

SOME GEOMETRICAL ASPECTS OF RIEMANNIAN MANIFOLDS WITH A POLE

(dedicated to Professor Isamu Mogi for his 60'th birthday)

By

Mitsuhiro ITOH

The aim of this paper is to describe some geometrical aspects of Riemannian manifolds with a pole. A point o of a Riemannian manifold is called a *pole*, if the exponential map \exp is a diffeomorphism at o . Simply connected complete Riemannian manifolds of nonpositive curvature (the Euclidean space, the hyperbolic space and a simply connected symmetric space of noncompact type, etc.) and a paraboloid of revolution are typical examples of Riemannian manifolds with a pole.

We give in **1** a sufficient condition on the existence of a pole in terms of curvature. Hessian comparison theorem, conformal changes of a metric and a generalization of Cartan's fixed point theorem are discussed in **2** ([6], [1] [2]). And we argue in **3** the order of a holomorphic function on a Kähler manifold with a pole ([7]).

1. As an easy consequence, a Riemannian manifold with a pole is diffeomorphic to the Euclidean space. On the contrary, any complete Riemannian manifold diffeomorphic to the Euclidean space does not necessarily have a pole.

The following proposition gives a sufficient condition on the existence of a pole.

PROPOSITION 1. *Let M be a connected complete Riemannian manifold and N be a complete surface with a pole p . Assume that M has a point o such that the sectional curvature $K(\Pi, t) \leq$ Gaussian curvature of N at a point with distance t from p for all $t > 0$, every normal geodesic γ issuing from o and every plane $\Pi(t)$ containing $\dot{\gamma}(t)$. Then \exp_o is of maximal rank. If, moreover, M is simply connected, then o is a pole.*

PROOF. It is sufficient to show that o has no conjugate point on each geo-

desic issuing from o ([4]). Let γ be a normal geodesic from o and J a Jacobi field along γ such that $J(0)=0$ and $\nabla_t J \neq 0$ at $t=0$. Suppose that $J(t_0)=0$ for $t_0>0$. Without loss of generality we may assume that $J(t) \neq 0$ for $0 < t < t_0$ and that J is perpendicular to γ . Since J satisfies the Jacobi's equation, $\langle \nabla_t^2 J, J \rangle = -\langle R(J, \dot{\gamma})\dot{\gamma}, J \rangle = -K(J \wedge \dot{\gamma})\|J\|^2$. On the other hand, by Schwarz' inequality $\langle \nabla_t^2 J, J \rangle = 1/2 d^2/dt^2 (\|J\|^2) - \|\nabla_t J\|^2 \leq 1/2 d^2/dt^2 (\|J\|^2) - (d/dt \|J\|)^2 = \|J\|(d^2/dt^2 \|J\|)$ for $0 < t < t_0$. Then we have

$$\frac{d^2}{dt^2} \|J\| + K(J(t) \wedge \dot{\gamma}(t))\|J\| \geq 0. \tag{1}$$

Note that $|d/dt \|J\||$ is bounded for $t \rightarrow +0$, since $|d/dt \|J\|| \leq \|\nabla_t J\|$ for $0 < t < t_0$.

Let γ' be a normal geodesic issuing from p in N . Since p is a pole, each nontrivial Jacobi field J' along γ' such that $J'(0)=0$ and $J' \perp \gamma'$ has no zero point for $t>0$. Since N is two dimensional, $J'(t)=h(t)E(t)$ where E is a parallel unit field along γ' and $h(t)$ is a smooth function such that $h(0)=0$ and $h(t)>0$ for $t>0$. By Jacobi's equation, we have

$$\frac{d^2}{dt^2} h(t) + K'(J'(t) \wedge \dot{\gamma}'(t))h(t) = 0. \tag{2}$$

On the other hand, by the curvature condition together with the lemma below, it follows that h has a zero for $0 < t < t_0$. Thus we have a contradiction.

