))\partial_j \vec{u}_H(\vec{u}_P), \vec{v}_P)$$
$$- \lambda(A_{PH}^{i0}(t)\partial_i \vec{u}_H(\vec{u}_P), \vec{v}_P) - \langle \nu_i A_{PH}^{ij}(t)\partial_j \vec{u}_H(\vec{u}_P), \vec{v}_P \rangle + \langle B_P(t)\vec{u}_P, \vec{v}_P \rangle$$
$$+ \langle B_{PH}^i(t)\partial_i \vec{u}_H(\vec{u}_P)\vec{v}_P \rangle + \lambda \langle B_{PH}^0(t)\vec{u}_H(\vec{u}_P), \vec{v}_P \rangle .$$

In the same way as in Lemma 4.2, we have,

(4.23) $$|\langle (B^j_{PH}(t)-\nu_i A^{ij}_{PH}(t))\partial_j \vec{u}_H(\vec{u}_P),\ \vec{v}_P\rangle|$$
$$\leqq C(\lambda,\ \mathcal{H}(0))(\langle\!\langle \vec{u}_P\rangle\!\rangle\langle\!\langle \vec{v}_P\rangle\!\rangle+\langle\!\langle \vec{u}_H(\vec{u}_P)\rangle\!\rangle\langle\!\langle \vec{v}_P\rangle\!\rangle+\|\vec{u}_H(\vec{u}_P)\|_1\|\vec{v}_P\|_1).$$

From (4.22) and (4.23), it follows that

(4.24) $$|D_P(t,\ \vec{u}_P,\ \vec{v}_P)|\leqq C(\lambda,\ \mathcal{M}(0),\ \varepsilon)\|\vec{u}_P\|_1\|\vec{v}_P\|_1$$

for any $\vec{u}_P,\ \vec{v}_P\in H^1(\Omega)$. Furthermore, we see that there exist a constant C and $\lambda_P=\lambda_P(\mathcal{M}(1))$ such that

(4.25) $$D_P(t,\ \vec{u}_P,\ \vec{u}_P)\geqq C\|\vec{u}_P\|_1^2 \qquad \text{for any } \lambda>\lambda_P,\ \vec{u}_P\in H^1(\Omega).$$

Employing the same argument as before, we see that there exists a unique solution $\vec{u}_P\in H^2(\Omega)$ to (4.17b). Therefore, $(\vec{u}^0_H+\vec{u}_H(\vec{u}_P),\ \vec{u}_P)$ is a unique solution to (4.16), which completes the lemma. ■

LEMMA 4.6. *Assume that* (A.1)-(A.4′) *are valid. Then,* $\mathcal{D}(t)$ *is dense in* $\mathcal{H}(t)$.

PROOF. Since $H^\infty(\Omega)$ is dense in $H^1(\Omega)$ and $C^\infty_0(\Omega)$ is dense in $L^2(\Omega)$, for any $\mathcal{U}=(\vec{u}_{H0},\ \vec{u}_{H1},\ \vec{u}_P)\in\mathcal{H}(t)$, there exist $\vec{v}^l_{H0}\in H^\infty(\Omega)$, $\vec{u}^l_{H1}\in C^\infty_0(\Omega)$ and $\vec{v}^l_P\in C^\infty_0(\Omega)$ such that

$$\|\vec{v}^l_{H0}-\vec{u}_{H0}\|_1+\|\vec{u}^l_{H1}-\vec{u}_{H1}\|+\|\vec{v}^l_P-\vec{u}_P\|\longrightarrow 0 \qquad \text{as } l\to\infty.$$

$\vec{v}^l_{H0}$ and $\vec{v}^l_P$ will be modified so that the boundary condition can be satisfied. Let $\vec{w}^l_{H0}$ be a vector of function satisfying the following:

(4.26a) $$B^1_H(t,\ x,\ \partial)\vec{w}^l_{H0}=-B^1_H(t,\ x,\ \partial)\vec{v}^l_{H0}, \qquad \vec{w}^l_{H0}=0 \quad \text{on } \Gamma,$$
$$\|\vec{w}^l_{H0}\|\leqq 1/l.$$

The existence of $\vec{w}^l_{H0}\in H^\infty(\Omega)$ is guaranted by Lemma 3.8 of [6]. And let $\vec{w}^l_P$ be a vector of functions satisfying the following:

(4.26b) $$B^1_P(t,\ x,\ \partial)\vec{w}^l_P=-B^1_P(t,\ x,\ \partial)\vec{v}^l_P-B^1_{PH}(t,\ x,\ \partial)(\vec{v}^l_{H0}+\vec{w}^l_{H0}) \qquad \text{on } \Gamma,$$
$$\vec{w}^l_P=0 \quad \text{on } \Gamma, \qquad \|\vec{w}^l_P\|\leqq 1/l.$$

The existence of $\vec{w}^l_P\in H^\infty(\Omega)$ is also guaranteed by Lemma 4.8 of [6]. If we put $\mathcal{U}^l=(\vec{u}^l_{H0},\ \vec{v}^l_{H1},\ \vec{u}^l_P)$ where $\vec{u}^l_{H0}=\vec{v}^l_{H0}+\vec{w}^l_{H0}$ and $\vec{u}^l_P=\vec{v}^l_P+\vec{w}^l_P$, $\mathcal{U}^l$ satisfies that $\mathcal{B}\mathcal{U}^l=0$ on Γ and $\|\mathcal{U}^l-\mathcal{U}\|_{\mathcal{H}(t)}\to 0$ as $l\to\infty$, which completes the proof of the lemma. ■

In view of Lemmas 4.5 and 4.6, an application of the Hille-Yoshida theorem yields the following theorem;

THEOREM 4.7. *Assume that* (A.1)-(A.4′) *are valid. Let* t_0, t_0 *and* $t_2 \in [0, T]$ *such that* $t_1 < t_2$. *Then, for any* $\mathcal{U}_0 \in \mathcal{D}(t_0)$ *and* $\mathcal{F}(t) \in C^0)[t_1, t_2]; \mathcal{H}(t_0))$ *there exists a unique* $\mathcal{U}(t) \in C^1([t_1, t_2]; \mathcal{H}(t_0)) \cap C^0([t_1, t_2]; \mathcal{D}(t_0))$ *such that*

$$(4.27) \qquad \frac{d}{dt}\mathcal{U}(t) = \mathcal{A}(t_0)\mathcal{U}(t) + \mathcal{F}(t), \quad \mathcal{U}(t) \in \mathcal{D}(t_0) \qquad \text{for any } t \in [t_1, t_2],$$

$$\mathcal{U}(t_1) = \mathcal{U}_0 .$$

If we put $\mathcal{U}_0 = (\vec{u}_{H0}, \vec{u}_{H1}, \vec{u}_{P0})$ and $\mathcal{F}(t) = (0, \vec{f}_H(t), \vec{f}_P(t))$, then the first and the third components of $\mathcal{U}(t)$ of Theorem 4.7 are solutions to (4.8). Summing up, we have proved the following theorem.

THEOREM 4.8. *Assume that* (A.1)-(A.4′) *are valid. Let* t_0, t_1 *and* $t_2 \in [0, T]$ *such that* $t_1 < t_2$. *If* $\vec{u}_{H0} \in H^2(\Omega)$, $\vec{u}_{H1} \in H^1(\Omega)$, $\vec{u}_{P0} \in H^2(\Omega)$, $\vec{f}_H, \vec{f}_P \in C^0([t, t_2]; L^2(\Omega))$ *and*

$$(4.28) \qquad B_H^1(t_0, \cdot, \partial)\vec{u}_{H0} + B_{HP}(t_0, \cdot)\vec{u}_{P0} + B_H^0(t_0, \cdot)\vec{u}_{H1} = 0 \qquad \text{on } \Gamma;$$

$$B_P^1(t_0, \cdot, \partial)\vec{u}_{P0} + B_{PH}^1(t_0, \cdot, \partial)\vec{u}_{P0} + B_{PH}^0(t_0, \cdot)\vec{u}_{H1} = 0 \qquad \text{on } \Gamma,$$

then there exists a unique solution $\vec{u} = (\vec{u}_H, \vec{u}_P) \in X^{2,0}([t_1, t_2]; \Omega) \times Z^{1,1}([t_1, t_2]; \Omega)$ *to* (4.8). ∎

Next, we shall get the estimate of the second energy.

LEMMA 4.9. *Assume that* (A.1)-(A.4′) *are valid. For* $\vec{u} = (\vec{u}_H, \vec{u}_P) \in E^2([0, T]; \Omega)$, *we put* $\vec{f}_E(t, x) = \mathcal{A}_E(t)[\vec{u}(t, x)]$ *and* $\vec{g}_E(t, x) = \mathcal{B}_E(t)[\vec{u}(t, x)]$ $(E = H, P)$. *If* $\vec{f}_E \in R^2([0, T]; \Omega)$ *and* $\vec{g}_E \in R^2([0, T]; \Gamma)$ $(E = H, P)$, *then there exists a constant* $C > 0$ *independent of* $\vec{u}$ *such that*

$$(4.29) \qquad |||\vec{u}(t)|||_2^2 \leqq C\{\|\vec{u}_H(0)\|_2^2 + \|\partial_t\vec{u}_H(0)\|_1^2 + \|\vec{u}_P(0)\|_2^2 + \sum_{E=H,P}(\|\vec{f}_E(0)\|^2 + \langle\!\langle \vec{g}_E(0)\rangle\!\rangle_{1/2}^2) + \sum_{E=H,P}\sum_{k=1}^{1}\int_0^t(\|\partial_t^k\vec{f}_E(s)\|^2 + \langle\!\langle \partial_s^k\vec{g}_E(s)\rangle\!\rangle_{1/2}^2)ds\} \qquad \text{for } 0 \leqq t \leqq T.$$

LEMMA 4.10. *Assume that* (A.1)-(A.4′) *are valid. Let* t_0, t_1 *and* $t \in [0, T]$ *such that* $t_1 < t_2$. *For* $\vec{u} = (\vec{u}_H, \vec{u}_P) \in X^{2,0}([t_1, t_2]; \Omega) \times Z^{1,1}([t_1, t_2]; \Omega)$, *we put* $\vec{f}_E(t, x) = \mathcal{A}_E(t_0)[\vec{u}(t, x)]$ *and* $\vec{g}_E(t, x) = \mathcal{B}_E(t_0)[\vec{u}(t, x)]$ $(E = H, P)$. *If* $\vec{f}_E \in R^2([t_1, t_2]; \Omega)$ *and* $\vec{g}_E \in R^2([t_1, t_2]; \Gamma)$ $(E = H, P)$, *then*

(4.30)
$$\|\partial_t^2\vec{u}_H(t)\|^2+\|\partial_t\vec{u}_H(t)\|^2_{\mathcal{G}(t_0)}+\|\partial_t\vec{u}_P(t)\|^2_{\mathcal{G}(t_0)}$$
$$\leqq e^{C_3(t-t_1)}\{\|\partial_t^2\vec{u}_H(t_1)\|^2+\|\partial_t\vec{u}_H(t_1)\|^2_{\mathcal{G}(t_0)}+\|\partial_t\vec{u}_P(t_1)\|^2_{\mathcal{G}(t_0)}$$
$$+C_4\sum_{E=H,P}\int_{t_1}^{t}(\|\partial_s\vec{f}_E(s)\|^2+\langle\langle\partial_s\vec{g}_E(s)\rangle\rangle^2_{1/2})ds\} \quad \textit{for any } t\in[t_1, t_2].$$

PROOFS OF LEMMA 4.9 AND 4.10. In the case that $\vec{u}=(\vec{u}_H, \vec{u}_P)$ is smooth in t, differentiating the equations once with respect to t, applying (E.1) to the resulting equations and using Theorem 3.1 to estimate the second derivatives with respect to x, we have (4.29). To remove the smoothness assumption with respect to t, we use the mollifier with respect to t. (4.30) can be obtained by use of (E.2) instead of (E.1) in the same manner. For details, see Lemmas 4.1 and 4.2 of [6]. ■

By Theorem 4.8 and Lemma 4.10, we can prove an existence theorem for $\mathcal{A}_E(t_0)$ and $\mathcal{B}_E(t_0)$ $(E=H, P)$ with the inhomogeneous boundary condition.

LEMMA 4.11. *Assume that* (A.1)-(A.4') *are valid.* t_0, t_1 *and* $t_2\in[0, T]$ *such that* $t_1<t_2$. *If* $\vec{u}_{H0}\in H^2(\Omega)$, $\vec{u}_{H1}\in H^1(\Omega)$, $\vec{u}_{P0}\in H^2(\Omega)$, $\vec{f}_E\in R^2([t_1, t_2]; \Omega)$ *and* $\vec{g}_E\in R^2([t_1, t_2]; \Gamma)$ $(E=H, P)$, *and they satisfy the compatibility condition of order* 0 *in the following sense*:

(4.31)
$$B_H^1(t_0, \cdot, \partial)\vec{u}_{H0}+B_{HP}(t_0, \cdot)\vec{u}_{P0}+B_H^0(t_0, \cdot)\vec{u}_{H1}=\vec{g}_H(t_1, x) \quad \textit{on } \Gamma;$$
$$B_P^1(t_0, \cdot, \partial)\vec{u}_{P0}+B_{PH}^1(t_0, \cdot, \partial)\vec{u}_{H0}+B_{PH}^0(t_0, \cdot)\vec{u}_{H1}=\vec{g}_P(t_1, x) \quad \textit{on } \Gamma,$$

then there exists a unique solution $\vec{u}=(\vec{u}_H, \vec{u}_P)\in X^{2,0}([t_1, t_2]; \Omega)\times Z^{1,1}([t_1, t_2]; \Omega)$ *to the equations*:

(4.32)
$$\mathcal{A}_E(t_0)\vec{u}[(t, x)]=\vec{f}_E(t, x) \quad \textit{in } [t_1, t_2]\times\Omega \ (E=H, P),$$
$$\mathcal{B}_E(t_0)[\vec{u}(t, x)]=\vec{g}_E(t, x) \quad \textit{on } [t_1, t_2]\times\Gamma \ (E=H, P),$$
$$\vec{u}_H(t_1, x)=\vec{u}_{H0}(x), \quad \partial_t\vec{u}_H(t_1, x)=\vec{u}_{H1}(x), \quad \vec{u}_P(t_1, x)=\vec{u}_{P0}(x) \quad \textit{in } \Omega.$$

PROOF. Since we know that (E.2) holds from Lemma 4.3, employing the same argument as in Lemma 4.3 of [6], we can prove the lemma. ■

LEMMA 4.12. *Assume that* (A.1)-(A.4') *are valid. If* $\vec{f}_E\in R^2([0, T]; \Omega)$ *and* $\vec{g}_E\in R^2([0, T]; \Gamma)$ $(E=H, P)$ *satisfy*

(4.33)
$$\vec{f}_E(0, x)=0 \quad \textit{in } \Omega, \qquad \vec{g}_E(0, x)=0 \quad \textit{on } \Gamma \ (E=H, P),$$

then there exists a unique solution $\vec{u}=(\vec{u}_H, \vec{u}_P)\in E^2([0, T]; \Omega)$ *to* (N) *with zero initial data and right members* $\vec{f}_E$ *and* $\vec{g}_E$ $(E=H, P)$.

PROOF. Since we see that (E.1) and (E.2) are valid by Lemma 4.3, using Lemma 4.11 and the method of Cauchy's polygonal line, we can prove the lemma in the same manner as in Lemma 4.4 of [6]. ∎

Using Lemmas 4.9 and 4.12, we can complete the existence theorem under the additional condition (A.4').

THEOREM 4.13. *Assume that* (A.1)-(A.4') *are valid. If* $\vec{u}_{H0}\in H^2(\Omega)$, $\vec{u}_{H1}\in H^1(\Omega)$, $\vec{u}_{P0}\in H^2(\Omega)$, $\vec{f}_E\in R^2([0, T]; \Omega)$ *and* $\vec{g}_E\in R^2([0, T]; \Omega)$, *and if they satisfy compatibility condition of order* 0, *i.e.*

$$\text{(4.34)}\qquad B_H^1(0, \cdot, \partial)\vec{u}_{H0}+B_{HP}(0, \cdot)\vec{u}_{P0}+B_H^0(0, \cdot)\vec{u}_{H1}=\vec{g}_H(0) \qquad on\ \Gamma;$$

$$B_P^1(0, \cdot, \partial)\vec{u}_{P0}+B_{PH}^0(0, \cdot, \partial)\vec{u}_{H0}+B_{PH}^0(0, \cdot)\vec{u}_{H1}=\vec{g}_P(0) \qquad on\ \Gamma,$$

then (N) *admits a unique solution* $\vec{u}=(\vec{u}_H, \vec{u}_P)\in E^2([0, T]; \Omega)$.

