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ON THE SOLVABILITY OF CONVOLUTION EQUATIONS IN $\mathcal{K}_M'$

By

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Abstract. Let $\mathcal{K}_M'$ be the space of distributions on R^n which grow no faster than $e^{M(kx)}$ for some $k>0$ where M is an increasing continuous function on R^n, and let $\mathcal{O}_C'(\mathcal{K}_M'; \mathcal{K}_M')$ be the space of convolution operators in $\mathcal{K}_M'$. We show that, for $S\in\mathcal{O}_C'(\mathcal{K}_M'; \mathcal{K}_M')$, $S*\mathcal{K}_M'=\mathcal{K}_M'$ is equivalent to the following: Every distribution $u\in\mathcal{O}_C'(\mathcal{K}_M'; \mathcal{K}_M')$ with $S*u\in\mathcal{K}_M$ is in $\mathcal{K}_M$.

1. Introduction.

Let $\mathcal{K}_M'$ be the space of distributions on R^n which grow no faster than $e^{M(kx)}$ for some $k>0$, where M is an increasing continuous functions on R^n; $\mathcal{K}_M'$ is the dual space of $\mathcal{K}_M$, which we describe later. We denote by $\mathcal{O}_C'(\mathcal{K}_M'; \mathcal{K}_M')$ the space of convolution operators in $\mathcal{K}_M'$.

In [1], S. Abdullah proved that, if S is a distributions in $\mathcal{O}_C'(\mathcal{K}_M'; \mathcal{K}_M')$ and $\hat{S}$ is its Fourier transform, the following conditions are equivalent:

(a) There exist positive constants A, C and a positive integer N such that

$$\sup_{\substack{z\in C^n \\ |z|\leq A\Omega^{-1}(\log(2+|\xi|))}} |\hat{S}(z+\xi)|\geq\frac{C}{(1+|\xi|)^N}, \qquad \xi\in R^n$$

where Ω^{-1} is the inverse of Ω, which is the dual to M in the sense of Young.

(b) $S*\mathcal{K}_M'=\mathcal{K}_M'$.

In this paper we prove that, for $S\in\mathcal{O}_C'(\mathcal{K}_M'; \mathcal{K}_M')$, the statements (a) and (b) are equivalent to the following: Every distribution $u\in\mathcal{O}_C'(\mathcal{K}_M'; \mathcal{K}_M')$ satisfying $S*u\in\mathcal{K}_M$ is in $\mathcal{K}_M$.

The motivation for this problem comes from the paper [5]. Here S. Sznaider and Z. Zielezny proved that, if S is a distribution in $\mathcal{O}_C'(\mathcal{K}_1'; \mathcal{K}_1')$ and $\hat{S}$ is its Fourier transform, the following statements are equivalent:

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(i) There exist positive constants N, r, C such that

$$\sup_{z\in C^n, |z|\leq r} |\hat{S}(\xi+z)| \geq \frac{C}{(1+|\xi|)^N}, \qquad \xi\in R^n,$$

(ii) $S*\mathcal{K}_1'=\mathcal{K}_1'$

(iii) If $u\in\mathcal{O}_C'(\mathcal{K}_1'; \mathcal{K}_1')$ and $S*u\in\mathcal{K}_1$, then $u\in\mathcal{K}_1$.

In view of this result it is natural to think the property (iii) in the space $\mathcal{K}_M'$ of distributions on R^n which grow no faster than $\exp(M(kx))$ for some $k>0$. Before presenting our theorems we recall briefly the basic facts about the spaces $\mathcal{K}_M'$, $\mathcal{O}_C'(\mathcal{K}_M'; \mathcal{K}_M')$ and K_M', for further details, we refer to [3].

