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PROPERTIES OF AN L-CARDINAL

By

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When we study the set theory ZF(aa), (Ref. [1] or [3]) it may be natural to consider a cardinal κ such that for every formula in the language of usual set theory,

$$R(\kappa)\models aa\alpha\phi \longleftrightarrow V\models aa\alpha\phi .$$

Let κ be measurable, M a transitive isomorph of V^{κ}/U where U is a normal ultrafilter on κ, and j the canonical elementary embedding of V into M. If "aa" is interpreted by the closed umbounded filter of κ and $j(\kappa)$ respectively, in M,

$$R(\kappa)\models aa\alpha\phi \longleftrightarrow R(j(\kappa))\models aa\alpha\phi .$$

Therefore measurability is sufficient to show the consistency of the desired situation. But when we want κ to have this property in full V, a new cardinal axiom is needed.

1. Definitions of an L-cardinal and its basic properties.

DEFINITION. Let ϕ be a formula in set theory whose constants are all in $R(\kappa)$, and λ be an ordinal$\geqq\kappa$.

a) A cardinal κ *is a* $(\phi\text{-}\lambda)$*-cardinal,* if there exists an elementary embedding $j: V\rightarrow M$ such that

(i) $j(\kappa)>\lambda$ and κ is the least ordinal moved by j,

(ii) for every x in $R(j(\kappa))^{M}$, $M\models\phi(x)\rightarrow V\models\phi(x)$.

b) κ *is a* ϕ*-cardinal* if for every $\lambda>\kappa$, κ is a $(\phi-\lambda)$-cardinal.

c) κ *is a* Σ_{n}*-cardinal* if for every Σ_{n} formula ϕ, κ is a ϕ-cardinal.

d) Let A be a set of formulas, κ *is a* $(A-\lambda)$*-cardinal* if for every formula in A, κ is a $(\phi-\lambda)$-cardinal.

e) κ *is an* L*-cardinal* if for every formula ϕ, κ is a ϕ-cardinal.

The axiom of an L-cardinal definitely cannot be formulated in ZFC. However, all the arguments can be carried out in ZFC within some $R(\kappa)$ where κ is inaccessible.

The first lemma is trivial but basic in the development.

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LEMMA 1.1. *Let ϕ be a formula such that $\{\alpha<\kappa \mid \phi(\alpha)\}$ is unbounded in κ and all constants are in $\boldsymbol{R}(\kappa)$. If κ is a $(\phi-\lambda)$-cardinal, there exists an ordinal $\beta>\lambda$ such that $\phi(\beta)$ holds.*

PROOF. Let j be an elementary embedding from $\boldsymbol{V}$ into $\boldsymbol{M}$ as in the definition of $(\phi-\lambda)$-cardinal. By assumption,

$$V \models \forall\alpha<\kappa \exists\beta<\kappa \{\alpha<\beta \wedge \phi(\beta)\}$$

Hence,

$$M \models \forall\alpha<j(\kappa) \exists\beta<j(\kappa) \{\alpha<\beta \wedge \phi(\beta)\}$$

Since $\lambda<j(\kappa)$, there is a $\beta<j(\kappa)$ such that $M\models\phi(\beta)$ and $\beta>\lambda$. Also $V\models\phi(\beta)$ because $\beta\in \boldsymbol{R}(j(\kappa))^{M}$.

COROLLARY 1.2. *Let κ be a $\{\phi, \neg\phi\}$-cardinal. If $\{\alpha<\kappa \mid \phi(\alpha)\}$ is closed unbounded in κ, then $\{\alpha \mid \phi(\alpha)\}$ is closed unbounded in $\boldsymbol{OR}$.*

PROOF. Assume $\{\alpha<\kappa \mid \phi(\alpha)\}=\boldsymbol{C}$ is closed unbounded in κ. By LEMMA 1.1. $\{\alpha \mid \phi(\alpha)\}=\boldsymbol{C}'$ is an unbounded class. Let α be a limit point of $\boldsymbol{C}'$ and $j: \boldsymbol{V}\to\boldsymbol{M}$ be an elementary embedding that satisfies the definition of $(\{\phi, \neg\phi\}-\alpha)$-cardinal.

$$V \models \forall\beta<\alpha \exists\gamma<\alpha \{\beta<\gamma \wedge \phi(\gamma)\} \quad \text{implies}$$

$$M \models \forall\beta<\alpha \exists\gamma<\alpha \{\beta<\gamma \wedge \phi(\gamma)\}.$$

As $\boldsymbol{C}$ is closed, $M\models\forall\alpha<j(\kappa)\{\forall\beta<\alpha\exists\gamma<\alpha\{\beta<\gamma\wedge\phi(\gamma)\}\to\phi(\alpha)\}$. Hence $M\models\phi(\alpha)$. Also $V\models\phi(\alpha)$.