PROOF. Since the uniqueness follows from (E.1), we have only to prove the existence. First, assuming that

$$(*)\qquad \vec{u}_{H0}, \vec{u}_{H1}, \vec{u}_{P0}\in H^\infty(\Omega), \qquad \vec{f}_H, \vec{f}_P\in C^\infty([0, T]; H^\infty(\Omega)).$$

Put $\vec{u}_{El}={}^t(u_{El1}, \cdots, u_{Elm_E})$ for $0\leq l\leq k_E$ $(E=H, P)$, where $k_H=2$ and $k_P=1$. By u'_{Ela} we denote the extension of u_{Ela} from Ω to $\boldsymbol{R}^n$. We put

$$\hat{U}_{Ea}(t, x)=\sum_{l,m=0}^{kE}\Big(\exp\sqrt{-1}a_m(1+|\xi|^2)^{1/2}t\Big)b_{ml}\hat{u}'_{Ela}(\xi)(1+|\xi|^2)^{-l/2},$$

where a_m and b_{ml} are chosen in such a way that $\sum_{m=l}^{k_E}(\sqrt{-1}a_m)^h b_{ml}=\delta_{hl}$ for $0\leq h,\ l\leq k_E$. Obviously, $\partial_t^h U_{Ea}(0, x)=u'_{Eha}(x)$ for $0\leq h\leq k_E$, and $U_{Ea}\in C^\infty(\boldsymbol{R}; H^\infty(\boldsymbol{R}^n))$ $(E=H, P)$. Put $\vec{U}=(\vec{U}_H, \vec{U}_P)$, $\vec{U}_H={}^t(U_{H1}, \cdots, U_{Hm_H})$, $\vec{U}_P={}^t(U_{P1}, \cdots, U_{Pm_P})$. Let $\vec{v}=(\vec{v}_H, \vec{v}_P)\in E^2([0, T]; \Omega)$ be a solution to the following problem:

$$\text{(4.35)}\qquad \mathcal{A}_H(t)[\vec{v}]=\vec{f}_H-\mathcal{A}_H(t)[\vec{U}],\quad \mathcal{A}_P(t)[\vec{v}]=\vec{f}_P-\mathcal{A}_P(t)[\vec{U}] \quad \text{in } [0, T]\times\Omega,$$

$$\mathcal{B}_H(t)[\vec{v}]=\vec{g}_H-\mathcal{B}_H(t)[\vec{U}],\quad \mathcal{B}_P(t)[\vec{v}]=\vec{g}_P-\mathcal{B}_P(t)[\vec{U}] \quad \text{on } [0, T]\times\Gamma,$$

$$\vec{v}_H(0, x)=\partial_t\vec{v}_H(0, x)=0,\quad \vec{v}_P(0, x)=0 \qquad \text{in } \Omega.$$

By the definitions of $\vec{u}_{H2}$, $\vec{u}_{P1}$, $\vec{U}$ and (4.34), $\vec{f}_E-\mathcal{A}_E(\cdot)[\vec{U}]\in C^0([0, T]; L^2(\Omega))$, $\partial_t(\vec{f}_E-\mathcal{A}_E(\cdot)[\vec{U}])\in L^2([0, T]; L^2(\Omega))$, $\vec{g}_E-\mathcal{B}_E(\cdot)[\vec{U}]\in C^0([0, T]; H^{1/2}(\Gamma))$ and $\partial_t(\vec{g}_E-\mathcal{B}_E(\cdot)[\vec{U}])\in L^2([0, T]; L^2(\Gamma))$ $(E=H, P)$ and (4.33) is satisfied. Therefore the existence of the solution $\vec{v}$ is guaranteed by Lemma 4.12. If we put $\vec{u}=\vec{v}+\vec{U}$, then $\vec{u}$ is in $E^2([0, T]; \Omega)$ and a solution to (N) with initial data $\vec{u}_{H0}$, $\vec{u}_{H1}$, $\vec{u}_{P0}$ and right members $\vec{f}_E$ and $\vec{g}_E$ $(E=H, P)$. Employing the same argument as in Theorem 4.5 of [6], and using Theorem 3.1 and Lemma 4.9, we can remove the additional assumption (*), which completes the proof of the theorem. ∎

THEOREM 4.14. *Let L be an integer $L\geqq 3$. Assume that* (A.1)-(A.4′) *are valid. If* $\vec{u}_{H0}\in H^L(\Omega)$, $\vec{u}_{H1}\in H^{L-1}(\Omega)$, $\vec{u}_{P0}\in H^L(\Omega)$, $\vec{f}_E\in R^L([0, T]; \Omega)$ *and* $\vec{g}_E\in R^L([0, T]; \Gamma)$, *and if* (2.3) *and* (2.4) *are satisfied, then the problem* (N) *admits a unique solution* $\vec{u}=(\vec{u}_H, \vec{u}_P)\in E^L([0, T]; \Omega)$.

PROOF. We can prove this theorem in the same manner as in Ikawa [[2], p. 364-367] or [[1], p. 604-607], so that we may omit the proof. ∎

§5. *A priori* estimate in half-space.

From now to §6, we assume that $n\geqq 2$. Our purpose in this section is to derive some *a priori* estimate for the following problem:

(5.1)
$$\mathcal{P}_H[\vec{u}]=\vec{f}_H,\quad \mathcal{P}_P[\vec{u}]=\vec{f}_P \quad \text{in } \boldsymbol{R}\times\boldsymbol{R}^n_+,$$
$$Q_H[\vec{u}]=\vec{g}_H,\quad Q_P[\vec{u}]=\vec{g}_P \quad \text{on } \boldsymbol{R}\times\boldsymbol{R}^n_0,$$

where $\boldsymbol{R}^n_+=\{x\in\boldsymbol{R}^n\,|\,x_n>0\}$; $\boldsymbol{R}^n_0=\{x\in\boldsymbol{R}^n\,|\,x_n=0\}$;

$$\mathcal{P}_H[\vec{u}]=-D_t^2\vec{u}_H+D_j(P_H^{jk}(t, x)D_k\vec{u}_H)+P_H^{j0}(t, x)D_jD_t\vec{u}_H-iP_{HP}^j(t, x)D_j\vec{u}_P;$$

$$\mathcal{P}_P[\vec{u}]=iP_P^0(t, x)D_t\vec{u}_P+D_j(P_P^{jk}(t, x)D_k\vec{u}_P)-iP_P^j(t, x)D_j\vec{u}_P$$
$$+P_{PH}^{jk}(t, x)D_jD_k\vec{u}_H+P_{PH}^{j0}(t, x)D_jD_t\vec{u}_H;$$

$$Q_H[\vec{u}]=-iP_H^{nk}(t, x)D_k\vec{u}_H+Q_{HP}(t, x')\vec{u}_P+iQ_H^0(t, x')D_t\vec{u}_H;$$

$$Q_P[\vec{u}]=-iP_P^{nk}(t, x)D_k\vec{u}_P+iQ_{PH}^0(t, x')D_t\vec{u}_H+iQ_{PH}^j(t, x')D_j\vec{u}_H;$$

$$i=\sqrt{-1},\quad D_t=-i\partial/\partial t,\quad D_j=-i\partial/\partial x_j,\quad x'=(x_1, \cdots, x_{n-1}).$$

From now, the functions in general are assumed to be complex-valued and $D^1u=(D_tu, D_1u, \cdots, D_nu)$. Let γ be an any real number $\geqq 1$, and q and r be integers $\in[1, n-1]$. For any integer $L\geqq 0$, $s\in\boldsymbol{R}$, scalar functions, $\vec{u}$, $\vec{v}$ and vector valued functions $\vec{u}$, $\vec{v}$, put

$$\mathcal{H}^L_\gamma=\{\vec{u}={}^t(u_1, \cdots, \vec{u}_{m_H})\,|\,u_i\in H^L_{loc}(\boldsymbol{R}\times\boldsymbol{R}^n_+),\ u_i(t, \cdot)\in H^L(\boldsymbol{R}^n_+) \text{ for all } t\in\boldsymbol{R},$$
$$|\vec{u}|_{L,\gamma}=\sum_{k+|\alpha|\leq L}\int_{\boldsymbol{R}\times\boldsymbol{R}^n_+}e^{-2\gamma t}|\partial_t^k\partial_x^\alpha\vec{u}(t, x)|^2dtdx<\infty\};$$

$$\mathcal{H}^{1,1}_\gamma=\{\vec{u}={}^t(u_1, \cdots, u_{m_P})\,|\,u_i\in C^0(\boldsymbol{R}; H^2(\boldsymbol{R}^n_+)),\ \partial_tu_i\in L^2_{loc}(\boldsymbol{R}\times\boldsymbol{R}^n_+),$$
$$|\partial^2\vec{u}|^2_{0,\gamma}+|\partial_t\vec{u}|^2_{0,\gamma}<\infty\};$$

$$\langle D'\rangle^s u(x')=(2\pi)^{-(n-1)}\int_{R^{n-1}} e^{ix'\xi'}(1+|\xi'|^2)^{s/2}\hat{u}(\xi')d\xi' ,$$

where $\xi'=(\xi_1, \cdots, \xi_{n-1})$ and $\hat{u}$ denotes the Fourier transform of $u(x')$. Put

$$\langle D'\rangle^s\vec{u}(x')={}^t(\langle D'\rangle^s u_1(x'), \cdots, \langle D'\rangle^s u_m(x'));$$

$$\langle u\rangle_{s,\gamma}^2=\int_{R^n} e^{-2\gamma t}|\langle D'\rangle^s u(t, x')|^2 dtdx';$$

$$(u, v)_\gamma=\int_{R\times R_+^n} e^{-2\gamma t}u(t, x)\overline{v(t, x)}dtdx, \quad \langle u, v\rangle_\gamma=\int_{R^n} e^{-2\gamma t}u(t, x')\overline{v(t, x')}dtdx',$$

$$(\vec{u}, \vec{v})_\gamma=\sum_{a=1}^m (u_a, v_a)_\gamma, \qquad \langle\vec{u}, \vec{v}\rangle_\gamma=\sum_{a=1}^m \langle u_a, v_a\rangle_\gamma,$$

$$\|\vec{u}\|^2=\int_{R_+^n}|\vec{u}|^2 dx, \quad `u(x')`=u(x', 0), \quad `\vec{u}`={}^t(`u_1`, \cdots, `u_m`);$$

$$\begin{aligned}B(l)=&\sum_{E=H,P}\sum_{j,k=1}^n |P_E^{jk}|_{\infty,l,R\times R_+^n}+|P_P^0|_{\infty,l,R\times R_+^n}+\sum_{j,k=1}^m |P_{PH}^{jk}|_{\infty,l,R\times R_+^n}\\ &+\sum_{j=1}^n(|P_H^{j0}|_{\infty,l,R\times R_+^n}+|P_{HP}^j|_{\infty,l,R\times R_+^n}+|P_P^j|_{\infty,l,R\times R_+^n}+|P_{PH}^{j0}|_{\infty,l,R\times R_+^n})\\ &+|Q_{HP}|_{\infty,l,R^n}+|Q_H^0|_{\infty,l,R^n}+\sum_{j=0}^n |Q_{PH}^j|_{\infty,l,R^n} \qquad \text{for } l=0, 1;\end{aligned}$$

$$\begin{aligned}B(1+\mu)=&\sum_{E=H,P}\sum_{j,k=1}^n(|P_H^{jk}|_{\infty,1+\mu,R\times R_+^n}+|P_P^{jk}|_{\infty,1,R\times R_+^n}+|P_{PH}^{jk}|_{\infty,1,R\times R_+^n})\\ &+\sum_{j=1}^n(|P_H^{j0}|_{\infty,1+\mu,R\times R_+^n}+|P_{HP}^j|_{\infty,1,R\times R_+^n}+|P_P^j|_{\infty,1,R\times R_+^n}+|P_{PH}^{j0}|_{\infty,1,R\times R_+^n})\\ &+|P_P^0|_{\infty,1,R\times R_+^n}+|Q_{HP}|_{\infty,1,R^n}+|Q_H^0|_{\infty,1,R^n}+\sum_{j=0}^n |Q_{PH}^j|_{\infty,1,R^n}\\ &\text{for } 0<\mu<1 .\end{aligned}$$

Throughout this section, we assume that:

(A.5.1) P_H^{jk} and P_H^{j0} are $m_H\times m_H$ matrices, P_{HP}^j are $m_H\times m_P$ matrices, P_P^0, P_P^{jk} and P_P^j are $m_P\times m_P$ matrices, P_{pH}^{jk} and P_{PH}^{j0} are $m_P\times m_H$ matrices and the elements of these matrices are real valued functions in $\mathcal{B}^\infty(\boldsymbol{R}\times\boldsymbol{R}_+^n)$. Q_{HP} is an $m_H\times m_P$ matrix, B_H^0 is an $m_H\times m_H$ matrix, Q_{PH}^0 and Q_{PH}^j are $m_P\times m_H$ matrices, and the elements of these matrices are real-valued functions in $\mathcal{B}^\infty(\boldsymbol{R}^n)$;

(A.5.2) $\quad {}^tP_E^{jk}=P_E^{kj}\ (E=H, P), \quad {}^tP_H^{j0}=P_H^{j0}, \quad {}^tP_P^0=P_P^0, \quad {}^tQ_H^0=Q_H^0$;

(A.5.3) there exist positive constants d_0, d_1 and d_2 such that

$$P_p(t, x) \geqq d_0 I;$$

$$\int_{R_+^n} P_E^{jk}(t, x) D_j \vec{u}_E(x) \overline{D_k \vec{u}_E(x)} dx \geqq d_1 \|\bar{\partial}^1 \vec{u}_E\|^2 - d_2 \|\vec{u}_E\|^2$$

for any $\vec{u}_E \in H^1(\boldsymbol{R}_+^n)$, $t \in \boldsymbol{R}$, $x \in \bar{\boldsymbol{R}}_+^n$ $(E=H, P)$;

(A.5.4) $\quad Q_H^0(t, x') + \frac{1}{2} P_H^{n0}(t, x', 0) \geqq 0 \quad$ for any $(t, x') \in \boldsymbol{R}^n$;

(A.5.5) $P_{Ea}^{jkb}(t, x) = \delta_{jk}\delta_{ab}$ $(E=H, P)$, $P_H^{n0}(t, x)$ is a positive constant matrix and other matrices vanish for $|t| > T_0$ with some $T_0 > 0$.

At first, we get the following Green's formula.