The space $\mathcal{K}_M'$. Let $\mu(\xi)(0\leq\xi\leq\infty)$ denote a continuous increasing function such that $\mu(0)=0$, $\mu(\infty)=\infty$. For $x\geq 0$, we define

$$M(x)=\int_0^x \mu(\xi)d\xi .$$

The functions $M(x)$ is an increasing, convex and continuous function with $M(0)=0$, $M(\infty)=\infty$. For $x<0$, we define $M(x)$ to be $M(-x)$ and for $x=(x_1, \cdots, x_n)\in R^n$, $n\geq 2$, we define $M(x)$ to be $M(x_1)+\cdots+M(x_n)$.

Now we list some properties of $M(x)$ which will be used later;

(i) $M(x)+M(y)\leq M(x+y)$ for all $x, y\geq 0$

(ii) $M(x+y)\leq M(2x)+M(2y)$ for all $x, y\geq 0$.

Let $\mathcal{K}_M$ be the space of all C^∞-functions ϕ in R^n such that

$$\nu_k(\phi)=\sup_{\substack{x\in R^n\\ |\alpha|\leq k}} e^{M(kx)}|D^\alpha\phi(x)|<\infty, \qquad k=0, 1, 2, \cdots,$$

where $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ and $D_j=i^{-1}(\partial/\partial x_j)$. Provided with the topology defined by the seminorms ν_k, $\mathcal{K}_M$ is a Frechet space. The dual $\mathcal{K}_M'$ of $\mathcal{K}_M$ is the space of all continuous linear functionals on $\mathcal{K}_M$. Then a distribution u is in $\mathcal{K}_M'$ if and only if there exist $m\in N^n$, $k\in N$ and a bounded continuous function $f(x)$ on R^n such that

$$u=D^m(e^{M(kx)}f(x)).$$

$\mathcal{K}_M'$ is endowed with the topology of uniform convergence on all bounded sets in $\mathcal{K}_M$.

The space $\mathcal{O}_C'(\mathcal{K}_M'; \mathcal{K}_M')$. If $u\in\mathcal{K}_M'$ and $\phi\in\mathcal{K}_M$, then the convolution $u*\phi$ is a C^∞-function defined by

$$u*\phi(x)=\langle u_y, \phi(x-y)\rangle .$$

where $\langle u, \phi\rangle = u(\phi)$.

The space $\mathcal{O}'_C(\mathcal{K}'_M;\mathcal{K}'_M)$ of convolution eperators in $\mathcal{K}'_M$ consists of distributions $S\in\mathcal{K}'_M$ such that $S*u\in\mathcal{K}'_M$ for every $u\in\mathcal{K}'_M$, where $\langle S*u, \phi\rangle = \langle u, \check{S}*\phi\rangle$ for every $\phi\in\mathcal{K}_M$. Then the space is the set of distributions S which satisfy the following equivalent conditions [3]:

(i) The distributions $S_k=\gamma_k S$, $k=1, 2, \cdots$ are in tempered distribution space, where $\gamma_k = e^{M(kx)}$.

(ii) For every integer $k\geqq 0$, there exists an integer $m\geqq 0$ such that

$$S=\sum_{|\alpha|\leq m} D^\alpha f_\alpha$$

where f_α are continuous functions in R^n whose products with $e^{M(kx)}$ are bounded.

(iii) For every $\phi\in\mathcal{K}_M$, the convolution $S*\phi$ is in $\mathcal{K}_M$.

The space K'_M. For $\phi\in\mathcal{K}_M$, the Fourier transform

$$\hat{\phi}(\xi)=\int_{R^n} e^{-i\langle x,\xi\rangle}\phi(x)dx$$

can be continued in C^n as an entire function of $\zeta=\xi+i\eta$ such that

$$\omega_k(\hat{\phi})=\sup_{\zeta\in C^n}(1+|\xi|)^k e^{-\Omega(\eta/k)}|\hat{\phi}(\zeta)|<\infty\,, \qquad k=1, 2, \cdots \tag{1}$$