LEMMA (Sturm's Comparison Theorem [3]). Let u_i be C^2 -functions defined on $[0, a]$, $i=1, 2$, which satisfy

$$\begin{aligned} \frac{d^2}{dt^2} u_1(t) + A_1(t)u_1(t) &\geq 0 \\ \frac{d^2}{dt^2} u_2(t) + A_2(t)u_2(t) &= 0, \\ u_1(0) = u_2(0) = 0, \quad \dot{u}_1(0) > 0 \quad \text{and} \quad \dot{u}_2(0) > 0, \end{aligned} \tag{3}$$

where A_i are C^0 -functions on $[0, a]$. Assume that $A_1(t) \leq A_2(t)$ for $0 \leq t \leq a$ and u_2 never vanishes on $(0, a]$. Then u_1 also never vanishes on $(0, a]$.

PROOF OF LEMMA. Note that $u_2(t) > 0$ for $t > 0$ from the initial condition. Suppose that $u_1(t_0) = 0$ for some $t_0 \in (0, a]$. Without loss of generality we may assume that $u_1(t) > 0$ for $0 < t < t_0$. From (3), we have, for $0 < t < t_0$,

$$0 < \int_0^t \{u_2(\ddot{u}_1 + A_1 u_1) - u_1(\ddot{u}_2 + A_2 u_2)\} dt$$

$$\begin{aligned}
 &= (u_2 \dot{u}_1 - u_1 \dot{u}_2) \Big|_0^t - \int_0^t (A_2 - A_1) u_1 u_2 dt \\
 &< u_2(t) \dot{u}_1(t) - u_1(t) \dot{u}_2(t),
 \end{aligned}$$

hence $\dot{u}_2(t)/u_2(t) < \dot{u}_1(t)/u_1(t)$. Then we have, for sufficiently small positive number c , $\log \{u_2(t)/u_2(c)\} = \int_c^t \{\dot{u}_2(t)/u_2(t)\} dt \leq \int_c^t \{\dot{u}_1(t)/u_1(t)\} dt = \log \{u_1(t)/u_1(c)\}$ for $c < t < t_0$. Since $u_i(c) > 0$, $i=1, 2$, $u_2(t_0) = \lim_{t \rightarrow t_0-0} u_2(t) < \lim_{t \rightarrow t_0-0} \{u_2(c)/u_1(c)\} u_1(t) = 0$. This leads a contradiction.

2. Let M be a Riemannian manifold with a pole o . The distance function $\rho(\cdot) = d(\cdot, o)$ has singularity only at o . By comparing the radial curvatures, Siu and Yau [6] and also Greene and Wu [1] showed the comparison theorem on Hessian of the distance functions. By *radial curvature* $K(t)$ for a normal geodesic $\gamma: [0, \infty) \rightarrow M$, $\gamma(0) = o$, we mean the sectional curvature of a plane which contains the tangent vector $\dot{\gamma}$ at $\gamma(t)$. *Hessian* of a smooth function f is defined by $\text{Hess}(f)(X, Y) = X(\tilde{Y}f) - (\nabla_X \tilde{Y})f$, where \tilde{Y} is a local extension of Y .

By using Schwarz' inequality again, we have a description of the comparison theorem in a free manner on any dimensional condition.

PROPOSITION 2 (*Hessian Comparison Theorem*). *Let (M, o) and (N, p) be Riemannian manifolds with poles o and p respectively. Assume that for all $t > 0$, the radial curvatures satisfy $K_M(t) \leq K_N(t)$ for each normal geodesics γ and σ issuing from the poles. Then*

$$\text{Hess}_M(\rho_M)(X, X) \geq \text{Hess}_N(\rho_N)(Y, Y),$$

where X and Y are unit vectors at $\gamma(t)$ and $\sigma(t)$ such that $X \perp \dot{\gamma}(t)$ and $Y \perp \dot{\sigma}(t)$, $t > 0$, respectively.

Note that if f is an increasing smooth function on $(0, \infty)$, then

$$\text{Hess}_M(f \circ \rho_M)(X, X) \geq \text{Hess}_N(f \circ \rho_N)(Y, Y),$$

since $\text{Hess}(f \circ \rho) = f' \cdot \text{Hess}(\rho) + f'' d\rho \otimes d\rho$.