LEMMA 5.1. *Assume that* (A.5.1)-(A.5.4) *are valid. For any* $\vec{u} = (\vec{u}_H, \vec{u}_P) \in \mathcal{H}_\gamma^2 \times \mathcal{H}_\gamma^{1,1}$, *the following identities are valid;*

(5.2)
$$\begin{aligned} &-i\{(\mathcal{P}_H[\vec{u}], D_n \vec{u}_H)_\gamma - (D_n \vec{u}_H, \mathcal{P}_H[\vec{u}])_\gamma\} \\ &\cong \langle `D_t \vec{u}_H` \rangle_\gamma^2 + \langle `P_H^{nn} D_n \vec{u}_H`, `D_n \vec{u}_H` \rangle_\gamma - \langle `P_H^{qr} D_r \vec{u}_H`, `D_q \vec{u}_H` \rangle_\gamma \\ &\quad - \langle `P_H^{q0} D_q \vec{u}_H`, `D_t \vec{u}_H` \rangle_\gamma \\ &\quad + 2\gamma\{(D_t \vec{u}_H, D_n \vec{u}_H)_\gamma + (D_n \vec{u}_H, D_t \vec{u}_H)_\gamma - (P_H^{q0} D_q \vec{u}_H, D_n \vec{u}_H)_\gamma\}; \end{aligned}$$

(5.3)
$$\begin{aligned} &-i\{(\mathcal{P}_H[\vec{u}], D_t \vec{u}_H)_\gamma - (D_t \vec{u}_H, \mathcal{P}_H[\vec{u}])_\gamma\} \\ &\cong 2\gamma\{|D_t \vec{u}_H|_{0,\gamma}^2 + (P_H^{jk} D_k \vec{u}_H, D_j \vec{u}_H)_\gamma\} + \langle (2Q_H^0 + `P_H^{n0}`) `D_t \vec{u}_H`, `D_t \vec{u}_H` \rangle_\gamma \\ &\quad - i\{\langle `D_t \vec{u}_H`, `Q_H[\vec{u}]` \rangle_\gamma - \langle `Q_H[\vec{u}]`, `D_t \vec{u}_H` \rangle_\gamma \\ &\quad + \langle Q_{HP} `\vec{u}_P`, `D_t \vec{u}_H` \rangle_\gamma - \langle `D_t \vec{u}_H`, Q_{HP} `\vec{u}_P` \rangle_\gamma\}; \end{aligned}$$

(5.4)
$$\begin{aligned} &(\mathcal{P}_P[\vec{u}], \vec{u}_P)_\gamma + (\vec{u}_P, \mathcal{P}_P[\vec{u}])_\gamma \\ &\simeq 2\gamma (P_P^0 \vec{u}_P, \vec{u}_P)_\gamma + 2(P_P^{kj} D_j \vec{u}_P, D_k \vec{u}_P)_\gamma \\ &\quad + \langle `\vec{u}_P`, `Q_P[\vec{u}]` \rangle_\gamma - \langle Q_P[\vec{u}]`, `\vec{u}_P` \rangle_\gamma \\ &\quad + i\{\langle Q_{PH}^0 `D_t \vec{u}_H`, `\vec{u}_P` \rangle_\gamma + \langle `\vec{u}_P`, Q_{PH}^0 `D_t \vec{u}_H` \rangle_\gamma \\ &\quad + \langle Q_{PH}^j `D_j \vec{u}_H`, `\vec{u}_P` \rangle_\gamma + \langle `\vec{u}_P`, Q_{PH}^j `D_j \vec{u}_H` \rangle_\gamma \\ &\quad + \langle `P_{PH}^{nn} D_n \vec{u}_H`, `\vec{u}_P` \rangle_\gamma + \langle `\vec{u}_P`, `P_{PH}^{nn} D_n \vec{u}_H` \rangle_\gamma \\ &\quad + \langle `P_{PH}^{n0} D_t \vec{u}_H`, `\vec{u}_P` \rangle_\gamma + \langle `\vec{u}_P`, `P_{PH}^{n0} D_t \vec{u}_H` \rangle_\gamma\}, \end{aligned}$$

where in (5.2) *and* (5.3), $A \cong B$ *means that*

$$|A - B| \leqq C(\boldsymbol{B}(1))\{|\vec{u}_H|_{1,\gamma}^2 + |\vec{u}_H|_{1,\gamma} |\bar{\partial}^1 \vec{u}_P|_{0,\gamma}\}$$

and in (5.4), $A \simeq B$ *means that*

$$|A-B|\leqq C(\boldsymbol{B}(1))\{|\vec{u}_P|_{0,\gamma}^2+|\partial^1\vec{u}_P|_{0,\gamma}|\vec{u}_P|_{0,\gamma}+|\vec{u}_H|_{1,\gamma}|\partial^1\vec{u}_P|_{0,\gamma}\}.$$

PROOF. By the integration by parts, we get

$$(D_t u, v)_\gamma=-2i\gamma(u, v)_\gamma+(u, D_t v)_\gamma; \tag{5.5a}$$

$$(D_n u, v)_\gamma=i\langle u, v\rangle_\gamma+(u, D_n v)_\gamma; \tag{5.5b}$$

$$(D_q u, v)_\gamma=(u, D_q v)_\gamma. \tag{5.5c}$$

Using (5.5) and (A.5.2), we can obtain (5.2), (5.3) and (5.4). ■

LEMMA 5.2. *Assume that* (A.5.1)-(A.5.4) *are valid. Then, the following estimates are valid.*

(1) *There exists a* $\gamma_0\geqq 1$ *depending only on* d_0, d_1, d_2 *and* $\boldsymbol{B}(1)$ *such that*

$$\begin{aligned}\gamma(|\vec{u}_H|_{1,\gamma}^2+|\vec{u}_P|_{0,\gamma}^2)+|\partial^1\vec{u}_P|_{0,\gamma}^2 &\leqq C\{\gamma^{-1}|\mathcal{P}_H[\vec{u}]|_{0,\gamma}^2+\gamma^{-1}|\mathcal{P}_P[\vec{u}]|_{0,\gamma}^2+\langle `Q_P[\vec{u}]`\rangle_{-1/2,\gamma}^2\\ &+|\langle `Q_H[\vec{u}]`, `D_t\vec{u}_H`\rangle_\gamma|+|\langle Q_{HP}`\vec{u}_P`, `D_t\vec{u}_H`\rangle_\gamma|\\ &+|\langle Q_{PH}^0`D_t\vec{u}_H`, `\vec{u}_P`\rangle_\gamma|+|\langle Q_{PH}^j`D_j\vec{u}_H`, `\vec{u}_P`\rangle_\gamma|\\ &+|\langle `P_{PH}^{nn}D_n\vec{u}_H`, `\vec{u}_P`\rangle_\gamma|+|\langle `P_{PH}^{n0}D_t\vec{u}_H`, `\vec{u}_P`\rangle_\gamma|\}\end{aligned} \tag{5.6}$$

for any $\gamma\geqq\gamma_0$ *and* $\vec{u}\in\mathcal{H}_\gamma^2\times\mathcal{H}_\gamma^{1,1}$, *where* $C=C(\boldsymbol{B}(1))$.

(2) *For any* $\gamma\geqq 1$ *and* $\vec{u}\in\mathcal{H}_\gamma^2\times\mathcal{H}_\gamma^{1,1}$,

$$\begin{aligned}\langle `D_t\vec{u}_H`\rangle_{0,\gamma}^2+\langle `P_H^{nn}D_n\vec{u}_H`, `D_n\vec{u}_H`\rangle_\gamma &\leqq C(\boldsymbol{B}(1))\{\gamma^{-1}|\mathcal{P}_H[\vec{u}]|_{0,\gamma}^2+\gamma^{-1}|\partial^1\vec{u}_P|_{0,\gamma}^2+\gamma|\vec{u}_H|_{1,\gamma}^2\}\\ &+C(\boldsymbol{B}(0))\{\langle `D_q\vec{u}_H`\rangle_{0,\gamma}\langle `D_t\vec{u}_H`\rangle_{0,\gamma}+\langle `D_q\vec{u}_H`\rangle_{0,\gamma}^2\}.\end{aligned} \tag{5.7}$$

PROOF. (1) Combining (5.3) and (5.4) implies that

$$\begin{aligned}&2\min(d_1, 1)\gamma|\vec{u}_H|_{1,\gamma}^2+2\gamma d_0|\vec{u}_P|_{0,\gamma}^2+2d_1|\partial^1\vec{u}_P|_{0,\gamma}^2\\ &\leqq 2|\mathcal{P}_H[\vec{u}]|_{0,\gamma}|D_t\vec{u}_H|_{0,\gamma}+2|\mathcal{P}_P[\vec{u}]|_{0,\gamma}|\vec{u}_P|_{0,\gamma}+2\langle `Q_P[\vec{u}]`\rangle_{-1/2,\gamma}|\partial^1\vec{u}_P|_{0,\gamma}\\ &+C(\boldsymbol{B}(1), d_2)\{|\vec{u}_H|_{1,\gamma}^2+|\vec{u}_P|_{0,\gamma}^2+|\vec{u}_H|_{1,\gamma}|\partial^1\vec{u}_P|_{0,\gamma}+|\vec{u}_P|_{0,\gamma}|\partial^1\vec{u}_P|_{0,\gamma}\}\\ &+2\gamma d_2|\vec{u}_H|_{0,\gamma}^2+2|\langle `Q_H[\vec{u}]`, `D_t\vec{u}_H`\rangle_\gamma|+2|\langle Q_{HP}`\vec{u}_P`, `D_t\vec{u}_H`\rangle_\gamma|\\ &+2|\langle Q_{PH}^0`D_t\vec{u}_H`, `\vec{u}_P`\rangle|+2|\langle Q_{PH}^j`D_j\vec{u}_H`, `\vec{u}_P`\rangle_\gamma|\\ &+2|\langle `P_{PH}^{nn}D_n\vec{u}_H`, `\vec{u}_P`\rangle_\gamma|+2|\langle `P_{PH}^{n0}D_t\vec{u}_H`, `\vec{u}_P`\rangle_\gamma|.\end{aligned} \tag{5.8}$$

Noting that $e^{-2\gamma t}=-(2\gamma)^{-1}(d/dt)e^{-2\gamma t}$, by integration by parts we have

$$|\vec{u}_H|_{0,\gamma}\leqq\gamma^{-1}|D_t\vec{u}_H|_{0,\gamma}. \tag{5.9}$$

Applying (5.9) to (5.8), we have (5.6).

(2) Noting that $|\langle {}^t P_H^{qr} D_r \vec{u}_H{}^t, {}^t D_q \vec{u}_H{}^t \rangle_\gamma| \leqq C(\boldsymbol{B}(0)) \langle {}^t D_q \vec{u}_H{}^t \rangle_{0,\gamma}^2$ and that $|\langle {}^t P_H^{q0} D_q \vec{u}_H{}^t, {}^t D_t \vec{u}_H{}^t \rangle_\gamma| \leqq C(\boldsymbol{B}(0)) \langle {}^t D_q \vec{u}_H{}^t \rangle_{0,\gamma} \langle {}^t D_t \vec{u}_H{}^t \rangle_{0,\gamma}$, we can obtain (5.7) from (5.2). ■

THEOREM 5.3. *Assume that* (A.5.1)-(A.5.4) *are valid. Let* $\gamma_0 \geqq 1$ *be the same constant as in Lemma* 5.2, *and* $\mu \in (0, 1)$. *Then, there exists a constant* $C = C(\mu, \boldsymbol{B}(1+\mu)) > 0$ *such that*

$$\text{(5.10)} \qquad \gamma(|\vec{u}_H|_{1,\gamma}^2 + |\vec{u}_P|_{0,\gamma}^2) + |\partial^1 \vec{u}_P|_{0,\gamma}^2 + \langle {}^t \bar{D}^1 \vec{u}_H{}^t \rangle_{-1/2,\gamma}^2$$
$$\leqq C\{\gamma^{-1} |\mathcal{P}_H[\vec{u}]|_{0,\gamma}^2 + \gamma^{-1} |\mathcal{P}_P[\vec{u}]|_{0,\gamma}^2 + \langle {}^t Q_H[\vec{u}]^t \rangle_{1/2,\gamma}^2 + \langle {}^t Q_P[\vec{u}]^t \rangle_{-1/2,\gamma}^2\},$$

for any $\gamma \geqq \gamma_0$ *and* $\vec{u} \in \mathcal{H}_\gamma^2 \times \mathcal{H}_\gamma^{1,1}$.

PROOF. Regarding x_n as a parameter, we use weighted pseudo- differential operators. Let κ be a small number determined later, and choose $\varphi_0(\sigma, \gamma, \xi')$ and $\varphi_1(\sigma, \gamma, \xi') \in C^\infty(\boldsymbol{R}^{n+1} - \{0, 0, 0\})$ (we will consider the case of $\gamma \geqq \gamma_0 \geqq 1$) so that $0 \leqq \varphi_0, \varphi_1 \leqq 1$, $\varphi_0 + \varphi_1 = 1$ and

$$\text{(5.11)} \qquad supp\, \varphi_0 \subset \{(\sigma, \gamma, \xi') \,|\, 2\kappa^2(\sigma^2+\gamma^2) \geqq |\xi'|^2\};$$
$$supp\, \varphi_1 \subset \{(\sigma, \gamma, \xi') \,|\, \kappa^2(\sigma^2+\gamma^2) \leqq |\xi'|^2\}.$$

Let Φ_0 and Φ_1 be weighted pseudo-differential operators with symbols φ_0 and φ_1 respectively. Namely

$$\Phi_l u = (2\pi)^{-n} e^{\gamma t} \iint e^{i(x'-y')\xi' + i(t-s)\sigma} \varphi_l(\sigma, \gamma, \xi') e^{-\gamma s} u(s, y', x_n) dy' ds d\xi' d\sigma$$
$$= (2\pi)^{-n} \int e^{i\{x'\xi' + s(\sigma - i\gamma)\}} \varphi_l(\sigma, \gamma, \xi') \hat{u}(\sigma - i\gamma, \xi', x_n) d\sigma d\xi' \qquad l=0, 1.$$

Let Λ_γ^s be weighted pseudo-differential operator with symbol $(\sigma^2+\gamma^2+|\xi'|^2)^{s/2}$. Put $\Phi_l \vec{u} = {}^t(\Phi_l \vec{u}_1, \cdots, \Phi_l \vec{u}_m)$, $l=0, 1$. We shall estimate $\langle {}^t \bar{D}^1 \vec{u}_H{}^t \rangle_{-1/2,\gamma}$. At first, we shall consider $\langle D^1 \Phi_0 \vec{u}_H \rangle_{0,\gamma}$. Since P_H^{nn} is non-singular as follows from (A.5.3), we have

$$\text{(5.12)} \qquad P_H[\Phi_0 \vec{u}] = P_H^{nn} \Phi_0 (P_H^{nn})^{-1} \mathcal{P}_H[\vec{u}] + F_0 \qquad \text{in } \boldsymbol{R} \times \boldsymbol{R}_+^n,$$

where

$$F_0 = P_H^{nn}[\Phi_0, (P_H^{nn})^{-1}] D_t^2 \vec{u}_H - P_H^{nn}[\Phi_0, (P_H^{nn})^{-1}(P_H^{nq} + P_H^{qn})] D_n D_q \vec{u}_H$$
$$- P_H^{nn}[\Phi_0, (P_H^{nn})^{-1} P_H^{qr}] D_q D_r \vec{u}_H - P_H^{nn}[\Phi_0, (P_H^{nn})^{-1} P_H^{j0}] D_j \vec{u}_H$$
$$- P_H^{nn}[\Phi_0, (P_H^{nn})^{-1} P_H^{j0}] D_j D_t \vec{u}_H + i P_H^{nn}[\Phi_0, (P_H^{nn})^{-1} P_{HP}^j] D_j \vec{u}_P,$$

and $[A, B]$ means a commutator of A and B, i.e. $[A, B]\vec{u} = A(B\vec{u}) - B(A\vec{u})$.

