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ON A MEAN-VALUE THEOREM CONCERNING DIFFERENCES OF TWO K-TH POWERS

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1. Introduction. For positive integers k, r, let $t_k(r)$ denote the number of pairs $(m, n)\in N\times Z$ with $m^k-|n|^k=r$. To study the average order of $t_k(r)$, one considers the summatory function $T_k(x)=\sum_{1\leq r\leq x} t_k(r)$ (x a large real variable). It has been proved by E. Krätzel [3] that, for $k\geqq 3$ and some small $\varepsilon_0>0$,

$$T_k(x)=c_1(k)x^{2/k}+c_2(k)x^{1/(k-1)}+c_3(k)F_k(x)x^{1/k-1/k^2}+O(x^{2/(3k)-\varepsilon_0}), \qquad (1)$$

where

$$c_1(k)=\Gamma^2\left(\frac{1}{k}\right)\left(2k\cos\left(\frac{\pi}{k}\right)\Gamma\left(\frac{2}{k}\right)\right)^{-1}, \qquad c_2(k)=2\zeta\left(\frac{1}{k-1}\right)k^{-1/(k-1)},$$

$$c_3(k)=\pi^{-1-1/k}\left(\frac{k}{2}\right)^{1/k-1}\Gamma\left(\frac{1}{k}\right), \qquad F_k(x)=\sum_{n=1}^{\infty} n^{-1-1/k}\sin\left(2\pi n x^{1/k}+\frac{\pi}{2k}\right),$$

hence $F_k(x)=O(1)$ and $F_k(x)=\Omega_{\pm}(1)$ as $x\to\infty$. For $k=2$, the problem is essentially equivalent to the Dirichlet divisor problem, since (cf. e.g. [4])

$$T_2(x)=D(x)-2D\left(\frac{x}{2}\right)+2D\left(\frac{x}{4}\right), \qquad D(x):=\sum_{0<mn\leq x} 0.$$

2. Statement of result. In this note, we apply the modern technique for the estimation of exponential sums (the "discrete Hardy-Littlewood method", due to Bombieri, Iwaniec, Mozzochi, Huxley and Watt), together with a refined analysis of the special functions involved, in order to improve the error term in the above estimate.

THEOREM. *For any real number* $k\geqq 2$, let $T_k(x)$ *denote the number of lattice points* $(m, n)\in N\times Z$ *with* $0<m^k-|n|^k\leqq x$. *If* $k\geqq 38/13$, *we have the asymptotic*

$$T_k(x)=c_1(k)x^{2/k}+c_2(k)x^{1/(k-1)}+c_3(k)F_k(x)x^{1/k-1/k^2}+\Delta_k(x)$$

with

$$\Delta_k(x)=O(x^{25/(38k)+\varepsilon}) \qquad \textit{for any } \varepsilon>0.$$

Consequently, for $k>38/13$,

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$$T_k(x)=c_1(k)x^{2/k}+c_2(k)x^{1/(k-1)}+(c_3(k)F_k(x)+o(1))x^{1/k-1/k^2}.$$

3. Proof of the theorem. We start from Krätzel's formula ([3], p. 112)

$$\Delta_k(x)=2\sum_{x<m^k\leq 2x}\psi_1((m^k-x)^{1/k})-2\sum_{m^k\leq ax}\psi(N_k(m,x))+O(1). \tag{2}$$

Here $a=(2^{1/k}-1)^k$, $\psi(y)=y-[y]-1/2$, $m\in N$ throughout and the function $v=N_k(u,x)$ is implicitly defined (for positive real u and x) by the equation

$$(v+u)^k-v^k=x \quad (v>0). \tag{3}$$

(In the first sum of Krätzel's formula, ψ is corrected to ψ_1, where $\psi_1(y)=1/2$ for integer y and $\psi_1(y)=\psi(y)$ otherwise (cf. the evaluation of S_3 in [3], p. 116). Furthermore we note, that the proof of (2) does not require the supposition that k is an integer.)

Our first step is to reduce the length of the second sum in (2), in order to make it accessible to the method of exponent pairs. Throughout the sequel the abbreviation $X:=x^{1/k}$ is used.