PROOF. We shall prove this by following [6]. Since o is a pole, there is a global vector field \tilde{X} on M such that (1) $\tilde{X}(o) = 0$, (2) $\tilde{X}(\gamma(t)) = X$, (3) $[\tilde{X}, \partial/\partial \rho] = 0$ and (4) \tilde{X} is a Jacobi field along $\gamma|_{[0, t]}$ perpendicular to $\dot{\gamma}$. Then we have

$$\begin{aligned}
 \text{Hess}_M(\rho_M)(X, X) &= \int_0^t \{ \|\nabla_{\partial/\partial \rho} \tilde{X}(s)\|^2 - K_M(\tilde{X}(s) \wedge \dot{\gamma}(s)) \|\tilde{X}(s)\|^2 \} ds \\
 &= I_0^t(\tilde{X}).
 \end{aligned}$$

There is also a global field \tilde{Y} satisfying the similar condition and $Hess_N(\rho_N)(Y, Y) = I_0^t(\tilde{Y})$. Let $Z(s)$ be a vector field along σ defined by $Z(s) = \|X(s)\|E(s)$, where E is a unit parallel field along σ such that $E(t) = Y$. Then $\|Z\| = \|X\|$, $Z(0) = 0$ and $Z(t) = Y$. By Schwarz' inequality, we have $\|\nabla_{\partial/\partial\rho_N} Z\| \leq \|\nabla_{\partial/\partial\rho_M} X\|$. From the curvature condition, $K_M(X(s) \wedge \dot{\gamma}(s))\|X(s)\|^2 \leq K_N(Z(s) \wedge \dot{\sigma}(s))\|Z(s)\|^2$, hence

$$Hess_M(\rho_M)(X, X) = I_0^t(\tilde{X}) \geq I_0^t(\tilde{Z}).$$

From the property of the quadratic form I_0^t , we have $I_0^t(\tilde{Z}) \geq I_0^t(\tilde{Y}) = Hess_N(\rho_N)(Y, Y)$.

A C^2 -function f is called *convex (strictly convex)* if and only if $Hess(f) \geq 0$ (> 0). Note that f is convex (strictly convex) if and only if $(f \circ \gamma)'' \geq 0$ ($(f \circ \gamma)'' > 0$) for every geodesic γ . The Hessian comparison theorem gives an estimation on the (strictly) convexity of a radial function. A function f on M is called *radial* if and only if f is a composition of ρ_M and a function defined on \mathbf{R}^+ .

COROLLARY 3. *Let (M, o) and (N, p) be as in Proposition 2. If the curvature assumption in the proposition is satisfied and there is an increasing function $f: \mathbf{R}^+ \rightarrow \mathbf{R}$, $f' > 0$ such that $f \circ \rho_N$ is (strictly) convex, then $f \circ \rho_M$ is also (strictly) convex.*

The Hessian of a radial function of a manifold with a pole is not necessarily positive definite. The above corollary gives an estimation of the convexity. By construction of a surface of revolution with Gaussian curvature $K(s)$, the following theorem is obtained [1]: Suppose $\int_0^\infty s \bar{K}(s) ds < 1$, where $\bar{K}(s) = \max\{0, \text{radial curvature at } x \text{ with } \rho(x) = s\}$. Then $(\mu/t)(g - d\rho \otimes d\rho)(X, X) \leq Hess(\rho)(X, X)$ at x with $\rho(x) = t$, $t > 0$ for a positive constant μ such that $1 - \int_0^\infty s \bar{K}(s) ds \leq \mu \leq 1$.

Since $Hess(\rho^2) = 2\rho \cdot Hess(\rho) + 2d\rho \otimes d\rho$, we have a crucial estimation for the strictly convexity of ρ^2 .