For $l=0, 1, \cdots, n-1$,

$$|[\Phi_0, B]D_l\vec{u}|_{0,\gamma}\leqq|\Phi_0(D_lB)\vec{u}|_{0,\gamma}+|[\Phi_0D_l, B]\vec{u}|_{0,\gamma},$$

where $D_0=D_t$ and $B\in\mathscr{B}^\infty(\boldsymbol{B}\times\bar{\boldsymbol{R}}^n_+)$. Since the order of Φ_0 is zero, by Theorem Ap.5 of [6] we have

$$|[\Phi_0D_l, B]\vec{u}|_{0,\gamma}\leqq C(\mu)|B|_{\infty,1+\mu,\boldsymbol{R}\times\boldsymbol{R}^n_+}|\vec{u}|_{0,\gamma}. \tag{5.13}$$

Therefore,

$$|F_0|_{0,\gamma}\leqq C(\mu, \boldsymbol{B}(1+\mu))\{|\vec{u}_H|_{1,\gamma}+|\bar{\partial}^1\vec{u}_P|_{0,\gamma}\}. \tag{5.14}$$

Applying (5.7) to (5.12) and noting that there exists a $d>0$ such that $P^{nn}_H\vec{v}\cdot\vec{v}\geqq d|\vec{v}|^2$ for any $\vec{v}\in\boldsymbol{R}^{m_H}$, we have

$$\begin{aligned}\langle' D_t\Phi_0\vec{u}_H'\rangle^2_{0,\gamma}+d\langle' D_n\Phi_0\vec{u}_H'\rangle^2_{0,\gamma}\\ \leqq C(\mu, \boldsymbol{B}(1+\mu))\{\gamma^{-1}|\mathscr{P}_H[\vec{u}]|^2_{0,\gamma}+\gamma^{-1}|\bar{\partial}^1\vec{u}_P|^2_{0,\gamma}+\gamma|\vec{u}_H|^2_{1,\gamma}\}\\ +C(\boldsymbol{B}(0))\{\langle' D_q\Phi_0\vec{u}_H'\rangle_{0,\gamma}\langle' D_t\Phi_0\vec{u}_H'\rangle_{0,\gamma}+\langle' D_q\Phi_0\vec{u}_H'\rangle^2_{0,\gamma}\}.\end{aligned} \tag{5.15}$$

From (5.11) it follows that

$$\langle D_q\Phi_0\vec{u}_H\rangle^2_{0,\gamma}\leqq C\kappa^2\langle D_t\Phi_0\vec{u}_H\rangle^2_{0,\gamma} \qquad \text{for } q=1, \cdots, n-1. \tag{5.16}$$

Substituting (5.16) into (5.15) and choosing $\kappa>0$ so small, we have

$$\langle D^1\Phi_0\vec{u}_H\rangle^2_{0,\gamma}\leqq C(\mu, \boldsymbol{B}(1+\mu))\{\gamma^{-1}|\mathscr{P}_H[\vec{u}]|^2_{0,\gamma}+\gamma^{-1}|\bar{\partial}^1\vec{u}_P|^2_{0,\gamma}+\gamma|\vec{u}_H|^2_{1,\gamma}\}. \tag{5.17}$$

Next, we shall estimate $\langle D^1\Phi_1\vec{u}_H\rangle_{-1/2,\gamma}$. Considering (5.11), we have

$$\begin{aligned}\langle D_t\Phi_1\vec{u}_H\rangle^2_{-1/2,\gamma}\\ =\left(\frac{1}{2\pi}\right)^n\int(1+|\xi'|^2)^{-1/2}(\sigma^2+\gamma^2)|\varphi_1(\sigma,\gamma,\xi')|^2|\hat{\vec{u}}_H(\sigma-i\gamma,\xi',0)|^2d\sigma d\xi'\\ \leqq C(\kappa)(2\pi)^{-n}\int(1+|\xi'|^2)^{1/2}|\hat{\vec{u}}_H(\sigma-i\gamma, \xi', 0)|^2d\sigma d\xi'\\ =C(\kappa)\langle\vec{u}_H\rangle^2_{1/2,\gamma}\leqq C(\kappa)|\vec{u}_H|^2_{1,\gamma}.\end{aligned} \tag{5.18}$$

And

$$\langle D_q\Phi_1\vec{u}_H\rangle_{-1/2,\gamma}\leqq\langle\Phi_1\vec{u}_H\rangle_{1/2,\gamma}\leqq C|\vec{u}_H|_{1,\gamma}. \tag{5.19}$$

From the fact that $(|\xi'|^2+1)^{-1/2}\leqq(1+1/\kappa^2)^{1/2}(\sigma^2+\gamma^2+|\xi'|^2)^{-1/2}$ on $supp\,\varphi_1$, it follows that

$$\begin{aligned}\langle' D_n\Phi_1\vec{u}_H'\rangle^2_{-1/2,\gamma}\leqq C(\kappa)\langle' \Lambda_\gamma^{-1/2}D_n\Phi_1\vec{u}_H'\rangle^2_{0,\gamma}\\ =-C(\kappa)\int_0^\infty\partial_n\langle\Lambda_\gamma^{-1/2}D_n\Phi_1\vec{u}_H(\cdot, x_n)\rangle^2_{0,\gamma}dx_n\end{aligned} \tag{5.20}$$

$$\leqq 2C(\kappa)\left|\int_0^\infty \langle \Lambda_\gamma^{-1/2} D_n \Phi_1 \vec{u}_H(\cdot, x_n), \Lambda_\gamma^{-1/2} D_n^2 \Phi_1 \vec{u}_H(\cdot, x_n)\rangle_{0,\gamma}^2 dx_n\right|$$

$$\leqq C(\kappa)|D_n \vec{u}_H|_{0,\gamma} |\Lambda_\gamma^{-1} D_n^2 \vec{u}_H|_{0,\gamma}.$$

We have

$$(5.21)\qquad |\Lambda_\gamma^{-1} D_n^2 \vec{u}_H|_{0,\gamma} \leqq C\{\gamma^{-1}|\mathcal{L}_H[\vec{u}]|_{0,\gamma} + |\vec{u}_H|_{1,\gamma} + \gamma^{-1}|\bar{\partial}^1 \vec{u}_P|_{0,\gamma}\},$$

where $C=C(\boldsymbol{B}(1))$.

Since

$$D_n^2 \vec{u}_H = (P_H^{nn})^{-1}\{\mathcal{L}_H[\vec{u}] + D_t^2 \vec{u}_H - P_H^{qr} D_q D_r \vec{u}_H - (P_H^{nq} + P_H^{qn}) D_q D_n \vec{u}_H$$

$$-(D_j P_H^{jk}) D_k \vec{u}_H - P_H^{j0} D_j D_t \vec{u}_H + i P_{HP}^j D_j \vec{u}_P\};$$

$$|\Lambda_\gamma^{-1} A D_l u|_{0,\gamma} \leqq |\Lambda_\gamma^{-1} D_l(Au)|_{0,\gamma} + |\Lambda_\gamma^{-1}(D_l A)u|_{0,\gamma}$$

$$\leqq C|A|_{\infty,1,\boldsymbol{R}\times\boldsymbol{R}_+^n}|u|_{0,\gamma} \qquad \text{for any } A \in \mathcal{B}^\infty(\boldsymbol{R}\times\boldsymbol{R}_+^n);$$

$$|A_\gamma u|_{0,\gamma} \leqq C\gamma^{-1}|u|_{0,\gamma} \qquad \text{for any weighted p.d.o. } A_\gamma \text{ of order } -1,$$

we can get (5.21) immediately. Applying (5.21) to (5.20), we have

$$(5.22)\qquad \langle ' D_n \Phi_1 \vec{u}_H ' \rangle_{-1/2,\gamma}^2 \leqq C(\kappa, \boldsymbol{B}(1))\{|\vec{u}_H|_{1,\gamma}^2 + \gamma^{-2}|\mathcal{L}_H[\vec{u}]|_{0,\gamma}^2 + \gamma^{-2}|\bar{\partial}^1 \vec{u}_P|_{0,\gamma}^2\}.$$

From (5.18), (5.19) and (5.22) it follows that

$$(5.23)\qquad \langle ' D^1 \Phi_1 \vec{u}_H ' \rangle_{-1/2,\gamma}^2 \leqq C(\kappa, \boldsymbol{B}(1))\{|\vec{u}_H|_{1,\gamma}^2 + \gamma^{-2}|\mathcal{L}_H[\vec{u}]|_{0,\gamma}^2 + \gamma^{-2}|\bar{\partial}^1 \vec{u}_P|_{0,\gamma}^2\}.$$

By (5.17) and (5.23) we have

$$(5.24)\qquad \langle ' \bar{D}^1 \vec{u}_H ' \rangle_{-1/2,\gamma} \leqq C\{\gamma|\vec{u}_H|_{1,\gamma}^2 + \gamma^{-1}|\mathcal{L}_H[\vec{u}]|_{0,\gamma}^2 + \gamma^{-1}|\bar{\partial}^1 \vec{u}_P|_{0,\gamma}^2\},$$

which together with (5.6) implies (5.10). ■

By using Theorem 5.3, we shall get the energy estimate of the same type as (E.1). Note that we can rewrite the operators as follows:

$$\mathcal{L}_H[\vec{u}] = \partial_t^2 \vec{u}_H - \partial_j(P_H^{jk} \partial_k \vec{u}_H) - P_H^{j0} \partial_j \partial_t \vec{u}_H - P_{HP}^j \partial_j \vec{u}_P;$$

$$\mathcal{L}_P[\vec{u}] = P_P^0 \partial_t \vec{u}_P - \partial_j(P_P^{jk} \partial_k \vec{u}_P) - P_P^j \partial_j \vec{u}_P - P_{PH}^{jk} \partial_j \partial_k \vec{u}_H - P_{PH}^{j0} \partial_j \partial_t \vec{u}_H;$$

$$Q_H[\vec{u}] = -P_H^{nk} \partial_k \vec{u}_H + Q_{HP} \vec{u}_P + Q_H^0 \vec{u}_H;$$

$$Q_P[\vec{u}] = -P_P^{nk} \partial_k \vec{u}_P + Q_{PH}^0 \partial_t \vec{u}_H + Q_{PH}^j \partial_j \vec{u}_H.$$

Let $\varepsilon \in [0, 1]$ and put

$$\mathcal{L}_E^\varepsilon(t)[\vec{u}] = \begin{cases} \mathcal{L}_H(t)[\vec{u}] - 2\varepsilon \partial_n \partial_t \vec{u}_H & \text{if } E=H, \\ \mathcal{L}_P(t)[\vec{u}] & \text{if } E=P, \end{cases}$$

$$P_H^{n0\varepsilon}(t) = P_H^{n0}(t) + 2\varepsilon I_{m_H}, \qquad P_H^{q0\varepsilon}(t) = P_H^{q0}(t), \quad q=1, \cdots, n-1.$$

Hereafter, we assume that all functions are real-valued, and we use the notation $D^1u=(\partial_t u, \partial_1 u, \cdots, \partial_n u)$ again.

LEMMA 5.4. *Let* $T>0$. *Assume that* (A.5.1)-(A.5.5) *are valid and that* $0<\varepsilon\leqq 1$. *Then, for any* $\vec{u}\in C^\infty([0, T]; H^2(\boldsymbol{R}^n_+))$ *such that* $\partial_t^k\vec{u}(0, x)=0$ *in* $\boldsymbol{R}^n_+$ *for any* $k\geqq 0$, *there exists a constant* $C=C(\boldsymbol{B}(1+\mu))>0$ *such that*

$$\text{(5.25)}\quad \int_0^t(\|\bar{D}^1\vec{u}_H(s)\|^2+\|\vec{u}_P(s)\|_1^2)ds+\int_0^t\langle\bar{D}^1\vec{u}_H(s)\rangle_{-1/2}^2 ds$$
$$\leqq Ce^{Ct}\int_0^t\{\|\mathcal{P}_H^\varepsilon(s)[\vec{u}(s)]\|^2+\|\mathcal{P}_P(s)[\vec{u}(s)]\|^2$$
$$+\langle\!\langle Q_H(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2+\langle\!\langle Q_P(s)[\vec{u}(s)]\rangle\!\rangle_{-1/2}^2\}\,ds \quad \textit{for any } 0\leqq t\leqq T,$$

where C *is a constant independent of* ε.

PROOF. Let t_0 be any time in $[0, T]$ and fixed. Put

$$\vec{f}_E(s, x)=\begin{cases}\mathcal{P}_E^\varepsilon(s)[\vec{u}(s, x)] & \text{for } 0\leqq s\leqq t_0,\\ 0 & \text{for } s<0,\end{cases}$$

$$\vec{g}_E(s, x)=\begin{cases}Q_E(s)[\vec{u}(s, x)] & \text{for } 0\leqq s\leqq t_0,\\ 0 & \text{for } s<0\ (E=H, P).\end{cases}$$

We known that $\vec{f}_E\in C^1((-\infty, t_0]; L^2(\boldsymbol{R}^n_+))$ and $\vec{g}_E\in C^1((-\infty, t_0]; H^{1/2}(\boldsymbol{R}^n_0))$ $(E=H, P)$. Choose a_0, a_1, b_0 and b_1 such that $b_0(-a_0)^k+b_1(-a_1)^k=1$ for $k=0, 1$. (i.e. $a_0=1$, $a_1=2$, $b_0=3$, $b_1=-2$). If we put

$$\text{(5.26a)}\quad F_E(t, x)=\begin{cases}\vec{f}_E(t, x) & \text{for } t\leqq t_0,\\ \sum_{l=0}^1 b_l\vec{f}_E(t_0-a_l(t-t_0), x) & \text{for } t>t_0,\end{cases}$$

$$\text{(5.26b)}\quad G_E(t, x)=\begin{cases}\vec{g}_E(t, x) & \text{for } t\leqq t_0,\\ \sum_{l=0}^1 b_l\vec{g}_E(t_0-a_l(t-t_0), x) & \text{for } t\leqq t_0\ (E=H, P),\end{cases}$$

we know that

(5.27a) $\quad F_E\in C^1(\boldsymbol{R}; L^2(\boldsymbol{R}^1_+)),\quad G_E\in C^1(\boldsymbol{R}; H^{1/2}(\boldsymbol{R}^n_0))\quad (E=H, P);$

(5.27b) $\quad F_E(t, x)=\mathcal{P}_E^\varepsilon(t)[\vec{u}(t, x)],\quad G_E(t, x)=Q_E(t)[\vec{u}(t, x)]\quad$ for $0\leqq t\leqq t_0$;

(5.27c) $\quad F_E(t, x)=0,\quad G_E(t, x)=0\quad (E=H, P)\quad$ for $t\leqq 0$, or $t\geqq 2t_0$.

Let $\vec{v}=(\vec{v}_H, \vec{v}_P)\in X^{2,0}([0, \infty); \boldsymbol{R}^n_+)\times Z^{1,1}([0, \infty); \boldsymbol{R}^n_+)$ be a solution to the following problem:

(5.28a) $\quad \mathcal{P}_H^\varepsilon(t)[\vec{v}]=F_H,\quad \mathcal{P}_P(t)[\vec{v}]=F_H\quad$ in $[0, \infty)\times\boldsymbol{R}^n_+$,

(5.28b) $\quad Q_H(t)[\vec{v}]=G_H,\quad Q_P(t)[\vec{v}]=G_P\quad$ on $[0, \infty)\times\boldsymbol{R}^n_0$, (5.28c) $$\ddot{v}_H(0, x)=\partial_t\ddot{v}_H(0, x)=0, \quad \ddot{v}_P(0, x)=0 \quad \text{in } \boldsymbol{R}^n_+.$$

In view of (5.27c), (5.28) satisfies the compatibility condition of order 0. If $\varepsilon>0$, then the assumptions of § 4 ((A.1)-(A.4′)) are satisfied. Therefore, the existence of $\ddot{v}$ is guaranteed by Theorem 4.13. Put $\ddot{v}^0(t, x)=\ddot{v}(t, x)$ for $t\geqq 0$ and $=0$ $t<0$. Since we know that $\partial_t^2\ddot{v}_H(0, x)=0$, $\partial_t\ddot{v}_P(0, x)=0$ from (5.27c) and (5.28c), $\ddot{v}^0_H\in X^{2,0}(\boldsymbol{R}; \boldsymbol{R}^n_+)$ and $\ddot{v}^0_P\in Z^{1,1}(\boldsymbol{R}; \boldsymbol{R}^n_+)$. Put $T_1=\max(T_0, 2t_0)$. By (A.5.5) we have

(5.29a) $$\partial_t^2\ddot{v}^0_H-\sum_{j=1}^n\partial_j^2\ddot{v}^0_H-2\varepsilon\partial_n\partial_t\ddot{v}^0_H=0 \quad \text{in } [T_1, \infty)\times\boldsymbol{R}^n_+,$$

$$-\partial_n\ddot{v}^0_H=0 \quad \text{on } [T_1, \infty)\times\boldsymbol{R}^n_0,$$

(5.29b) $$\partial_t\ddot{v}^0_P-\sum_{j=1}^n\partial_j^2\ddot{v}^0_P=0 \quad \text{in } [T_1, \infty)\times\boldsymbol{R}^n_+,$$

$$-\partial_n\ddot{v}^0_P=0 \quad \text{on } [T_1, \infty)\times\boldsymbol{R}^n_0.$$

Multiply (5.29a) and $\ddot{v}^0_H$ by $\partial_t\ddot{v}^0_H$, and (5.29b) by $\ddot{v}^0_P$. Integrating the resulting formula, by Gronwall's inequality we have

(5.30a) $$\|\bar{D}^1\ddot{v}^0_H(t)\|^2\leqq\exp(t-T_1)\|\bar{D}^1\ddot{v}^0_H(T_1)\|^2 \quad \text{for } t>T_1;$$