LEMMA 1. *For* $0<\delta<a^{1/k}$ *define* $\lambda(\delta)$ *by* $\lambda^k-(\lambda-\delta)^k=1$, $\lambda>\delta$, *(hence* $\lambda\to\infty$ *for* $\delta\to 0$*), then*

$$\sum_{\delta^k x<m^k\leq ax}\psi(N_k(m,x))=-\sum_{2x<m^k\leq\lambda^k x}\psi_1((m^k-x)^{1/k})+O(1). \tag{4}$$

PROOF. Consider the planar domains

$$D_1=\{(u,v)\in\boldsymbol{R}^2\,|\,\delta X<u\leq\lambda X,\ 0\leq v\leq u-\delta X\},$$

$$D_2=\{(u,v)\in\boldsymbol{R}^2\,|\,2^{1/k}X<u\leq\lambda X,\ 0\leq v<(u^k-X^k)^{1/k}\},$$

$$D_3=\{(u,v)\in\boldsymbol{R}^2\,|\,a^{1/k}X<u\leq 2^{1/k}X,\ 0\leq v\leq u-a^{1/k}X\},$$

$$D_4=D_1\setminus(D_2\cup D_3)$$

and let M_j denote the number of lattice points in D_j, counting points at the u-axis with weight 1/2; denote by A_j the area of D_j ($j=1,\cdots,4$). Applying the Euler summation formula and estimating the remainder integrals to $O(1)$ by the second mean-value theorem, we obtain after straightforward computations

$$M_1=A_1-(\lambda-\delta)\psi(\lambda X)X-(\lambda-\delta)\psi(-\delta X)X+O(1),$$

$$M_2=A_2-(\lambda-\delta)\psi(\lambda X)X+\psi(2^{1/k}X)X\sum_{2^{1/k}<u\leq\lambda X}\psi_1((u^k-X^k)^{1/k})+O(1),$$

$$M_3=A_3-\psi(-a^{1/k}X)X-\psi(2^{1/k}X)X+O(1),$$

$$M_4=\sum_{-a^{1/k}X<m\leq-\delta X}N_k(-m,x)-\sum_{\delta X\leq m<a^{1/k}X}\psi(N_k(m,x))$$

$$=A_4+\psi(-a^{1/k}X)X-(\lambda-\delta)\psi(-\delta X)X-\sum_{\delta X<m\leq a^{1/k}X}\psi(N_k(m,\ x))+O(1).$$

Since $M_1=M_2+M_3+M_4$, this yields (4). ///

Thus (2) may be rewritten as

$$\Delta_k(x)=2\sum_{X<m\leq\lambda X}\psi_1((m^k-X^k)^{1/k})-2\sum_{m\leq\delta X}\psi(N_k(m,\ x))+O(1), \tag{5}$$

where δ is some sufficiently small positive constant and $\lambda=\lambda(\delta)$ as before.

To estimate the second sum we need a close analysis of the function $N_k(u,\ x)$ for $ux^{-1/k}$ small.

LEMMA 2. *For some* $\varepsilon_1>0$, *we have a series representation*

$$N_k(u,\ x)x^{-1/k}=\gamma_0(ux^{-1/k})^{-1/(k-1)}+\sum_{j=1}^{\infty}\gamma_j(ux^{-1/k})^{(jk-1)/(k-1)} \tag{6}$$

(*with* $\gamma_0=k^{-1/(k-1)}$) *valid for* $0<ux^{-1/k}\leqq\varepsilon_1$, *permitting iterated termwise differentiation with respect to* u *in this range.*