Consider a paraboloid of revolution, $2z = x^2 + y^2$. Then the origin is a pole. The Gaussian curvature $K(p)$ at $p = (x, y, z)$ and $\rho(p)$ are written as $K(p) = 1/\{(1 + |p|^2)^3\}$ and $\rho(p) = 1/2\{|p|\sqrt{1 + |p|^2} + \log(|p| + \sqrt{1 + |p|^2})\}$, $|p|^2 = x^2 + y^2$. $\rho^2(p)$ is not convex, on the other hand $Hess(|p|^2) = 2/(1 + |p|^2) \cdot (dx^2 + dy^2)$, that is, $|p|^2$ is strictly convex, Note that $K(p)$ has the same order as $1/\{\rho(p)^3\}$ at infinity ($\rho(p) \rightarrow \infty$). Hence $\int_0^\infty s \cdot \bar{K}(s) ds$ diverges.

We observed that ρ^2 is not always strictly convex. However, we can find a new metric g^* from a conformal change of the given g such that ρ^{*2} is strictly convex.

PROPOSITION 4. *Let (M, g, o) be a Riemannian manifold with a pole. Assume that the radial curvature K is bounded above by a suitable smooth function of ρ . Then there is a continuous function $f: \mathbf{R} \rightarrow \mathbf{R}$ such that $f \circ \rho \in C^\infty(M)$ and (1) (M, g^*, o) is also a Riemannian manifold with a pole o , where $g^* = e^{2f \circ \rho} g$, (2) the radial curvature of g^* is nonpositive and hence (3) the square of the distance function ρ^{*2} is strictly convex with respect to g^* .*

Before proving this, we show following two statements by considering geometrical aspects of a metric g^* defined by $g^* = e^{2f \circ \rho} g$.

PROPOSITION 5. *Let γ is a normal g -geodesic issuing from o . Then a curve: $s \rightarrow \gamma(t(s))$ is a normal g^* -geodesic, where $t(s)$ is the inverse function of $s(t) = \int_0^t e^{f(u)} du$.*

PROOF. We apply the formulae of the covariant derivative with respect to a conformal change $g^* = e^{2f \circ \rho} g$ ([5]) to the curve $c(s) = \gamma(t(s))$:

$$\begin{aligned} \nabla_X^* Y &= \nabla_X Y + d\sigma(X)Y + d\sigma(Y)X - g(X, Y) \text{grad } \sigma, \\ \frac{\nabla^*}{ds} Y &= \frac{\nabla}{ds} Y + \frac{d}{ds} \sigma(c(s))Y + (Y\sigma)\dot{c}(s) - g(\dot{c}(s), Y) \text{grad } \sigma \end{aligned} \tag{4}$$

$(\sigma = f \circ \rho)$.

Since $g^*(\dot{c}(s), \dot{c}(s)) = 1$, we have $(\nabla^*/ds)\dot{c} \perp \dot{c}$ by covariantly differentiating both sides. Let $Y_i, 1 \leq i \leq n$, be orthonormal g -parallel vector fields such that $Y_n = \dot{\gamma}$. We shall show $(\nabla^*/ds)\dot{c}(s) \perp Y_i(t(s)), 1 \leq i \leq n-1$. By covariantly differentiating $g^*(\dot{c}(s), Y_i(t(s))) = 0$, we have

$$0 = g^*\left(\frac{\nabla^*}{ds}\dot{c}(s), Y_i(t(s))\right) + g^*\left(\dot{c}(s), \frac{\nabla^*}{ds} Y_i(t(s))\right).$$

From (4) together with the relations $\dot{c}(s) = (dt/ds)\dot{\gamma}$ and $\text{grad } \rho(s(t)) = \dot{\gamma}(t)$,

$$\frac{\nabla^*}{ds} Y_i(t(s)) = \frac{d}{ds}(f \circ \rho)(c(s))Y_i(s).$$

Hence we derive that $g^*((\nabla^*/ds)\dot{c}(s), Y_i(s)) = -g^*(\dot{c}(s), (\nabla^*/ds)Y_i(s)) = 0$. Thus, we have $(\nabla^*/ds)\dot{c}(s) \perp Y_i(s), 1 \leq i \leq n$, that is, $(\nabla^*/ds)\dot{c}(s) = 0$.