(5.30b) $$\|\ddot{v}^0_P(t)\|^2\leqq\exp(t-T_1)\|\ddot{v}^0_P(T_1)\|^2 \quad \text{for } t>T_1.$$

Differentiating (5.29) with respect to t, we have

(5.31a) $$\|D^1\partial_t\ddot{v}^0_H(t)\|^2\leqq\exp(t-T_1)\|D^1\partial_t\ddot{v}^0_H(T_1)\|^2 \quad \text{for } t>T_1;$$

(5.31b) $$\|\partial_t\ddot{v}^0_P(t)\|^2\leqq\exp(t-T_1)\|\partial_t\ddot{v}^0_P(T_1)\|^2 \quad \text{for } t>T_1.$$

From the properties of Laplacian, we have

(5.32) $$\|\ddot{v}^0_H(t)\|_2\leqq C\{\|\partial_t^2\ddot{v}^0_H(t)\|+\|\partial_t\ddot{v}^0_H(t)\|_1+\|\ddot{v}^0_H(t)\|\},$$

$$\|\ddot{v}^0_P(t)\|_2\leqq C\{\|\partial_t\ddot{v}^0_P(t)\|+\|\ddot{v}^0_P(t)\|\} \quad \text{for } t>T_1.$$

Combining (5.30)-(5.32), we have

(5.33) $$\|\bar{D}^2\ddot{v}^0_H(t)\|^2+\|\partial_t\ddot{v}^0_P(t)\|^2+\|\ddot{v}^0_P(t)\|_2$$

$$\leqq C(\exp(t-T_1))\{\|\bar{D}^2\ddot{v}^0_H(T_1)\|^2+\|\partial_t\ddot{v}^0_P(T_1)\|^2+\|\ddot{v}^0_P(T_1)\|^2\}$$

for any $t>T_1$. Therefore, for any $\gamma>1$, we see that $\ddot{v}^0\in\mathcal{H}^2_\gamma(\boldsymbol{R}\times\boldsymbol{R}^n_+)\times\mathcal{H}^{1,1}_\gamma(\boldsymbol{R}\times\boldsymbol{R}^n_+)$. From (5.27c) and (5.28a), it follows that $\mathcal{P}^\varepsilon_E[\ddot{v}^0]=F_E$ in $\boldsymbol{R}\times\boldsymbol{R}^n_+$, $Q_E[\ddot{v}^0]=G_E$ on $\boldsymbol{R}\times\boldsymbol{R}^n_+$ $(E=H, P)$. By Theorem 5.3, we obtain

(5.34) $$\gamma(|\ddot{v}^0_H|^2_{1,\gamma}+|\ddot{v}^0_P|^2_{0,\gamma})+|\bar{\partial}^1\ddot{v}^0_P|^2_{0,\gamma}+\langle\text{‘}\bar{D}^1\ddot{v}^0_H\text{’}\rangle^2_{-1/2,\gamma}$$

$$\leqq\{\gamma^{-1}|F_H|^2_{0,\gamma}+\gamma^{-1}|F_P|^2_{0,\gamma}+\langle G_H\rangle^2_{1/2,\gamma}+\langle G_P\rangle^2_{-1/2,\gamma}\},$$

for any $\gamma\leqq\gamma_0$. By the definition of F_H, F_P, G_H, G_P, we have

$$|F_H|_{0,\gamma}^2 \leqq C\int_0^{t_0} \|\mathcal{L}_H^\varepsilon(s)[\vec{u}(s)]\|^2 ds\,;\ |F_P|_{0,\gamma}^2 \leqq C\int_0^{t_0} \|\mathcal{L}_P(s)[\vec{u}(s)]\|^2 ds\,;$$

$$\langle G_H \rangle_{1/2,\gamma}^2 \leqq C\int_0^{t_0} \langle\!\langle Q_H(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2 ds\,;$$

$$\langle G_P \rangle_{-1/2,\gamma}^2 \leqq C\int_0^{t_0} \langle\!\langle Q_P(s)[\vec{u}(s)]\rangle\!\rangle_{-1/2}^2 ds\,.$$

By (5.27b) and the uniqueness of solutions, we see that $\vec{v}_E^0(t, x)=\vec{u}_E(t, x)$ for $0\leqq t\leqq t_0$ $(E=H, P)$. Therefore, the lemma follows from (5.34). ■

To get the same estimate as in Lemma 5.4 for any $\vec{u}=(\vec{u}_H, \vec{u}_P)\in X^{2,0}([0, T]\,;\boldsymbol{R}_+^n)\times Z^{1,1}([0, T]\,;\boldsymbol{R}_+^n)$ such that $\vec{u}_H(0, x)=\partial_t\vec{u}_H(0, x)=0$, $\vec{u}_P(0, x)=0$ in $\boldsymbol{R}_+^n$, we need the approximation of $\vec{u}$:

LEMMA 5.5. *Let $\vec{u}=(\vec{u}_H, \vec{u}_P)\in X^{2,0}([0, T]\,;\boldsymbol{R}_+^n)\times Z^{1,1}([0, T]\,;\boldsymbol{R}_+^n)$ be a pair such that $\vec{u}_H(0, x)=\partial_t\vec{u}_H(0, x)=0$, $\vec{u}_P(0, x)=0$ in $\boldsymbol{R}_+^n$. Then, there exist $\vec{u}^k=(\vec{u}_H^k, \vec{u}_P^k)\in C^\infty([0, T]\,;H^2(\boldsymbol{R}_+^n))$ $k=1, 2, \cdots$ satisfying the following properties;*

(a) $$\partial_t^l\vec{u}^k(0, x)=0 \qquad \textit{for any } k\geqq 1 \textit{ and } l\geqq 0\,;$$

(b) $$\|\bar{D}^1(\vec{u}_H^k(t)-\vec{u}_H(t))\| \longrightarrow 0 \qquad \textit{as } k\to\infty,\ \textit{for any } t\in[0, T]\,;$$

$$\|\bar{\partial}^1(\vec{u}_P^k(t)-\vec{u}_P(t))\| \longrightarrow 0 \qquad \textit{as } k\to\infty,\ \textit{for any } t\in[0, T]\,;$$

(c) $$\int_0^T \|\bar{D}^2(\vec{u}_H^k(t)-\vec{u}_H(t))\|^2 dt \longrightarrow 0 \qquad \textit{as } k\to\infty\,;$$

$$\int_0^T (\|\partial_t(\vec{u}_P^k(t)-\vec{u}_P(t))\|^2+\|\bar{\partial}^2(\vec{u}_P^k(t)-\vec{u}_P(t))\|^2)dt \longrightarrow 0 \qquad \textit{as } k\to\infty.$$

PROOF. In the same manner as in Lemma 6.7 of [6], we can construct $\vec{u}^k$, so that we may omit the proof. ■

THEOREM 5.6. *Assume that* (A.5.1)-(A.5.5) *are valid and that* $0\leqq\varepsilon\leqq 1$. *Then for any $\vec{u}=(\vec{u}_H, \vec{u}_P)\in X^{2,0}([0, T]\,;\boldsymbol{R}_+^n)\times Z^{1,1}([0, T]\,;\boldsymbol{R}_+^n)$ such that $\vec{u}_H(0, x)=\partial_t\vec{u}_H(0, x)=0$, $\vec{u}_P(0, x)=0$ in $\boldsymbol{R}_+^n$ and for any $T>0$, there exists a $C=C(\boldsymbol{B}(1+\mu))>0$ such that*

(5.35) $$|||\vec{u}(t)|||_1^2+\int_0^t \|\bar{D}^1\vec{u}_H(s)\|^2 ds$$
$$\leqq Ce^{Ct}\int_0^t \{\|\mathcal{L}_H^\varepsilon(s)[\vec{u}(s)]\|^2+\|\mathcal{L}_P(s)[\vec{u}(s)]\|^2$$
$$+\langle\!\langle Q_H(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2+\langle\!\langle Q_P(s)[\vec{u}(s)]\rangle\!\rangle_{-1/2}^2\}\,ds$$

for $0\leqq t\leqq T$.

PROOF. For $\varepsilon\in(0, 1]$, by Lemmas 5.4 and 5.5, we have

$$\int_0^t \{\|\bar{D}^1\vec{u}_H(s)\|^2+\|\vec{u}_P(s)\|_1^2+\langle\!\langle\bar{D}^1\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2\}\,ds \tag{5.36}$$

$$\leqq Ce^{Ct}\int_0^t \{\|\mathcal{P}_H^\varepsilon[\vec{u}(s)]\|^2+\|\mathcal{P}_P[\vec{u}(s)]\|^2$$

$$+\langle\!\langle Q_H[\vec{u}(s)]\rangle\!\rangle_{1/2}^2+\langle\!\langle Q_P[\vec{u}(s)]\rangle\!\rangle_{-1/2}^2\}\,ds \qquad 0\leqq t\leqq T$$

for any pair $\vec{u}=(\vec{u}_H, \vec{u}_P)\in X^{2,0}([0, T]; \boldsymbol{R}_+^n)\times Z^{1,1}([0, T]; \boldsymbol{R}_+^n)$ satisfying $\vec{u}_H(0, x)=\partial_t\vec{u}_H(0, x)=0$, $\vec{u}_P(0, x)=0$ in $\boldsymbol{R}_+^n$. Since the constant C in (5.36) is independent of ε, letting $\varepsilon\downarrow 0$, we have that (5.36) is also valid for $\varepsilon=0$. Calculating $(\mathcal{P}_H^\varepsilon[\vec{u}(t)]+d_2\vec{u}_H(t), \partial_t\vec{u}_H(t))$ and $(\mathcal{P}_P[\vec{u}(t)], \vec{u}_P(t))$ and combining the resulting formulas, we have

$$\|\partial_t\vec{u}_H(t)\|^2+\|\vec{u}_H(t)\|_1^2+\|\vec{u}_P(t)\|^2 \tag{5.37}$$

$$\leqq C\int_0^t \{\|\mathcal{P}_H^\varepsilon[\vec{u}(s)]\|^2+\|\mathcal{P}_P[\vec{u}(s)]\|^2+\langle\!\langle Q_H[\vec{u}(s)]\rangle\!\rangle_{1/2}^2+\langle\!\langle Q_P[\vec{u}(s)]\rangle\!\rangle_{-1/2}^2\}\,ds$$

$$+C\int_0^t \{\|\bar{D}^1\vec{u}_H(s)\|^2+\|\vec{u}_P(s)\|_1^2+\langle\!\langle\bar{D}^1\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2\}\,ds\,,$$

where $C=C(\mathcal{M}(1))$ $(0\leqq\varepsilon\leqq 1)$. Combining (5.36) and (5.37), we obtain (5.35). ■

Considering the adjoint problem, we can get the estimate in the case of non-zero initial data. At first, the following Green's formula is got.

LEMMA 5.7. *Assume that* (A.5.1)-(A.5.4) *are valid. For any pair* $\vec{u}=(\vec{u}_H, \vec{u}_P)$ *and* $\vec{v}=(\vec{v}_H, \vec{v}_P)\in X^{2,0}([0, T]; \boldsymbol{R}_+^n)$, *we have the following identity*:

$$\int_0^t (\mathcal{P}_H^\varepsilon(s)[\vec{u}(s)], \partial_t\vec{v}_H(s))ds+\int_0^t (\mathcal{P}_P(s)[\vec{u}(s)], \vec{v}_P(s))ds \tag{5.38}$$

$$\cong(\partial_t\vec{u}_H(t), \partial_t\vec{v}_H(t))-(\partial_t\vec{u}_H(0), \partial_t\vec{v}_H(0))$$

$$+(P_H^{jk}(t)\partial_k\vec{u}_H(t), \partial_j\vec{v}_H(t))-(P_H^{jk}(0)\partial_k\vec{u}_H(0), \partial_j\vec{v}_H(0))$$

$$+(P_P^0(t)\vec{u}_P(t), \vec{v}_P(t))-(P_P^0(0)\vec{u}_P(0), \vec{v}_P(0))$$

$$-\int_0^t (\partial_t\vec{u}_H(s), \mathcal{P}_H^{*\varepsilon}(s)[\vec{v}(s)])ds+\int_0^t (\vec{u}_P(s), \mathcal{P}_P^*(s)[\vec{v}(s)])ds$$

$$-\int_0^t \langle\partial_t\vec{u}_H(s), Q_H^{*\varepsilon}(s)[\vec{v}(s)]\rangle ds+\int_0^t \langle\vec{u}_P(s), Q_P^*(s)[\vec{v}(s)]\rangle ds$$

$$-\int_0^t \langle Q_H(s)[\vec{u}(s)], \partial_t\vec{v}_H(s)\rangle ds-\int_0^t \langle Q_P(s)[\vec{u}(s)], \vec{v}_P(s)\rangle ds$$

for $0\leqq t\leqq T$, *where*

$$\mathcal{L}_H^{*\varepsilon}(t)[\vec{v}(t)]=\partial_t^2\vec{v}_H(t)-\partial_j(P_H^{jk}(t)\partial_k\vec{v}_H(t))-P_H^{j0\varepsilon}(t)\partial_j\partial_t\vec{v}_H(t)\,;$$

$$\mathcal{L}_P^{*}(t)[\vec{v}(t)]=-P_P^0(t)\partial_t\vec{v}_P(t)-\partial_j(P_P^{jk}(t)\partial_k\vec{v}_P(t))$$
$$+{}^tP_P^j(t)\partial_j\vec{v}_P(t)+{}^tP_{HP}^j(t)\partial_j\partial_t\vec{v}_H(t)\,;$$

$$Q_H^{*\varepsilon}(t)[\vec{v}(t)]=-P_H^{nj}(t)\partial_j\vec{v}_H(t)-(Q_H^0+P_H^{n0\varepsilon})(t)\partial_t\vec{v}_H(t)+Q_{HP}^*(t)\vec{v}_P(t)\,;$$

$$Q_P^{*}(t)[\vec{v}(t)]=-P_P^{nj}(t)\partial_j\vec{v}_P(t)-Q_{PH}^{0*}(t)\partial_t\vec{v}_H\,;$$

$$Q_{HP}^*(t,\ x)=[-{}^tP_{PH}^{n0}-{}^t(Q_{PH}^0(P_H^{nn})^{-1}Q_H^0)-{}^t(P_H^{nn}(P_H^{nn})^{-1}Q_H^0)](t,\ x)\,;$$

$$Q_{PH}^{0*}(t,\ x)=[-{}^tP_{HP}^n-{}^tQ_{HP}](t,\ x)$$

and where $A\cong B$ *means*

$$|A-B|\leq C(\boldsymbol{B}(1))\Big\{\Big(\int_0^t\|\bar{D}^1\vec{u}_H\|^2ds\Big)^{1/2}+\Big(\int_0^t\|\vec{u}_P\|^2ds\Big)^{1/2}\Big(\int_0^t\langle\!\langle\vec{u}_P\rangle\!\rangle^2ds\Big)^{1/2}$$
$$+\Big(\int_0^t\langle\!\langle Q_H[\vec{u}]\rangle\!\rangle_{1/2}^2ds\Big)^{1/2}\Big\}\times\Big\{\Big(\int_0^t\|\bar{D}^1v_H\|^2ds\Big)^{1/2}+\Big(\int_0^t\|v_P\|_1^2ds\Big)^{1/2}\Big\}.$$

PROOF. Noting that $P_H^{nk}\partial_k\vec{u}_H=Q_H[\vec{u}]-Q_{HP}\vec{u}_P-Q_H^0\partial_t\vec{u}_H$ and $-P_P^{nk}\partial_k\vec{u}_P=Q_P[\vec{u}]-Q_{PH}^0\partial_t\vec{u}_H-Q_{PH}^j\partial_j\vec{u}_H$ on Γ, we can obtain (5.38) by integration by parts. ∎

THEOREM 5.8. *Assume that* (A.5.1)-(A.5.5) *are valid and* $T>0$. *Then, for any* $\vec{u}=(\vec{u}_H,\ \vec{u}_P)\in X^{2,0}([0,\ T]\ ;\ \boldsymbol{R}_+^n)\times Z^{1,1}(0,\ T]\ ;\ \boldsymbol{R}_+^n)$ *there exists a* $C=C(\boldsymbol{B}(1+\mu),\ T)$ *such that*

(5.39)
$$|\!|\!|\vec{u}(t)|\!|\!|_1^2\leq C\,\{\|\bar{D}^1\vec{u}_H(0)\|^2+\|\vec{u}_P(0)\|^2$$
$$+\int_0^t(\|\mathcal{L}_H^\varepsilon(s)[\vec{u}(s)]\|^2+\|\mathcal{L}_P(s)[\vec{u}(s)]\|^2$$
$$+\langle\!\langle Q_H(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2+\langle\!\langle Q_P(s)[\vec{u}(s)[d(s)]\rangle\!\rangle_{-1/2}^2)ds\}$$

for $t\in[0,\ T]$ *and* $0\leq\varepsilon\leq 1$.