For simplicity we consider a fixed formula $\Phi_0(x)\equiv\exists\alpha\{x=R(\alpha)\}$. If κ is a $(\Phi_0-\lambda)$-cardinal, $\boldsymbol{R}(j(\kappa))^{M}=\boldsymbol{R}(j(\kappa))$. Of course some results follow by the weaker assumption that κ is a $(\phi-\lambda)$-cardinal which is reduced from the fact that κ is a $(\Phi_0-\lambda)$-cardinal.

LEMMA 1.3. *If κ is a Φ_0-cardinal, $\{\alpha \mid \alpha$ is strongly inaccessible$\}$ is a proper class.*

PROOF. Since κ is measurable, $\{\alpha<\kappa \mid \alpha$ is strongly inaccessible$\}$ is unbounded in κ. Since $\boldsymbol{R}(j(\kappa))^{M}=\boldsymbol{R}(j(\kappa))$, the strongly inaccessibles in $\boldsymbol{M}$ which are less than $j(\kappa)$ are also strongly inaccessible in $\boldsymbol{V}$. Now the conclusion is clear by Corollary 1.2.

LEMMA 1.4. *If κ is a $(\Phi_0-\kappa)$-cardinal, κ is the κ-th measurable.*

PROOF. Let $j: \boldsymbol{V}\to\boldsymbol{M}$ be an associated embedding. $j(\kappa)>\kappa$ and $\lim(\kappa)$ implies

$j(\kappa)>\kappa+\omega$. Hence $R(\kappa+2)^M=R(\kappa+2)$. Let U be a normal ultrafilter on κ. Since $U\in R(\kappa+2)$, $M\models(U$ is a normal ultrafilter on $\kappa)$. Hence $M\models(\kappa$ is measurable). As usual, define U'

$$X\in U' \text{ iff } X\subset\kappa\wedge\kappa\in j(X)$$

U' is a normal ultrafilter on κ and $\{\alpha<\kappa|\alpha$ is measurable$\}\in U'$.

COROLLARY 1.5. *If κ is a Φ_0-cardinal, $\{\alpha|\alpha$ is measurable$\}$ is a proper class.*

PROOF. By Corollary 1.2 and Lemma 1.4.

1.3 and 1.5 follow from the assumption of an extendible cardinal. 1.4 follows from 2^κ-supercompactness. As our definition of $(\phi-\lambda)$-cardinal does not assume ${}^\lambda M\subset M$, it is not known whether κ carries a normal ultrafilter on $P_\kappa\lambda$. Even it is not clear whether κ is λ-compact (Ref. [2]). The next definition is also due to [2].

DEFINITION. *Σ_n relativizes down to $R(\kappa)$ iff for each formula $\phi\in\Sigma_n$,*

$$\forall a\in R(\kappa)\{\phi(a)\to R(\kappa)\models\phi(a)\}.$$

LEMMA 1.6. *If κ is a Σ_n-cardinal, Σ_n relativizes down to $R(\kappa)$.*

PROOF. By induction on n. Π_2 relativizes down to $R(\kappa)$ since κ is strongly inaccessible (Ref. [4]). Thus Σ_1 relativizes down to $R(\kappa)$. Assume $\phi\in\Sigma_n$ and $a\in R(\kappa)$, $\phi(a)$ holds. $(n\geqq 2)$ There is a Σ_{n-1} formula ψ such that $\phi(a)\equiv\exists x\psi(x, a)$. Since $\phi(a)$, for some b, $V\models\psi(b, a)$. Choose λ large enough to $b\in R(\lambda)$. Let $j: V\to M$ be an associated embedding of $(\neg\psi-\lambda)$-ness of κ. (Note that $\neg\psi\in\Sigma_n$.) Since $b\in R(j(\kappa))^M$ and $V\models\psi(b, a)$, $M\models\psi(b, a)$. (As Φ_0 is a Σ_2 formula, we can assume $b\in M$.) But in V, Σ_{n-1} relativizes down to $R(\kappa)$. Hence in M, Σ_{n-1} relativizes down to $R(j(\kappa))$. Therefore $M\models(R(j(\kappa))\models\psi(b, a))$. Hence $M\models R(j(\kappa))\models\phi(a))$. By elementarity of j and $j(a)=a$, $R(\kappa)\models\phi(a)$ in V.

Now the followings are all clear.