NOTE. If $\lim_{t \rightarrow \infty} \int_0^t e^{f(u)} du = +\infty$, $c(s)$ is defined on the whole \mathbf{R} and therefore $\exp: T_o M \rightarrow M$ is a diffeomorphism with respect to g^* , that is, (M, g^*) has a pole o . The distance function $\rho^*(\cdot) = d^*(\cdot, o)$ with respect to g^* is given by

$$\rho^*(p) = \int_0^{\rho(p)} e^{f(u)} du, \quad p \in M.$$

Now we shall consider the radial curvature of g^* . Let R and R^* be the curvature tensors of g and g^* respectively. Then we have ([5])

$$\begin{aligned} g^*(R^*(X, Y)Y, X) &= e^{2f \circ \rho} g(R(X, Y)Y, X) + 2S_{f \circ \rho}(X, Y)g^*(X, Y) \\ &\quad - S_{f \circ \rho}(Y, Y)g^*(X, X) - S_{f \circ \rho}(X, X)g^*(Y, Y) \end{aligned} \tag{5}$$

where

$$S_\sigma = \text{Hess}(\sigma) - d\sigma \otimes d\sigma + \frac{1}{2} \|grad \sigma\|^2 g, \quad \sigma \in C^\infty(M).$$

The radial curvature $K^*(Y \wedge grad \rho)$ at p ($Y \perp grad \rho$) with respect to g^* is written as

$$\begin{aligned} K^*(Y \wedge grad \rho) &= e^{-2f \circ \rho} \{K(Y \wedge grad \rho) - f''(\rho(p)) \\ &\quad - f'(\rho(p)) \text{Hess}(\rho)(Y, Y) / \|Y\|^2\}. \end{aligned} \tag{6}$$

The above formula is obtained as follows. Since $Y \perp grad \rho$,

$$\begin{aligned} K^*(Y \wedge grad \rho) &= \frac{1}{e^{4f \circ \rho} \|Y\|^2 \cdot \|grad \rho\|^2} g^*(R^*(Y, grad \rho) grad \rho, Y) \\ &= e^{-2f \circ \rho} \{K(Y \wedge grad \rho) - S_{f \circ \rho}(grad \rho, grad \rho) / \|grad \rho\|^2 \\ &\quad - S_{f \circ \rho}(Y, Y) / \|Y\|^2\}. \end{aligned}$$

On the other hand, $S_{f \circ \rho} = f' \text{Hess}(\rho) + \{f'' - f'^2\} d\rho \otimes d\rho + 1/2(f'^2)g$, hence we have (6).

PROPOSITION 6. *There is a function $f \circ \rho \in C^\infty(M)$ such that the radial curvature is nonpositive everywhere with respect to $g^* = e^{2f \circ \rho} g$.*

PROOF. By the assumption of Proposition 4, we can choose smooth functions $\bar{K}(t)$ from \mathbf{R}^+ to \mathbf{R} which satisfies

$$\bar{K}(t) \geq \max\{0, \text{radial curvature at } x, \rho(x)=t\}.$$

Set $\bar{H}(t) = -\int_0^t \bar{K}(t) dt$, then \bar{H} is also smooth and satisfies that

$$\bar{H}(t) \leq \min\{\text{Hess}(\rho)(Y, Y) \text{ at } x, \rho(x)=t, Y \in M_x, \|Y\|=1\}.$$

The nonnegative function $\bar{u}(t) = \exp\left(-\int_0^t \bar{H} dt\right) \cdot \int_0^t \bar{K}(t) \exp\left(\int \bar{H} dt\right) dt$ is a solution of $d\bar{u}/dt + \bar{H}\bar{u} - \bar{K} = 0$. Then we have for \bar{u} ,

$$\begin{aligned} & \frac{d\bar{u}}{dt}(t) + \bar{u}(t) \text{Hess}(\rho)(Y, Y) / \|Y\|^2 - K(Y \wedge \text{grad } \rho) \\ & = \bar{u}(t) \{ \text{Hess}(\rho)(Y, Y) / \|Y\|^2 - \bar{H}(t) \} + \{ \bar{K}(t) - K(Y \wedge \text{grad } \rho) \} \leq 0, \end{aligned}$$

for each $Y \in M_x$, $\rho(x) = t$. Therefore, if we set $f(t) = \int_0^t \bar{u}(t) dt$, then (M, g^*) , $g^* = e^{2f \circ \rho} g$ has nonpositive radial curvature from (6).