PROOF. To estimate $\int_0^t\langle\partial_t\vec{u}_H(s)\rangle_{-1/2}^2ds$, we solve the problem for $\mathcal{L}_H^{*\varepsilon}$, $\mathcal{L}_P^*$, $Q_H^{*\varepsilon}$ and Q_P^* from $t=t_0$ to 0. Namely for any $\vec{g}_H(t)\in C_0^\infty([0,\ t_0]\times\boldsymbol{R}_0^n)$, $t_0\in[0,\ T]$ and $0<\varepsilon\leq 1$, we solve

(5.40)
$$\begin{aligned}
&\mathcal{L}_H^{*\varepsilon}(t)[\vec{v}(t)]=0, &&\mathcal{L}_P^*(t)[\vec{v}(t)]=0 &&\text{in } [0,\ t_0]\times\boldsymbol{R}_0^n,\\
&Q_H^{*\varepsilon}(t)[\vec{v}(t)]=\vec{g}_H(t), &&Q_P^*(t)[\vec{v}(t)]=0 &&\text{on } [0,\ t_0]\times\boldsymbol{R}_0^n,\\
&\vec{v}_H(t_0)=\partial_t\vec{v}_H(t_0)=0, &&\vec{v}_P(t_0)=0 &&\text{in } \boldsymbol{R}_+^n.
\end{aligned}$$

Since $\vec{g}_H(t_0)=0$ on $\boldsymbol{R}_0^n$, the compatibility condition of order 0 is satisfied at $t=t_0$, i.e.

$$Q_H^{*\varepsilon}(t)[\vec{v}(t)]|_{t=t_0}=\vec{g}_H(t_0)=0, \qquad Q_P^*(t)[\vec{v}(t)]|_{t=t_0}=0 \qquad \text{on } \Gamma.$$

Therefore, if we put $\vec{w}(t)=\vec{v}(t_0-t)$, we can rewrite (5.40a) as follows:

$$\text{(5.40b)} \qquad \begin{aligned} &\tilde{\mathscr{P}}_H^{*\varepsilon}[\vec{w}(t)]=0, && \tilde{\mathscr{P}}_P^*(t)[\vec{w}(t)]=0 && [0, t_0]\times \boldsymbol{R}_+^n,\\ &\tilde{Q}_H^{*\varepsilon}(t)[\vec{w}(t)]=\vec{g}_H(t_0-t), && \tilde{Q}_P^*(t)[\vec{w}(t)]=0 && \text{on } [0, t_0]\times \boldsymbol{R}_0^n,\\ &\vec{w}_H(0)=\partial_t\vec{w}_H(0)=0, && \vec{w}_P(0)=0 && \text{in } \boldsymbol{R}_+^n,\end{aligned}$$

where

$$\tilde{\mathscr{P}}_H^{*\varepsilon}(t)[\vec{w}(t)]=\partial_t^2\vec{w}_H(t)-\partial_j(P_H^{jk}(t_0-t)\partial_k\vec{w}_H(t))-P_H^{j0\varepsilon}(t_0-t)\partial_j\partial_t\vec{w}_H(t)\,;$$

$$\begin{aligned}\tilde{\mathscr{P}}_P^*(t)[\vec{w}(t)]=&-P_P^0(t_0-t)\partial_t\vec{w}_P(t)-\partial_j(P_P^{jk}(t_0-t)\partial_k\vec{w}_P(t))\\ &+{}^tP_P^j(t_0-t)\partial_j\vec{w}_P(t)+{}^tP_{HP}^j(t_0-t)\partial_j\partial_t\vec{w}_H(t)\,;\end{aligned}$$

$$\begin{aligned}\tilde{Q}_H^{*\varepsilon}(t)[\vec{w}(t)]=&-P_H^{nj}(t_0-t)\partial_j\vec{w}_H(t)-(Q_H^0+P_H^{n0\varepsilon})(t_0-t)\partial_t\vec{w}_H(t)\\ &+Q_{HP}^*(t_0-t)\vec{w}_P(t)\,;\end{aligned}$$

$$\tilde{Q}_P^*(t)[\vec{w}(t)]=-\mathscr{P}_P^{nj}(t_0-t)\partial_j\vec{w}_P(t)-Q_{PH}^{0*}(t_0-t)\partial_t\vec{w}_H\,.$$

(5.40) satisfies the compatibility condition order 0 at $t=0$, i.e.

$$\tilde{Q}_H^{*\varepsilon}(t)[\vec{w}(t)]|_{t=0}=\vec{g}_H(t_0)=0, \quad \tilde{Q}_P^*(t)[\vec{w}(t)]|_{t=0}=0 \qquad \text{on } \Gamma.$$

From the fact that

$$\text{(54.1)} \qquad (Q_H^0+P_H^{n0\varepsilon})(t_0-t)+\frac{1}{2}(-P_H^{n0\varepsilon}(t_0-t))\geqq\varepsilon\,,$$

and so on, (A.5.1)-(A.5.5) are valid for (5.40b). Assume that $\varepsilon>0$, then (A.1)-(A.4′) are valid. Therefore, by Theorem 4.13 there exists a solution $\vec{w}=(\vec{w}_H, \vec{w}_P)\in X^{2,0}\times Z^{1,1}([0, t_0]; \boldsymbol{R}_+^n)$ to (5.40b) for any $\vec{g}_H\in C_0^\infty([0, t_0]\times\boldsymbol{R}^{n-1})$ and $\varepsilon>0$. Moreover, by Theorem 5.6 there exists a constant $C=C(\boldsymbol{B}(1+\mu))>0$ independent of ε such that

$$\text{(5.43)} \qquad |||\vec{w}(t)|||_1^2+\int_0^t\|\bar{D}^1\vec{w}_H(s)\|^2ds\leqq Ce^{Ct}\int_0^t\langle\!\langle\vec{g}_H(t_0-s)\rangle\!\rangle_{1/2}^2ds\,.$$

Since $\vec{w}(t)=\vec{v}(t_0-t)$, if we put $t=t_0$, we have

$$\text{(5.44)} \qquad \begin{aligned}&\int_0^{t_0}(\|\bar{D}^1\vec{v}_H(s)\|^2+\|\vec{v}_P(s)\|_1^2+\langle\!\langle\bar{D}^1\vec{v}_H(s)\rangle\!\rangle_{-1/2}^2)ds\\ &\qquad+\|\bar{D}^1\vec{v}_H(0)\|^2+\|\vec{v}_P(0)\|_2\leqq Ce^{CT}\int_0^{t_0}\langle\!\langle\vec{g}_H(s)\rangle\!\rangle_{1/2}^2ds\,.\end{aligned}$$

Since t_0 is arbitrary, we can replace t_0 with t in (5.44). Substituting the solution $\vec{v}(t)=\vec{w}(t_0-t)$ of (5.40) into (5.38), and combining the resulting formula (5.44), we have

$$(5.45)\qquad \left|\int_0^t \langle \partial_t \vec{u}_H(s),\ \vec{g}_H(s)\rangle ds\right|$$

$$\leqq Ce^{CT}\Big\{\|\bar{D}^1\vec{u}_H(0)\|+\|\vec{u}_P(0)\|+\Big(\int_0^t \|\mathcal{P}_H^\varepsilon(s)[\vec{u}(s)]\|^2 ds\Big)^{1/2}$$

$$+\Big(\int_0^t \|\mathcal{P}_P(s)[\vec{u}(s)]\|^2 ds\Big)^{1/2}+\Big(\int_0^t \langle\!\langle Q_H(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2 ds\Big)^{1/2}$$

$$+\Big(\int_0^t \langle\!\langle Q_P(s)[\vec{u}(s)]\rangle\!\rangle_{-1/2}^2 ds\Big)^{1/2}+\Big(\int_0^t \|\bar{D}^1\vec{u}_H(s)\|^2 ds\Big)^{1/2}$$

$$+\Big(\int_0^t \|\vec{u}_P(s)\|^2 ds\Big)^{1/2}+\Big(\int_0^t \langle\!\langle \vec{u}_P(s)\rangle\!\rangle^2 ds\Big)^{1/2}\Big\}\times\Big(\int_0^t \langle\!\langle \vec{g}_H(s)\rangle\!\rangle_{1/2}^2 ds\Big)^{1/2}$$

for $0<t\leqq T$ and $\varepsilon>0$, where the constant C is independent of ε. Since $C_0^\infty([0, t]\times \boldsymbol{R}^{n-1})$ is dense in $L^2([0, t]; H^{1/2}(\boldsymbol{R}^{n-1}))$, since $L^2([0, t]; H^{1/2}(\boldsymbol{R}^{n-1}))$ and $L^2([0, t]; H^{-1/2}(\boldsymbol{R}^{n-1}))$ are dual, and since

$$\int_0^t \langle\!\langle \nabla\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2 ds \leqq C(\boldsymbol{B}(0))\Big\{\int_0^t \langle\!\langle Q_H(s)[\vec{u}(s)]\rangle\!\rangle_{-1/2}^2 ds$$

$$+\int_0^t \langle\!\langle \partial_t\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2 ds+\int_0^t \langle\!\langle \vec{u}_P(s)\rangle\!\rangle_{-1/2}^2 ds+\int_0^t \|\vec{u}_H(s)\|_1^2 ds\Big\},$$

we have

$$(5.46)\qquad \int_0^t \langle\!\langle \bar{D}^1\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2 ds \leqq Ce^{CT}\{\|\bar{D}^1\vec{u}_H(0)\|^2+\|\vec{u}_P(0)\|^2$$

$$+\int_0^t (\|\mathcal{P}_H^\varepsilon[\vec{u}]\|^2+\|\mathcal{P}_P[\vec{u}]\|^2+\langle\!\langle Q_H[\vec{u}]\rangle\!\rangle_{1/2}^2+\langle\!\langle Q_P[\vec{u}]\rangle\!\rangle_{-1/2}^2)ds$$

$$+\int_0^t (\|\bar{D}^1\vec{u}_H(s)\|^2+\|\vec{u}_P(s)\|^2+\langle\!\langle \vec{u}_P(s)\rangle\!\rangle^2)ds\}\qquad \text{for } \varepsilon>0.$$

On the other hand, calculating $(\mathcal{P}_H(t)[\vec{u}(t)]+d_2\vec{u}_H,\ \partial_t\vec{u}_H(t))$ and $(\mathcal{P}_P(t)[\vec{u}(t)],\ \vec{u}_P(t))$ and combining, for $0\leqq\varepsilon\leqq 1$ we have

$$(5.47)\qquad \frac{1}{2}\frac{d}{ds}\{\|\partial_s\vec{u}_H(s)\|^2+(P_H^{jk}(s)\partial_k\vec{u}_H(s)\partial_j\vec{u}_H(s))+d_2\|\vec{u}_H(s)\|^2$$

$$+(P_P^0(s)\vec{u}_P(s),\ \vec{u}_P(s))\}+\frac{d_2}{2}\|\vec{u}_P(s)\|_1^2$$

$$\leqq C(\boldsymbol{B}(1))(\|\mathcal{P}_H^\varepsilon[\vec{u}]\|^2+\|\mathcal{P}_P[\vec{u}]\|^2+\langle\!\langle Q_H[\vec{u}]\rangle\!\rangle_{1/2}^2+\langle\!\langle Q_P[\vec{u}]\rangle\!\rangle_{-1/2}^2$$

$$+\|\bar{D}^1\vec{u}_H(s)\|^2+\|\vec{u}_P(s)\|^2+\langle\!\langle \bar{D}^1\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2).$$

Integrating (5.47) from 0 to t, and combining the resulting formula and (5.46), by Gronwall's inequality we have (5.39) for $0<t\leqq T$ and $0<\varepsilon\leqq 1$. When $t=0$, obviously (5.39) is valid. Since the constant C is independent of ε, letting $\varepsilon\downarrow 0$, (5.39) is also valid, which completes the proof of the theorem.

§6. A proof of Theorem 2.1 and 2.2.

In this section, we consider the case that $n\geqq 2$, too. Let us introduce the following notation:

$$\mathcal{A}_H^{\varepsilon}(t)[\vec{u}]=\mathcal{A}_H(t)[\vec{u}]+2\varepsilon\sum_{i=1}^{n}\nu_i(x)\partial_i\partial_t\vec{u}_H\,,$$

$$A_H^{i0\varepsilon}(t)=A_H^{i0}(t)-2\varepsilon\nu_i(x)I_{m_H},\quad A_H^{1\varepsilon}(t,\,x,\,\partial)\partial_t\vec{u}_H=A_H^{i0\varepsilon}(t)\partial_i\partial_t\vec{u}_H\,.$$

When we replace $\mathcal{A}_H(t)$ with $\mathcal{A}_H^{\varepsilon}(t)$ in problem (N), we call the problem $(\mathrm{N})_\varepsilon$. Using the local coordinate system, we can reduce the problem $(\mathrm{N})_\varepsilon$ to the case that $\Omega=\boldsymbol{R}_+^n$, so that applying Theorem 5.8, we have the following theorem.

THEOREM 6.1. *Assume that* (A.1)-(A.4) *are valid. Then, there exists a constant* $C=C(\mathcal{M}(1+\mu),\,\Gamma,\,T)$ *such that*

$$\begin{aligned}(6.1)\qquad |||\vec{u}(t)|||_1^2\leqq C\{&\|\bar{D}^1\vec{u}_H(0)\|^2+\|\vec{u}_P(0)\|^2\\&+\int_0^t(\|\mathcal{A}_H^{\varepsilon}(s)[\vec{u}(s)]\|^2+\|\mathcal{A}_P(s)[\vec{u}(s)]\|^2\\&+\langle\!\langle\mathcal{B}_H(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2+\langle\!\langle\mathcal{B}_P(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2)ds\}\end{aligned}$$

for any $\vec{u}=(\vec{u}_H,\,\vec{u}_P)\in X^{2,0}([0,\,T]\,;\,\Omega)\times Z^{1,1}([0,\,T]\,;\,\Omega)$ *and* $0\leqq\varepsilon\leqq 1$.

Next theorem is concerning the higher order estimate.

THEOREM 6.2. *Assume that* (A.1)-(A.4) *are valid. Let L be an integer* $\geqq 2$. *For* $\vec{u}=(\vec{u}_H,\,\vec{u}_P)\in E^L([0,\,T]\,;\,\Omega)$, *we put* $\vec{f}_E(t,\,x)=\mathcal{A}_E(t)[\vec{u}(t,\,x)]$ *and* $\vec{g}_E(t,\,x)=\mathcal{B}_E(t)[\vec{u}(t,\,x)]$ $(E=H,\,P)$. *If* $\vec{f}_E\in R^L([0,\,T]\,;\,\Omega)$ *and* $\vec{g}_E\in R^L([0,\,T]\,;\,\Gamma)$ $(E=H,\,P)$, *then we have*

$$\begin{aligned}(6.2)\qquad |||\vec{u}(t)|||_L^2\leqq C\Big[&\|\bar{D}^1\vec{u}_H(0)\|^2+\|\bar{D}^{L-2}\vec{u}_P(0)\|_2^2+\|\partial_t^{L-1}\vec{u}_P(0)\|^2\\&+\sum_{E=H,P}\{\|\bar{D}^{L-2}\vec{f}_E(0)\|^2+\langle\!\langle\bar{D}^{L-2}\vec{g}_E(0)\rangle\!\rangle_{1/2}^2\\&+\sum_{j=1}^{L-2}(\|\partial_t^j\vec{f}_E(t)\|_{L-2-j}^2+\langle\!\langle\partial_t^j\vec{g}_E(t)\rangle\!\rangle_{L-j-3/2}^2\\&+\int_0^t(\|\partial_s^{L-1}\vec{f}_E(s)\|^2+\langle\!\langle\partial_s^{L-1}\vec{g}_E(s)\rangle\!\rangle_{1/2}^2)ds\}\Big]\end{aligned}$$

for $0 \leqq t \leqq T$.