PROOF. Let us define new positive real variables t, w by the substitution $u=x^{1/k}t^{1-1/k}$, $v=N_k(u,\ x)=x^{1/k}t^{-1/k}w$. Entering this into (3), we get

$$(t+w)^k-w^k=t. \tag{7}$$

We put $H(t,\ w)=t^{-1}((t+w)^k-w^k)-1$ for $w>0$, $t\neq 0$, $t+w>0$, and $H(0,\ w)=kw^{k-1}-1$ for $w>0$. Then H is analytic for $w>0$, $t+w>0$, and satisfies $H(0,\ \gamma_0)=0$, $H_w(0,\ \gamma_0)\neq 0$. Hence, by the implicit function theorem for analytic functions, (7) can be solved (in some small interval to the right of $t=0$) to

$$w=\gamma_0+\sum_{j=1}^{\infty}\gamma_j t^j.$$

Inverting the above substitution, we complete the proof of lemma 2. ///

Applying the inequalities of Koksma and Erdös/Turán (cf. [**1**], p. 104 and p. 107) to the function ψ and the sequence $f(m):=N_k(m,\ x)$, $m\leqq\delta X$, we obtain

$$\sum_{m\leq\delta X}\psi(N_k(m,\ x))\ll XH^{-1}+\sum_{h=1}^{H}\frac{1}{h}|S_1(h)|, \tag{8}$$

where $H\geqq 1$ is a free integer parameter and

$$S_1(h):=\sum_{m\leq\delta X}e(-hN_k(m,\ x))=\sum_{j=1}^{J-1}\sum_{m_j<m\leq m_{j+1}}e(-hN_k(m,\ x))+O(1). \tag{9}$$

Here the summation interval $[1,\ \delta X]$ is divided into J subintervals of the form $(m_j,\ m_{j+1}]$ with $m_j:=2^j$ for $j=1,\cdots,J-1$, $m_J:=\delta X$ $(m_{J-1}<\delta X\leqq 2m_{J-1})$. To estimate the partial sums we use the method of exponent pairs (cf. [**6**]). From (6) we obtain for every $r\geqq 0$ and $ux^{-1/k}\leqq\delta$, $\delta\to 0+$,

$$\frac{d^{r+1}}{du^{r+1}}(-hN_k(u, x))=(-1)^r h\frac{\gamma_0}{k-1}C_r x^{1/(k-1)}u^{-k/(k-1)-r}(1+o(1))$$

where $C_r=\prod_{j=0}^{r-1}(k/(k-1)+j)$. Therefore $-hN_k(u, x)$ satisfies condition (3) of [6], p. 214 with $s=k/(k-1)$, if δ is sufficiently small. From this we infer for every exponent pair (α, β):

$$\sum_{m_j<m\leq m_{j+1}} e(-hN_k(m, x))\ll(hX^{k/(k-1)})^\alpha m_j^{\beta-\alpha k/(k-1)}, \qquad S_1(h)\ll h^\alpha X^\beta,$$

and with the choice $H:=[X^{(1-\beta)/(1+\alpha)}]$ in (8)

$$\sum_{m\leq\delta x^{1/k}}\psi(N_k(m, x))\ll X^{(\alpha+\beta)/(1+\alpha)}. \tag{10}$$

Recently Huxley and Watt [2] have proved, that for any $\varepsilon'>0$, $(9/56+\varepsilon', 37/56+\varepsilon')$ is an exponent pair. Applying the "A-step" two times followed by a "B-step", we get the exponent pair $(51/139+\varepsilon, 74/139+\varepsilon)$, $\varepsilon>0$. Inserting into (10) yields the desired bound $x^{25/(38k)+\varepsilon}$.

To estimate the first sum in (5) we split it two parts:

$$\sum_{X<m\leq\lambda X}\psi_1(f(m))=\sum_{X<m\leq(1+\rho)X}\psi_1(f(m))+\sum_{(1+\rho)X<m\leq\lambda X}\psi_1(f(m))=:S_2+S_3. \tag{11}$$