From these propositions, we have a required function $f \circ \rho$ in Proposition 4, since $\lim_{t \rightarrow \infty} \int_0^t e^{f(a)} da = \infty$ by $f' = \bar{u} \geq 0$. Thus Proposition 4 is proved.

At the last part of 2, we find a necessary condition for the existence of a strictly convex radial function, by a group-theoretical version. The following proposition is a generalization of E. Cartan's fixed point theorem [2].

PROPOSITION 7 (Fixed Point Theorem). *Let (M, o) be a Riemannian manifold with a pole o . Let K be a compact Lie group which acts on M as isometries. If there is a strictly convex increasing radial function $f \circ \rho$, then K has a common fixed point.*

REMARK. If M is of negative curvature, then ρ^2 is strictly convex by comparing M with a Euclidean space. Thus we have the well known E. Cartan's fixed point theorem [2]: A compact Lie group which acts as isometries on a simply connected complete Riemannian manifold of negative curvature has a common fixed point.

PROOF. Let dk denote the Haar measure on K , normalized by $\int_K dk = 1$. Consider the real function F on M given by $F(x) = \int_K f \circ \rho(k \cdot x) dk$. Then F is a nonnegative continuous function. Since $f \circ \rho$ is exhaustion and the orbit of o is compact, there is a ball $B_r(o)$ such that $F(x) > F(o)$ for all $x \in B_r(o)$. The closure of $B_r(o)$ contains a minimum point x_o for F . The point x_o is also a minimum for F on M . Since $F(k \cdot x_o) = F(x_o)$ for $k \in K$, in order to prove the existence of the fixed point, it is sufficient to show that $F(x) > F(x_o)$ if $x \neq x_o$. But this is derived by the strictly convexity of F , since $F(\gamma(t))'' = \int_K \{ f \circ \rho(k \cdot \gamma(t)) \}'' dk$ for every geodesic γ .

3. Let M be a complete open Kähler manifold. As in function theory, the order $\gamma(f)$ of a holomorphic function f is defined by

$$\gamma(f) = \limsup_{r \rightarrow +\infty} \log M(f, r) / \log r,$$

where $M(f, r) = \sup\{|f(x)|; x \in M, d(o, x) = r, o \text{ is a fixed point}\}$ [7]. The definition of $\gamma(f)$ does not depend on the choice of o . If $\gamma(f)$ is positive finite, then for each $\varepsilon > 0$, there are $C > 0$ and $\nu > 0$ such that $\gamma(f) \leq \nu < \gamma(f) + \varepsilon$ and $|f(x)| \leq C(1 + \rho(x))^\nu$ for all $x \in M$ ($\rho(x) = d(x, o)$).

We discuss some aspects of $\gamma(f)$.

Let (M, o) be a Kähler manifold with a pole o and (N, p) a model space, $\dim M = \dim N = n$, which satisfy the radial curvature $K_M(t) \leq$ the radial curvature $K_{N(o)}$ for all $t > 0$. By a *model* we mean a Riemannian manifold (N, p) with a pole p such that every linear isometry $\phi: N_p \rightarrow N_p$ is realized as the differential of an isometry $\Phi: N \rightarrow N$ ([1]). Let $V_M(r)$ and $V_N(r)$ be the volumes of the open balls $B_M(r)$ and $B_N(r)$ of radius r around o and p in M and N respectively. Note that by the sub-mean value property, $V_M(r) \geq V_N(r)$.