PROOF. In the case that $L=2$, employing the same argument of the proof of Lemma 4.9, we have (6.2) by use of (6.1) instead of (E.1). In the case that $L \geqq 3$, we can prove (6.2) by induction on L. ∎

Now, we shall prove the existence of solutions ts (N) by using the existence theorem for $(N)_\varepsilon$ with $\varepsilon>0$ which was obtained in §4 and the high order estimate of Theorem 6.2. To this end, at first we give an approximation for initial data and right members of (N). Such approximation was given by Ikawa [3].

LEMMA 6.3. *Assume that* $\vec{u}_{H0} \in H^2(\Omega)$, $\vec{u}_{H1} \in H^1(\Omega)$, $\vec{u}_{P0} \in H^2(\Omega)$ *and* $\vec{f}_E \in R^2([0, T]; \Omega)$, $\vec{g}_E \in R^2([0, T]; \Omega)$ $(E=H, P)$, *and that they satisfy the compatibility condition of order* 0 *of* (N), *i.e.*

$$B_H^1(0, x, \partial)\vec{u}_{H0}+B_H^0(0, x)\vec{u}_{H1}+B_{HP}^0(0, x)\vec{u}_{P0}=\vec{g}_H(0, x) \quad \text{on } \Gamma; \tag{6.3}$$

$$B_P^1(0, x, \partial)\vec{u}_{P0}+B_{PH}^1(0, x, \partial)\vec{u}_{H0}+B_{PH}^0(0, x)\vec{u}_{H1}=\vec{g}_P(0, x) \quad \text{on } \Gamma.$$

Then, there exist $\vec{u}_{H0}^k$, $\vec{u}_{H1}^k \in H^\infty(\Omega)$, $\vec{u}_{P0} \in H^\infty(\Omega)$, $\vec{f}_H^k$, $\vec{f}_P^k \in C^\infty([0, T]; H^\infty(\Omega))$ *and* $\vec{g}_H^k$, $\vec{g}_P^k \in C^\infty([0, T]; H^\infty(\Gamma))$ *satisfying the compatibility condition of order* 1 *of* (N), *i.e.*

$$B_H^1(0, x, \partial)\vec{u}_{H0}^k+B_H^0(0, x)\vec{u}_{H1}^k+B_{HP}^0(0, x)\vec{u}_{P0}^k=\vec{g}_H^k(0, x) \quad \text{on } \Gamma; \tag{6.4$_{(0)}$}$$

$$B_P^1(6, x, \partial)\vec{u}_{P0}^k+B_{PH}^0(0, x)\vec{u}_{H1}^k+B_{PH}^1(0, x, \partial)\vec{u}_{H0}^k=\vec{g}_P^k(0, x) \quad \text{on } \Gamma;$$

$$\begin{aligned} &B_H^1(0, x, \partial)\vec{u}_{H1}^k+B_H^0(0, x)\vec{u}_{H2}^k+B_{HP}^0(0, x)\vec{u}_{P1}^k+B_H^{1(1)}(0, x, \partial)\vec{u}_{H0}^k \\ &\quad +B_H^{0(1)}(0, x)\vec{u}_{H1}^k+B_{HP}^{0(1)}(0, x)\vec{u}_{P1}^k=\partial_t\vec{g}_H^k(0, x) \quad \text{on } \Gamma, \\ &B_P^1(0, x, \partial)\vec{u}_{P1}^k+B_{PH}^0(0, x)\vec{u}_{H2}^k+B_{PH}^1(0, x, \partial)\vec{u}_{H1}^k+B_{PH}^{0(1)}(0, x)\vec{u}_{H1}^k \\ &\quad +B_P^{1(1)}(0, x, \partial)\vec{u}_{P0}^k+B_{PH}^{1(1)}(0, x, \partial)\vec{u}_{H0}^k=\partial_t\vec{g}_P^k(0, x) \quad \text{on } \Gamma, \end{aligned} \tag{6.4$_{(1)}$}$$

where $B^{(l)}(0, x, \partial)\vec{u}(x)=\partial_t^l[B(t, x, \partial)\vec{u}(x)]|_{t=0}$ *and*

$$\begin{aligned} \vec{u}_{H2}^k&=\vec{f}_H^k(0, x)+A_H^2(0, x, \partial)\vec{u}_{H0}^k+A_H^1(0, x, \partial)\vec{u}_{H1}^k \\ &\quad +A_{HP}^1(0, x, \partial)\vec{u}_{P0}^k \in H^1(\Omega); \\ \vec{u}_{P1}^k&=A_P^0(0, x)^{-1}\{\vec{f}_P^k(0, x)+A_P^2(0, x, \partial)\vec{u}_{P0}^k+A_{PH}^2(0, x, \partial)\vec{u}_{H0}^k \\ &\quad +A_{PH}^1(0, x, \partial)\vec{u}_{H1}^k\} \in H^2(\Omega). \end{aligned} \tag{6.5}$$

Moreover we have

(6.6) $$\|\vec{u}^k_{H0}-\vec{u}_{H0}\|_2^2+\|\vec{u}^k_{H1}-\vec{u}_{H1}\|_1^2+\|\vec{u}^k_{P0}-\vec{u}_{P0}\|_2^2$$
$$+\sum_{E=H,P}\Big\{\sup_{t\in[0,T]}\|\vec{f}^k_E(t)-\vec{f}_E(t)\|^2+\sup_{t\in[0,T]}\langle\!\langle\vec{g}^k_E(t)-\vec{g}_E(t)\rangle\!\rangle^2_{1/2}$$
$$+\sum_{E=H,P}\sum_{l=0}^{1}\int_0^T(\|\partial_t^l(\vec{f}^k_E(t)-\vec{f}_E(t))\|^2+\langle\!\langle\partial_t^l(\vec{g}^k_E(t)-\vec{g}_E(t))\rangle\!\rangle^2_{1/2})dt\Big\}$$
$$\longrightarrow 0 \qquad as\ k\to\infty\,.$$

PROOF. We can choose $\vec{v}^k_{H0}$, $\vec{v}^k_{H1}$, $\vec{v}^k_{P0}\in H^\infty(\Omega)$, $\vec{h}^k_H$, $\vec{h}^k_P\in C^\infty([0, T]; H^\infty(\Omega))$ and $\vec{g}^k_H$, $\vec{g}^k_P\in C^\infty([0, T]; H^\infty(\Gamma))$ such that $\vec{v}^k_{H0}\to\vec{u}_{H0}$ and $\vec{v}^k_P\to\vec{v}_{P0}$ in $H^2(\Omega)$, $\vec{v}^{k0}_{H1}\to\vec{u}_{H1}$ in $H^1(\Omega)$, $h^k_E\to\vec{f}_E$ in $R^2([0, T]; \Omega)$ and $\vec{g}^k_E\to\vec{g}_E$ in $R^2([0, T]; \Gamma)$ $(E=H, P)$ as $k\to\infty$. Let $\vec{w}^k_{H0}$ and $\vec{w}^k_{P0}\in H^\infty(\Omega)$ be solutions to the equations:

(6.7) $$-\partial_i(A^{ij}_H(0)\partial_j\vec{w}^k_{H0})+\lambda_H\vec{w}^k_{H0}=0 \qquad \text{in } \Omega,$$
$$-\partial_i(A^{ij}_P(0)\partial_j\vec{w}^k_{P0})+\lambda_P\vec{w}^k_{P0}=0 \qquad \text{in } \Omega,$$
$$B^1_H(0,\cdot,\partial)\vec{w}^k_{H0}+B_{HP}(0)\vec{w}^k_{P0}$$
$$=\vec{g}^k_H(0)-(B^1_H(0,\cdot,\partial)\vec{v}^k_{H0}+B_{HP}(0)\vec{v}^k_{P0}+B^0_H(0)\vec{v}^q_{H1}) \qquad \text{on } \Gamma,$$
$$B^1_P(0,\cdot,\partial)\vec{w}^k_{P0}+B^1_{PH}(0,\cdot,\partial)\vec{w}^k_{H0}$$
$$=\vec{g}^k_P(0)-(B^1_P(0, x,\partial)\vec{v}^k_{P0}+B^1_{PH}(0,\cdot,\partial)\vec{v}^k_{H0}+B^0_{PH}(0)\vec{v}^k_{H1}) \qquad \text{on } \Gamma.$$

Theorem 3.1 guarantees the existence of $\vec{w}^k_{H0}$ and $\vec{w}^k_{P0}\in H^\infty(\Omega)$, and using the estimate of Theorem 3.1 implies that

(6.8) $$\|\vec{w}^k_{H0}\|_2+\|\vec{w}^k_{P0}\|_2\longrightarrow 0 \qquad \text{as } k\to\infty\,.$$

Put $\vec{u}^k_{H0}=\vec{v}^k_{H0}+\vec{w}^k_{H0}$ and $\vec{u}^k_{P0}=\vec{v}^k_{P0}+\vec{w}^k_{P0}$. Then $\vec{u}^k_{H0}$ $\vec{u}^k_{P0}$, $\vec{v}^k_{H1}\in H^\infty(\Omega)$, $\vec{h}^k_E\in C^\infty([0, T]; H^\infty(\Omega))$ and $\vec{g}^k_E\in C^\infty([0, T]; H^\infty(\Gamma))$ $(E=H, P)$ satisfying the compatibility condition of order 0 of (N) such that $\vec{u}^k_{H0}\to\vec{u}_{H0}$, $\vec{u}^k_{P0}\to\vec{u}_{P0}$ in $H^2(\Omega)$, $\vec{v}^k_{H1}\to\vec{u}_{H1}$ in $H^1(\Omega)$, $\vec{h}^k_E\to\vec{f}_E$ in $R^2([0, T]; \Omega)$ and $\vec{g}^k_E\to\vec{g}_E$ in $R^2([0, T]; \Gamma)$ $(E=H, P)$ as $k\to\infty$. Put

(6.9) $$\vec{u}^k_{H2}=\vec{h}^k_H(0, x)+A^2_H(0, x,\partial)\vec{u}^k_{H0}+A^1_H(0, x,\partial)\vec{v}_{H1}+A^1_{HP}(0, x,\partial)\vec{u}^k_{P0}\,;$$
$$\vec{v}^k_{P1}=A^0_P(0, x)^{-1}\{\vec{h}^k_P(0, x)+A^2_P(0, x,\partial)\vec{u}^k_{P0}+A^2_{PH}(0, x,\partial)\vec{u}^k_{H0}$$
$$+A^1_{PH}(0, x,\partial)\vec{v}^k_{H1}\}\,.$$

Let $\vec{w}^k_{H1}$ and $\vec{w}^k_{P1}$ be functions such that

(6.10) $$B^1_H(0, x,\partial)\vec{w}^k_{H1}=(\partial_t\vec{g}^k_H)(0, x)-\{B^1_H(0, x,\partial)\vec{v}^k_{H1}+B^0_H(0, x)\vec{u}^k_{H2}$$
$$+B_{HP}(0, x)\vec{v}^k_{P1}+B^{1(1)}_H(0, x,\partial)\vec{u}^k_{H0}$$
$$+B^{0(1)}_H(0, x)\vec{v}^k_{H1}+B^{(1)}_{HP}(0, x)\vec{u}^k_{P0}\} \qquad \text{on } \Gamma,$$

$$\vec{w}^k_{H1}|_{\Gamma}=0, \qquad \|\vec{w}^k_{H1}\|_1 \leqq 1/k\,;$$

$$(6.11) \qquad B^1_P(0,x,\partial)\vec{w}^k_{P1}=\vec{g}^k_P(0,x)-\{B^1_P(0,x,\partial)\vec{v}^k_{P1}+B^0_{PH}(0,x)\vec{u}^k_{H2} +B^1_{PH}(0,x,\partial)(\vec{v}^k_{H1}+\vec{w}^k_{H1})+B^{1(1)}_P(0,x,\partial)\vec{u}^k_{P0} +B^{0(1)}_{PH}(0,x)(\vec{v}^k_{H1}+\vec{w}^k_{H1})+B^{1(1)}_{PH}(0,x,\partial)\vec{u}^k_{H0}\} \quad \text{on } \Gamma,$$

$$\vec{w}^k_{P1}|_{\Gamma}=0, \qquad \|\vec{w}^k_{P1}\|_1 \leqq 1/k\,.$$

The existence of $\vec{w}^k_{H1}$ and $\vec{w}^k_{P1}\in H^\infty(\Omega)$ is assured by Lemma 3.8 of [6]. We put $\vec{u}^k_{H1}=\vec{v}^k_{H1}+\vec{w}^k_{H1}$, $\vec{u}^k_{P1}=\vec{v}^k_{P1}+\vec{w}^k_{P1}$, $\vec{f}^k_H(t,x)=\vec{h}^k_H(t,x)-A^1_H(0,x,\partial^1)\vec{w}^k_{H1}$ and $\vec{f}^k_P(t,x)=\vec{h}^k_P(t,x)-A^1_{PH}(0,x,\partial^1)\vec{w}^k_{H1}+A^0_P(0,x)\vec{w}^k_{P1}$. Since $\vec{w}^k_{H1}|_\Gamma=0$, $\vec{u}^k_{H0}$, $\vec{u}^k_{H1}$, $\vec{u}^k_{P0}$, $\vec{f}^k_E$ and $\vec{g}^k_E$ satisfy $(6.4_{(0)})$. By the definition of $\vec{f}^k_H$ and $\vec{f}^k_P$, we see that $\vec{u}^k_{H2}$ and $\vec{u}^k_{P1}$ satisfy (6.5), and by (6.8) and (6.9) we know that $(6.4_{(1)})$ and (6.6) are valid. Therefore, these $\vec{u}^k_{H0}$, $\vec{u}^k_{H1}$, $\vec{u}^k_{H2}$, $\vec{u}^k_{P0}$, $\vec{u}^k_{P1}$, $\vec{f}^k_E$ and $\vec{g}^k_E$ $(E=H,P)$ are required approximations. ■

PROOF OF THEOREM 2.1. First, we assume that $\vec{u}_{H0}\in H^\infty(\Omega)$, $\vec{u}_{H1}\in H^\infty(\Omega)$, $\vec{u}_{P0}\in H^\infty(\Omega)$, $\vec{f}_E\in C^\infty([0,T]\,;H^\infty(\Omega))$ and $\vec{g}_E\in C^\infty([0,T]\,;H^\infty(\Gamma))$ $(E=H,P)$ satisfy the compatibility condition of order 1 for problem (N). Put

$$\vec{f}^\varepsilon_E(t,x)=\vec{f}_H(t,x)+2\varepsilon\sum_{i=1}^n \nu_i(x)\partial_i\vec{u}_{H1}\,.$$

Then $\vec{u}_{H0}$, $\vec{u}_{H1}$, $\vec{u}_{P0}$, $\vec{f}^\varepsilon_H$, $\vec{f}_P$, $\vec{g}_H$ and $\vec{g}_P$ satisfy the compatibility condition of order 1 for problem $(\mathrm{N})_\varepsilon$ for $0<\varepsilon\leqq 1$. Then, for $\varepsilon>0$ by Theorems 4.13 and 4.14 we see that there exists a solution $\vec{u}^\varepsilon\in X^{3,0}([0,T]\,;\Omega)\times Z^{2,1}([0,T]\,;\Omega)$ of problem $(\mathrm{N})_\varepsilon$ with initial data $\vec{u}_{H0}$, $\vec{u}_{H1}$, $\vec{u}_{P0}$ and right members $\vec{f}^\varepsilon_E$ and $\vec{g}_E$ $(E=H,P)$. Since

$$\|\bar{D}^1\vec{f}^\varepsilon_H(t)\|+\|\partial_t^2\vec{f}^\varepsilon_H(t)\|\leqq\|\bar{D}^1\vec{f}_H(t)\|+\|\partial_t^2\vec{f}_H(t)\|+C\varepsilon\|\vec{u}_{H1}\|_2\,,$$

by Theorem 6.2 we have

$$(6.12) \qquad \|\bar{D}^3\vec{u}^\varepsilon_H(t)\|^2+\|\partial_t^2\vec{u}^\varepsilon_P(t)\|^2+\sum_{j=0}^1\|\partial_t^j\vec{u}^\varepsilon_P(t)\|^2_{3-j}$$
$$\leqq C\Big\{\|\vec{u}_{H0}\|_4^2+\|\vec{u}_{H1}\|_3^2+\|\vec{u}_{P0}\|_4^2+\|\vec{f}_P(0)\|_2^2 +\sum_{E=H,P}(\|\bar{D}^1\vec{f}_E(0)\|^2+\langle\!\langle\bar{D}^1\vec{g}_E(0)\rangle\!\rangle^2_{1/2}) +\sum_{E=H,P}\sum_{j=0}^1(\|\partial_t^j\vec{f}_E(t)\|^2_{1-j}+\langle\!\langle\partial_t^j\vec{g}_E(t)\rangle\!\rangle^2_{3/2-j}) +\sum_{E=H,P}\int_0^t(\|\partial_s^2\vec{f}_E(s)\|^2+\langle\!\langle\partial_s^2\vec{g}_E(s)\rangle\!\rangle^2_{1/2})ds\Big\}\,.$$