Here ρ is a sufficiently small constant and $f(y)=(y^k-X^k)^{1/k}$. Again the method of exponent pairs can be used to deal with S_2. Like in (8) and (9) we first obtain

$$S_2\ll XH^{-1}+\sum_{h=1}^{H}\frac{1}{h}|S_2(h)|, \tag{12}$$

with

$$S_2(h)=\sum_{X<m\leq(1+\rho)X}e(hf(m))\ll V+\sum_{j=1}^{J-1}\sum_{v_j<m-[X]\leq v_{j+1}}e(hf(m)),$$

where $V>1$ is a suitable large constant, $v_j:=2^jV$, $j=1, \cdots, J-1$, $(v_{J-1}<\rho X\leqq 2v_{J-1})$ and $v_J:=\rho X$. The behaviour of $f(y)=X((y/X)^k-1)^{1/k}$ for small $t:=y/X-1$, $(t<\rho)$ is described by its (absolut convergent) series representation $(a_0=k,\ b_0=k^{1/k})$:

$$f(y)=X((1+t)^k-1)^{1/k}=X\Big(t\sum_{j=0}^{\infty}a_jt^j\Big)^{1/k}=X\sum_{j=0}^{\infty}b_jt^{j+1/k}.$$

Hence for $r\geqq 0$

$$\frac{d^{r+1}}{dy^{r+1}}f(y)=(-1)^r\frac{b_0}{k}X^{(k-1)/k}C_r'(y-X)^{-(k-1)/k-r}(1+o(1)),$$

with $C_r'=\prod_{j=0}^{r-1}((k-1)/k+j)$. Introducing a new variable $v=y-[X]>V$ this reads

$$\frac{d^{r+1}}{dv^{r+1}}f(v+[X])=(-1)^r\frac{b_0}{k}X^{(k-1)/k}C'_r v^{-(k-1)/k-r}\Big(1+o(1)+O\Big(\frac{1}{V}\Big)\Big).$$

Therefore $hf(v+[X])$ satisfies condition (3) of [6], p. 214 with $s=(k-1)/k$, if ρ is sufficiently small and V is sufficiently large. We thus conclude for any exponent pair (α, β):

$$\sum_{v_j<v\leq v_{j+1}} e(hf(v+[X]))\ll(hX^{(k-1)/k}v_j^{-(k-1)/k})^\alpha v_j^\beta \quad\text{and}\quad S_2(h)\ll h^\alpha X^\beta.$$

The choice $(\alpha, \beta)=(51/139+\varepsilon, 74/139+\varepsilon)$ together with $H:=[X^{13/38}]$ and (12) yields

$$S_2\ll x^{25/(38k)+\varepsilon}. \tag{13}$$

It remains to estimate S_3. We use the inequalities of Erdös/Turán and Koksma once more to abtain

$$S_3\ll XH^{-1}+\sum_{h=1}^{H}\frac{1}{h}|S_3(h)| \tag{14}$$

with $H:=[X^{13/38}]$ and

$$S_3(h)=\sum_{(1+\rho)X<m\leq\lambda X} e(hf(m)).$$

Transforming $S_3(h)$ by the "Van der Corput step" (e.g. [7], p. 75, theorem 4.9), we derive

$$\begin{aligned}S_3=e\Big(-\frac{1}{8}\Big)(k-1)^{1/2}h^{q/2}X^{1/2}\sum_{\xi<u\leq\eta}\Phi(u)e(F(u))\\ +O(h^{-1/2}X^{1/2})+O(\log x)+O(h^{2/5}X^{2/5}),\end{aligned} \tag{15}$$

where $q=k/(k-1)$, $\xi=hf'(\lambda X)\gg h$, $\eta=hf'((1+\rho)X)\ll h$,

$$\Phi(u)=u^{-(k-2)/2(k-1)}(u^q-h^q)^{-(k+1)/(2k)}\ll h^{-q/2-1/2} \quad\text{and}\quad F(u)=-X(u^q-h^q)^{1/q}.$$

The new exponential sum in (15) is now dealt with by the following lemma, which is an easy consequence (derived in [5]) of Huxley's and Watt's deep estimate in [2].