THEOREM 1.7. *If κ is an L-cardinal, Σ_ω relativizes down to $R(\kappa)$.*

COROLLARY 1.8. *If κ is an L-cardinal, $R(\kappa)$ is an elementary substructure of V.*

COROLLARY 1.9. *If κ is an L-cardinal and α is a definable cardinal, then $\alpha<\kappa$.*

Note that: If κ is supercompact, Σ_2 relativizes down to $R(\kappa)$. If κ is extendible, Σ_3 relativizes down to $R(\kappa)$. (Ref. [2])

Also recall the notion of "*ghost cardinal*" of M. Takahashi. It is the least cardinal not definable by $\tilde{\Delta}_1^2$-formula. Of course an L-cardinal is not definable in set theory, ZFC.

COROLLARY 1.10. *Let κ be an L-cardinal and ϕ be a formula whose constants are all in $R(\kappa)$. If there exists an ordinal $\gamma \geqq \kappa$, such that $\phi(\gamma)$ holds, then $\{\alpha \mid \phi(\alpha)\}$ is a proper class and $\{\alpha<\kappa \mid \phi(\alpha)\}$ is unbounded in κ.*

PROOF. By Corollary 1.2, it suffices to show $\{\alpha<\kappa \mid \phi(\alpha)\}$ is unbounded in κ. If not, there is an $\alpha<\kappa$ such that $\forall\beta<\kappa(\alpha<\beta\rightarrow\neg\phi(\beta))$. By the above Theorem, $\forall\beta<\kappa(\alpha<\beta\rightarrow R(\kappa)\models\neg\phi(\beta))$. Then $R(\kappa)\models\forall\beta(\alpha<\beta\rightarrow\neg\phi(\beta))$. Using Corollary 1.9, we have $\forall\beta(\alpha<\beta\rightarrow\neg\phi(\beta))$. Contradicting $\phi(\gamma)$.

1.7-1.10 are too strong and give us suspicion about consistency.

2. Cohen extension and an L-cardinal.

Once L-cardinal is defined, many problems are raised. If V is a model of ZFC$+\exists\kappa$: L-cardinal, is there a Cohen extension of V where ZFC$+\exists\kappa$: L-cardinal $+$G.C.H. hold? Is not $j(\kappa)$ necessarily measurable? (When κ is extendible, $j(\kappa)$ is always measurable.) Can an L-cardinal be strongly compact?

All these questions are unclear now. We need some technics to preserve an L-cardinal. We only get a quite easy fact that is not useful to solve the above problems.

LEMMA 2.1. *If $|P|<\kappa$ and κ is a $((\Vdash\phi)-\lambda)$-cardinal, then κ is a $(\phi-\lambda)$-cardinal in $V[G]$.*

PROOF. We can assume $P\in R(\kappa)$. Let $j: V\rightarrow M$ be an elementary embedding that witnesses κ is a $((\Vdash\phi)-\lambda)$-cardinal. We extend j to $\tilde{j}$ as usual.

$$\tilde{j}(K_G(\underset{\sim}{x}))=K_G(j(\underset{\sim}{x}))$$

(We use the notations of [7].) $\tilde{j}$ is an elementary embedding of $V[G]$ into $M[G]$. (Ref. [5]) If $x\in R(j(\kappa))^{M[G]}$, there is a name $\underset{\sim}{x}\in R(j(\kappa))^M$ such that $K_G(\underset{\sim}{x})=x$. $M[G]\models\phi(x)$ iff $M\models(p\Vdash\phi(\underset{\sim}{x}))$ for some $p\in G$. The latter implies $\exists p\in G(p\Vdash\phi(\underset{\sim}{x}))$ in V. Therefore $V[G]\models\phi(x)$.

THEOREM 2.2. *If κ is an L-cardinal in V and $|P|<\kappa$, κ is an L-cardinal in $V[G]$.*

Using Theorem 2.2, we consider the relation between an L-cardinal and strongly compact cardinals.

THEOREM 2.3. *Let κ be an L-cardinal.*
(i) *If κ is strongly compact, κ is the κ-th strongly compact.*
(ii) *If κ is not strongly compact, there is no strongly compact cardinal greater tnan κ, and it is consistent that there is an L-cardinal and there is no strongly compact cardinal.*

PROOF. (i) By Corollary 1.10.
(ii) At first we assert that κ is not a limit of strongly compacts. For a measurable cardinal that is a limit of strongly compacts is strongly compact and κ is clearly measurable. (Ref. [6])

Thus there is no strongly compact above κ by Corollary 1.10. And there exists a regular cardinal $\alpha<\kappa$ such that there is no strongly compact cardinal between α and κ. We use the forcing condition that collapses α to ω_1. In the extended universe there is no strongly compact cardinal and κ remains an L-cardinal by Theorem 2.2.

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