Since the constant C is independent of ε, the right-hand sice of (6.12) is independent of ε. Let us observe that $\vec{u}^{\varepsilon}-\vec{u}^{\varepsilon'}$ satisfies the following equations:

$$(6.13)\qquad \begin{aligned}
&\mathcal{A}_H^{\varepsilon}(t)[\vec{u}^{\varepsilon}-\vec{u}^{\varepsilon'}]=2(\varepsilon'-\varepsilon)\sum_{i=1}^{n}\nu_i(x)\partial_i\partial_t\vec{u}_H^{\varepsilon'}+\vec{f}_H^{\varepsilon}-\vec{f}_H^{\varepsilon'} && \text{in } [0, T]\times\Omega,\\
&\mathcal{A}_P(t)[\vec{u}^{\varepsilon}-\vec{u}^{\varepsilon'}]=0 && \text{in } [0, T]\times\Omega,\\
&\mathcal{B}_H(t)[\vec{u}^{\varepsilon}-\vec{u}^{\varepsilon'}]=0,\qquad \mathcal{B}_P(t)[\vec{u}^{\varepsilon}-\vec{u}^{\varepsilon'}]=0 && \text{on } [0, T]\times\Gamma,\\
&(\vec{u}_H^{\varepsilon}-\vec{u}_H^{\varepsilon'})(0)=\partial_t(\vec{u}_H^{\varepsilon}-\vec{u}_H^{\varepsilon'})(0)=0,\qquad (\vec{u}_P^{\varepsilon}-\vec{u}_P^{\varepsilon'})(0)=0 && \text{in } \Omega.
\end{aligned}$$

Since $\vec{f}_H^{\varepsilon}-\vec{f}_H^{\varepsilon'}=2(\varepsilon-\varepsilon')\sum_{i=1}^{n}\nu_i(x)\partial_i\vec{u}_{H1}$, applying Theorem 6.2 to (6.13) implies that

$$(6.14)\qquad |||(\vec{u}^{\varepsilon}-\vec{u}^{\varepsilon'})(t)|||_2^2\leqq C(\varepsilon'-\varepsilon)^2\{\|\vec{u}_{H1}\|^2+\|\partial_t\vec{u}_H^{\varepsilon'}(t)\|^2+\int_0^t\|\partial_s^2\vec{u}_H^{\varepsilon'}(s)\|_1^2ds\},$$

where C is independent of ε and ε'. From (6.12), letting ε and $\varepsilon'\downarrow 0$ in (6.14), we see that the right-hand-side of (6.14) tends to 0. Therefore $\{\vec{u}^{\varepsilon}\}$ is a Cauchy sequence in $E^2([0, T]; \Omega)$, so that there exists a $\vec{u}=(\vec{u}_H, \vec{u}_P)\in E^2(0, T]; \Omega)$ such that $\vec{u}^{\varepsilon}\to\vec{u}$ as $\varepsilon\downarrow 0$ in $E^2([0, T]; \Omega)$. The limit $\vec{u}$ satisfies (N).

To remove the additional assumption, we use the approximation constructed in Lemma 6.3: $\vec{u}_{H0}^k$, $\vec{u}_{H1}^k$, $\vec{u}_{P0}^k\in H^{\infty}(\Omega)$, $\vec{f}_E\in C^{\infty}([0, T]; H^{\infty}(\Omega))$ and $\vec{g}_E^k\in C^{\infty}([0, T]; H^{\infty}(\Gamma))$ $(E=H, P)$, which satisfy (6.3)-(6.6). Then, we already know that there exists a $\vec{u}^k=(\vec{u}_H^k, \vec{u}_P^k)\in E^2([0, T]; \Omega)$ satisfying (N) with initial data $\vec{u}_{H0}^k$, $\vec{u}_{H1}^k$, $\vec{u}_{P0}^k$ and right members $\vec{f}_E^k$ and $\vec{g}_E^k$ $(E=H, P)$. Let us apply Theorem 6.2 with $L=2$ to $\vec{u}^k-\vec{u}^{k'}$, and then we have

$$(6.15)\qquad \begin{aligned}
|||(\vec{u}^k-\vec{u}^{k'})(t)|||_2^2\leqq C\Big[&\|\vec{u}_{H0}^k-\vec{u}_{H0}^{k'}\|_2^2+\|\vec{u}_{H1}^k-\vec{u}_{H1}^{k'}\|_1^2+\|\vec{u}_{P0}^k-\vec{u}_{P0}^{k'}\|_2^2\\
&+\sum_{E=H,P}\Big\{\|(\vec{f}_E^k-\vec{f}_E^{k'})(0)\|^2+\langle\!\langle(\vec{g}_E^k-\vec{g}_E^{k'})(0)\rangle\!\rangle_{1/2}^2\\
&+\|(\vec{f}_E^k-\vec{f}_E^{k'})(t)\|^2+\langle\!\langle(\vec{g}_E^k-\vec{g}_E^{k'})(t)\rangle\!\rangle_{1/2}^2\\
&+\sum_{h=0}^{1}\int_0^t(\|\partial_s^h(\vec{f}_E^k-\vec{f}_E^{k'})(s)\|^2+\langle\!\langle\partial_s^h(\vec{g}_E^k-\vec{g}_E^{k'})(s)\rangle\!\rangle_{1/2}^2)ds\Big\}\Big].
\end{aligned}$$

From (6.6), it follows that the right-hand side of (6.15) tends to 0 as k and $k'\to\infty$. Therefore $\{\vec{u}^k\}$ is a Cauchy sequence in $E^2([0, T]; \Omega)$ so that there exists a lmit $\vec{u}=(\vec{u}_H, \vec{u}_P)\in E^2([0, T]; \Omega)$ satisfying (N) with initial data $\vec{u}_{H0}\in H^2(\Omega)$, $\vec{u}_{H1}\in H^1(\Omega)$, $\vec{u}_{P0}\in H^2(\Omega)$ and right members $\vec{f}_E\in R^2([0, T]; \Omega)$ and $\vec{g}_E\in R^2([0, T]; \Omega)$ $(E=H, P)$. The uniqueness follows from (6.1). Hence the assertion (1) in Theorem 2.1 has proved. (2.5) in Theorem 2.1 was already proved as Theorem 6.1, and (2.6) in Theorem 2.1 can be obtained from (4.1) immediately, which completes the proof of Theorem 2.1 with $n\geqq 2$.

Theorem 2.2 can be proved by Theorem 2.1 in the same manner as in [[2], p. 364-p. 367] or [[1], p. 604-p. 607]. ■

§ 7. Energy estimate for $n=1$.

When $n=1$, we may assume that $\Omega=(0, 1)$ or $=\boldsymbol{R}_+$ $(=\{x\in\boldsymbol{R}\mid x>0\})$. The case that $\Omega=(0, 1)$ can be reduced the case that $\Omega=\boldsymbol{R}_+$, so that we consider the case that $\Omega=\boldsymbol{R}_+$, below. We can write the problem (N) as follows.

$$\begin{aligned}
\mathcal{A}_H(t)[\vec{u}]&=\partial_t^2\vec{u}_H(t)-\partial_x(A_H^{11}(t)\partial_x\vec{u}_H(t))-A_H^{10}(t)\partial_x\partial_t\vec{u}_H(t)\\
&\quad -A_{HP}^1(t)\partial_x\vec{u}_P(t)=\vec{f}_H(t) && \text{in } [0, T]\times\boldsymbol{R}_+,\\
\mathcal{A}_P(t)[\vec{u}]&=A_P^0(t)\partial_t\vec{u}_P(t)-\partial(A_P^{11}(t)\partial_x\vec{u}_P(t))+A_P^1(t)\partial_x\vec{u}_P(t)\\
&\quad -A_{PH}^{11}(t)\partial_x^2\vec{u}_H(t)-A_{PH}^{10}(t)\partial_x\partial_t\vec{u}_H(t)=f_P(t) && \text{in } [0, T]\times\boldsymbol{R}_+,\\
\mathcal{B}_H(t)[\vec{u}]&=-A_H^{11}(t)\partial_x\vec{u}_H(t)+B_{HP}(t)\vec{u}_P(t)+B_H^0\partial_t\vec{u}_H(t)\\
&=\vec{g}_H(t) && \text{on } [0, T],\\
\mathcal{B}_P(t)[\vec{u}]&=-A_P^{11}(t)\partial_x\vec{u}_P(t)+B_P\vec{u}_P(t)+B_{PH}^1(t)\partial_x\vec{u}_H(t)\\
&\quad +B_{PH}^0(t)\partial_t\vec{u}_H(t)=\vec{g}_P(t) && \text{on } [0, T],\\
\vec{u}_H(0)&=\vec{u}_{H0},\quad \partial_t\vec{u}_H(0)=\vec{u}_{H1},\quad \vec{u}_P(0)=\vec{u}_{P0} && \text{in } \boldsymbol{R}_+.
\end{aligned}$$

Theorem 6.1 is a key to prove Theorem 2.1 for $n\geqq 2$. But, when $n=1$, we replace Theorem 6.1 by the following theorem.

THEOREM 7.1. *Assume that* (A.1)-(A.4) *are valid. Then, there exists a constant* $C=C(\mathcal{M}(1), T)$ *such that*

$$\text{(7.1)}\qquad \|\vec{u}(t)\|_1^2\leqq C\Big\{\|\bar{D}^1\vec{u}_H(0)\|^2+\|\vec{u}_P(0)\|^2 + \int_0^t(\|\mathcal{A}_H^\varepsilon(t)[\vec{u}]\|^2+\|\mathcal{A}_P(t)[\vec{u}]\|^2+|\mathcal{B}_H(t)[\vec{u}]|_{x=0}|^2+|\mathcal{B}_P(t)[\vec{u}]|_{x=0}|^2)ds\Big\}$$

for any $\vec{u}=(\vec{u}_H, \vec{u}_P)\in X^{2,0}([0, T]; \Omega)\times Z^{1,1}([0, T]; \Omega)$. *Here,* $\mathcal{A}_H^\varepsilon(t)[\vec{u}]=\mathcal{A}_H(t)[\vec{u}]-2\varepsilon\partial_x\partial_t\vec{u}$, $0\leqq\varepsilon\leqq 1$.

By Theorem 7.1 we get Theorem 2.1 (3). Replacing Theorem 6.1 by Theorem 7.1, we can prove Theorem 2.1 with $n=1$ by the same argument as in § 6.

PROOF OF THEOREM 7.1. Calculate $(\mathcal{A}_H(t)[\vec{u}]+\delta_0\vec{u}_H, \partial_t\vec{u}_H)$ and $(\mathcal{A}_P(t)[\vec{u}], \vec{u}_P)$ by iteration by parts and combine the resulting formulas, then we have

$$\frac{1}{2}\frac{d}{dt}\Big\{\|\partial_t\vec{u}_H(t)\|^2+(A_H^{11}(t)\partial_x\vec{u}_H(t),\ \partial_x\vec{u}(t))+\delta_0\|\vec{u}_H(t)\|^2+(A_P^0(t)\vec{u}_P(t),\ \vec{u}_P(t))$$
$$+\frac{\delta_1}{2}\|\partial_x\vec{u}_P(t)\|^2+(B_H^0(t,\ 0)+\frac{1}{2}A_H^{10}(t,\ 0))\partial_t\vec{u}_H(t,\ 0)\cdot\partial_t\vec{u}_H(t,\ 0)$$
$$\leqq\sum_{E=H,P}\|\mathcal{A}_E(t)[\vec{u}]\|^2+C(\mathcal{M}(1),\ \delta_1)\{\|\partial_t\vec{u}_H(t)\|^2+\|\vec{u}_H(t)\|_1^2+\|\vec{u}_P(t)\|^2$$
$$+C(\mathcal{M}(1))(|\vec{u}_P(t,\ 0)\|D^1\vec{u}_H(t,\ 0)|+|\vec{u}_P(t,\ 0)|^2)$$
$$+|\mathcal{B}_H(t)[\vec{u}]|_{x=0}\|\partial_t\vec{u}_H(t,\ 0)|+|\mathcal{B}_P(t)[\vec{u}]|_{x=0}\|\vec{u}_P(t,\ 0)|. \tag{7.2}$$

Calculating $(\mathcal{A}_H(t)[\vec{u}],\ \partial_x\vec{u}_H)$ by integration by parts, we have

$$\frac{d}{dt}\Big\{(\partial_t\vec{u}_H(t),\ \partial_x\vec{u}_H(t))-\frac{1}{2}(A_H^{10}(t)\partial_x\vec{u}_H(t),\ \partial_x\vec{u}_H(t))\Big\}$$
$$+\frac{1}{2}A_H^{11}(t,\ 0)\partial_x\vec{u}_H(t,\ 0)\cdot\partial_x\vec{u}_H(t,\ 0)+\frac{1}{2}|\partial_t\vec{u}_H(t,\ 0)|^2$$
$$\leqq\|\mathcal{A}_H(t)[\vec{u}]\|^2+C(\mathcal{M}(1))\{\|\vec{u}_H(t)\|_1^2+\|\partial_x\vec{u}_P(t)\|\|\partial_x\vec{u}_H(t)\|\}. \tag{7.3}$$

Multiplying (7.3) by $\sigma>0$ and combining the resulting formula and (7.2), we have

$$\frac{d}{dt}\Big\{\|\partial_t\vec{u}_H\|^2+(A_H^{11}(t)\partial_x\vec{u}_H(t),\ \partial_x\vec{u}_P(t))+\delta_0\|\vec{u}_H(t)\|^2+(A_P^0(t)\vec{u}_P(t),\ \vec{u}_P(t))$$
$$+\sigma(\partial_t\vec{u}_H(t),\ \partial_x\vec{u}_H(t))+\frac{\sigma}{2}(A_H^{10}(t)\partial_x\vec{u}_H(t),\ \partial_x\vec{u}_H(t))\Big\}$$
$$\frac{\sigma\delta_1}{4}|\partial_x\vec{u}_H(t,\ 0)|^2+\frac{\sigma}{4}|\partial_t\vec{u}_H(t,\ 0)|^2+\frac{\delta_1}{4}\|\partial_x\vec{u}_P(t)\|^2$$
$$\leqq\sum_{E=H,P}(\|\mathcal{A}_E(t)[\vec{u}]\|^2+|\mathcal{B}_E(t)[\vec{u}]|_{x=0}|^2)+\sigma\|\mathcal{A}_H(t)[\vec{u}]\|^2$$
$$+C(\mathcal{M}(1),\ \delta_1,\ \sigma)\{\|\partial_t\vec{u}_H(t)\|^2+\|\vec{u}_H(t)\|_1^2+\|\vec{u}_P(t)\|^2\}. \tag{7.4}$$

Here we have used that

$$|\vec{u}(t,\ 0)|^2\leqq C\|\partial_x\vec{u}(t)\|\|\vec{u}(t)\|. \tag{7.5}$$

Applying Gronwall's inequality to (7.4) and taking $\sigma>0$ sufficiently small, we have (7.1) in the case that $\varepsilon=0$. When $0<\varepsilon\leqq1$, by the same argument we can prove the theorem.

Theorem 2.2 with $n=1$ can be proved by Theorem 2.1 in the same manner as in [[2], p. 364-p. 367] or [[1], p. 604-p. 607], too.

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