Now we show the following

PROPOSITION 8. Assume that $V_M(r) \sim r^\alpha$, $V_N(r) \sim r^\beta$, $\beta \geq 1$ ($r \rightarrow \infty$). If a holomorphic function f has $\gamma(f) < 1 + (\beta - \alpha)/2$, then $df = 0$ at o .

REMARK. If (M, o) is of nonpositive curvature and $\alpha < 2n + 2$ in the above proposition, then a bounded holomorphic function is constant, since every point gives a pole. Note that $V_N(r) \sim r^{2n}$ for $(N, p) = (\mathbb{C}^n, o)$ with a flat metric.

Before the proof of the proposition, we have some lemmas.

LEMMA (Sub-mean-value Property). Let ϕ be a continuous nonnegative subharmonic function on M , then

$$\int_{B_M(r)} \phi \geq V_N(r)\phi(o) \quad \text{for all } r > 0.$$

For the proof, see Theorem B, [1].

LEMMA (Integral Inequality of the Laplacian). Assume that $(d/dr)V_N(r)$ is an increasing function. Let f be a nonnegative subharmonic function. Then for all λ , $0 < \lambda < 1$, there is a constant $\gamma = \gamma_\lambda > 0$ such that

$$\int_{B_M(\lambda r)} \Delta f \leq \frac{\gamma}{r^2} \int_{B_M(r)} f. \tag{7}$$

PROOF. Since $f \geq 0$, we have, from (3.6) in [1]

$$\int_{B_M(r)} \left[\Delta f \left(\int_{t=\rho}^{t=r} \frac{dt}{v_N(t)} \right) \right] dv \leq \frac{1}{v_N(r)} \int_{S_M(r)} f d\omega(r),$$

which implies

$$\int_0^r \int_t^r \left(\int_{S_M(t)} \Delta f \, d\omega(t) \right) \frac{ds}{v_N(s)} \, dt \leq \frac{1}{v_N(r)} \int_{S_M(r)} f \, d\omega(r),$$

where $v_N(r)=v(r)$ denotes the volume of the r -sphere $S_N(r)$ around p in N .

By using Fubini's theorem with respect to s and t on the left hand side, we have

$$\begin{aligned} \frac{1}{v(r)} \int_{S(r)} f \, d\omega(r) &\geq \int_0^r \left[\int_0^s \left(\int_{S(t)} \Delta f \, d\omega(t) \right) dt \right] \frac{ds}{v(s)} \\ &= \int_0^r \left(\frac{1}{v(s)} \int_{B(s)} \Delta f \right) ds. \end{aligned}$$

Multiply by $v(r)$ and integrate relative to r . Then

$$\int_{B(u)} f \leq \int_0^u v(r) \left[\int_0^r \left(\frac{1}{v(s)} \int_{B(s)} \Delta f \right) ds \right] dr.$$

Since $\Delta f \geq 0$,

$$\int_0^r \frac{1}{v(s)} \left(\int_{B(s)} \Delta f \right) ds \geq \int_{\sqrt{\lambda}r}^r \frac{1}{v(s)} \left(\int_{B(\sqrt{\lambda}r)} \Delta f \right) ds = \left(\int_{B(\sqrt{\lambda}r)} \Delta f \right) \cdot \int_{\sqrt{\lambda}r}^r \frac{ds}{v(s)}$$

and that

$$\begin{aligned} \int_0^u v(r) \left(\int_0^r \frac{1}{v(s)} \int_{B(s)} \Delta f \, ds \right) dr &\geq \int_0^u v(r) \left(\int_{B(\sqrt{\lambda}r)} \Delta f \right) \left(\int_{\sqrt{\lambda}r}^r \frac{ds}{v(s)} \right) dr \\ &\geq \int_{\sqrt{\lambda}u}^u \left(v(r) \cdot \int_{\sqrt{\lambda}r}^r \frac{ds}{v(s)} \right) dr \cdot \left(\int_{B(\lambda u)} \Delta f \right) \geq (1-\sqrt{\lambda})(1-\lambda) \frac{u^2}{2} \int_{B(\lambda u)} \Delta f, \end{aligned}$$

where the last inequality follows from $v_N(r)$ being increasing. Hence we obtain the inequality (7).