LEMMA 3. *Let $c\in N$, $M\geqq 1$ and $T\geqq 1$ real parameters, F a real function six times continuously differentiable on $[M/2, 2^cM]$, satisfying in this interval $M^{-r}T\ll|F^{(r)}|\ll M^{-r}T$, $r=4, 5, 6$. Suppose that $M\ll T^{4/15}$. Then for any real $M'\in[M, 2^cM]$ and any $\varepsilon>0$,*

$$\sum_{M\leq u\leq M'} e(F(u))=O(M^{116/139}T^{9/278+\varepsilon})+O(M^{1091/1668}T^{32/417+\varepsilon}).$$

In our case the derivatives of $F(u)$ are of the form

$$F^{(r)}(u)=(q-1)Xh^q u^{1-q-r}(1-(h/u)^q)^{1/q-r}P_r((h/u)^q),$$

with

$$P_r(x)=\sum_{i=0}^{r-2} K_{i,r}x^i,$$

and

$$K_{0,4}=(1+q)(2+q) \qquad K_{0,5}=-(1+q)(2+q)(3+q)$$

$$K_{1,5}=-(1+q)(7-4q) \qquad K_{1,5}=(1+q)(29-11q^2)$$

$$K_{2,4}=(2-q)(3-q) \qquad K_{2,5}=-(1+q)(2-q)(23-11q)$$

$$K_{3,5}=(2-q)(3-q)(4-q)$$

$$K_{0,6}=(1+q)(2+q)(3+q)(4+q)$$

$$K_{1,6}=-(1+q)(2+q)(73-7q-26q^2)$$

$$K_{2,6}=(1+q)(329-129q-146q^2+66q^3)$$

$$K_{3,6}=-(1+q)(2-q)(163-129q+26q^2)$$

$$K_{4,6}=(2-q)(3-q)(4-q)(5-q).$$

Note that $0<h/u\leqq(1-\lambda^{-k})^{1/q}<1$. Therefore that assumptions of lemma 3 are verified with M a constant multiple of h and $T=hX$, if the polynomials P_r, $r=4, 5, 6$, have no zeros in $[0, 1]$, for $1<q\leqq 38/25$. (This can be checked in the following way: For $1<q\leqq q_r$, where $q_4=5/3$, $q_5=3/2$ and $q_6=7/5$, all derivatives of P_r have constant sign, hence it suffices to consider $P_r(0)$ and $P_r(1)$; in the remaining cases the polynomials P_r can be bounded from zero by replacing each coefficient by its smallest or largest value.) We thus conclude for any subinterval I of $[\xi, \eta]$ that

$$\sum_{u\in I} e(F(u))\ll h^{116/139}(hX)^{9/278+\varepsilon}+h^{1091/1668}(hX)^{32/417+\varepsilon}.$$

Applying summation by parts in (14) and inserting the result into (13) we obtain

$$S_3(h)\ll h^{51/139+\varepsilon}X^{74/139+\varepsilon}+h^{385/1668+\varepsilon}X^{481/834+\varepsilon} \quad \text{and} \quad S_3\ll X^{25/38+\varepsilon}.$$

Together with (5), (11) and (13), this completes the proof of our theorem.

Added in proof. By an application of a still more advanced version of Huxley's method, the authors have meanwhile improved the error term in the Theorem to

$$\Delta_k(x)=O(x^{7/11k}(\log x)^{45/22})$$

(which is valid for any real $k\geqq 2$). This is to be published in a subsequent paper.

References

[1] Hlawka, E., "Theorie der Gleichverteilung," Bibl. Inst., Mannheim-Wien-Zürich, 1979.
[2] Huxley, M. N. and Watt, N., Exponential sums and the Riemann zeta function, Proc. London Math. Soc. **57** (1988), 1-24.
[3] Krätzel, E., Mittlere Darstellung natürlicher Zahlen als Differenz zweier k-ter Potenzen, Acta Arith. **16** (1969), 111-121.
[4] Krätzel, E., Primitive lattice points in special plane domains and a related three-dimensional lattice point problem I., Forschungsergebnisse Friedrich-Schiller-Univ., N/87/11 (1987), 16 p.
[5] Müller, W. and Nowak, W. G., Lattice points in domains $|x|^p+|y|^p \leqq R^p$, Arch. Math. **51** (1988), 55-59.
[6] Phillips, E., The zeta-function of Riemann; further developments of Van der Corput's method, Quart. J. Oxford **4** (1933), 209-225.
[7] Titchmarsh, E. C., revised by Heath-Brown D. R., "The theory of the Riemann zeta-function," 2nd ed., Oxford, 1987.

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