LEMMA (Cauchy's inequality for derivatives of holomorphic functions). For each holomorphic function f on M ,

$$\|df\|^2(0) \leq \frac{\gamma}{V_N(r/2)r^2} \int_{B_M(r)} |f|^2.$$

PROOF. Since $\Delta|f|^2 = \|df\|^2$ and $\Delta\|df\|^2 = \nabla^2 df\|^2$, from above lemmas,

$$\begin{aligned} \|df\|^2(0) &\leq \frac{1}{V_N(r/2)} \int_{B_M(r/2)} \|df\| \\ &= \frac{\gamma}{V_N(r/2)r^2} \int_{B(r)} |f|^2, \quad \text{where } \gamma = \gamma_{1/2}. \end{aligned}$$

PROOF OF PROPOSITION 8. Since $\gamma(f) < 1 + 1/2(\beta - \alpha)$, there is $\nu > 0$ such that

$\gamma(f) < \nu < 1 + (\beta - \alpha)/2$, hence we have $|f(x)| < C(1 + \rho(x))^\nu$ for some $C > 0$. Then, from the above,

$$\|df\|^2(o) \leq \frac{\gamma}{V_N(r/2)r^2} \int_{B(r)} |f|^2 \leq \frac{\gamma C^2}{V_N(r/2)r^2} (1+r)^{2\nu} V_M(r) \sim r^{(2\nu-2+\alpha-\beta)}.$$

Letting $r \rightarrow \infty$, we have $df=0$ at o .

As an application of the proposition, we have the following

COROLLARY 9. *Let $F=(f^1, \dots, f^N); M \rightarrow \mathbb{C}^N$ be a holomorphic mapping. If $\sum_{j=1}^n \gamma(f^{i_j}) < n - n(\alpha - \beta)/2$ for each $1 \leq i_1 < \dots < i_n \leq N$, then F is not of maximal rank at o .*

Moreover, if M is a Stein manifold and $F; M \rightarrow \mathbb{C}^N$ is a proper holomorphic imbedding, then $\sum_{j=1}^n \gamma(f^{i_j}) \geq n - n(\alpha - \beta)/2$ for some $1 \leq i_1 < \dots < i_n \leq N$.

PROOF. Consider the holomorphic n -forms $df^{i_1} \wedge \dots \wedge df^{i_n}$, $1 \leq i_1 < \dots < i_n \leq N$. From the proposition, we have an estimate of the norm of $df^{i_1} \wedge \dots \wedge df^{i_n}$;

$$\begin{aligned} \|df^{i_1} \wedge \dots \wedge df^{i_n}\|^2(o) &\leq \prod_{j=1}^n \|df^{i_j}\|^2(o) \leq \prod_{j=1}^n \frac{\gamma}{V_N(r/2)r^2} \int_{B(r)} |f^{i_j}|^2 \\ &= \frac{\gamma^n}{V_N(r/2)^n \cdot r^{2n}} \prod_j \int_{B(r)} |f^{i_j}|^2 \\ &\leq \gamma^n \prod C_j^2 \cdot \frac{(1+r)^{2\sum \nu_j} \cdot V_M(r)^n}{V_N(r/2)^n r^{2n}}, \end{aligned}$$

where $\nu_j > 0$, $j=1, \dots, n$, satisfy $\gamma(f^{i_j}) < \nu_j < \gamma(f^{i_j}) + \varepsilon_j$ and $\sum_j \nu_j < n - n(\alpha - \beta)/2$. By letting $r \rightarrow \infty$, we have $df^{i_1} \wedge \dots \wedge df^{i_n} = 0$ at o .

The last statement is easily derived from the above argument, since the F is of maximal rank everywhere.

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Institute of Mathematics
University of Tsukuba
Sakura-mura Niihari-gun
Ibaraki 305 JAPAN