where $\Omega(y)$ is the dual of $M(x)$ in the sense of Young. If K_M is the space of all entire functions with the property (1) and the topology in K_M is defined by the seminorms ω_k, then the Fourier transform is an isomorphism of $\mathcal{K}_M$ onto K_M. The dual K'_M of K_M is the space of the Fourier transforms of distributions in $\mathcal{K}'_M$. The Fourier transform $\hat{u}$ of a distribution $u\in\mathcal{K}'_M$ is defined by the Parseval formula

$$\langle\hat{u}, \hat{\phi}\rangle=(2\pi)^n\langle u_x, \phi(-x)\rangle\,.$$

Also if $S\in\mathcal{O}'_C(\mathcal{K}'_M;\mathcal{K}'_M)$ and $u\in\mathcal{K}'_M$, we have the formula

$$\widehat{S*u}=\hat{S}\cdot\hat{u}\,,$$

where the product on the right—hand side is defined by

$$\langle\hat{S}\hat{u}, \psi\rangle=\langle\hat{u}, \hat{S}\psi\rangle, \qquad \psi\in K_M\,.$$

The following lemma will be used in the next section. It's proof can be found in [3].

LEMMA (Paley-Wiener type theorem). *Let $\zeta=\xi+i\eta\in C^n$. An entire function $F(\zeta)$ is the Fourier transform of a distribution S in $\mathcal{O}'_C(\mathcal{K}'_M;\mathcal{K}'_M)$ if and only if for every $\varepsilon>0$, there exist constants N and C such that*

$$|F(\xi+i\eta)|\leqq C(1+|\zeta|)^N e^{\Omega(\varepsilon\eta)}.$$

2. Main Theorem

THEOREM. *If S is a distribution in $\mathcal{O}'_C(\mathcal{K}'_M\,;\mathcal{K}'_M)$ and $\hat{S}$ be its Fourier transform, then the following conditions are equivalent:*

(a) *There exist positive constants A, C and a positive integer N such that*

$$\sup_{\substack{z\in C^n \\ |z|\leqq A\Omega^{-1}(\log(2+|\xi|))}} |\hat{S}(z+\xi)|\geqq\frac{C}{(1+|\xi|)^N},\qquad \xi\in R^n.$$

(b) $S*\mathcal{K}'_M=\mathcal{K}'_M$

(c) *If $u\in\mathcal{O}'_C(\mathcal{K}'_M\,;\mathcal{K}'_M)$ and $S*u\in\mathcal{K}_M$, then $u\in\mathcal{K}_M$.*

PROOF. It suffices to show that (b)⇒(c)⇒(a).

(b)⇒(c). The proof goes along exactly the same lines as proof of Theorem 1 in [5]. For the completeness we give the proof. If S is a distribution in $\mathcal{O}'_C(\mathcal{K}'_M\,;\mathcal{K}'_M)$, then so is $T=\check{S}$ and, by (1), the mapping $S*: u\to S*u$ of $\mathcal{K}'_M$ into $\mathcal{K}'_M$ is the transpose of the mapping $T*: \phi\to T*\phi$ of $\mathcal{K}_M$ into $\mathcal{K}_M$. Condition (b) is satisfied if and only if $T*$ an isomorphism of $\mathcal{K}_M$ onto $T*\mathcal{K}_M$ (see e.g., [2, Corollary on p. 92]). In particular the inverse $T*\phi\to\phi$ must be continuous.

Suppose now that $S*u=\phi$ where $u\in\mathcal{O}'_C(\mathcal{K}'_M\,;\mathcal{K}'_M)$ and $\phi\in\mathcal{K}_M$. Since $\langle S*u,\varphi\rangle=\langle T*\check{u},\check{\varphi}\rangle$ for $\varphi\in\mathcal{K}_M$, then

$$T*\check{u}=(-1)^n\check{\phi}\tag{2}$$

and for the proof if suffices to show that $\check{u}\in\mathcal{K}_M$. If ψ is a C^∞-function with $\operatorname{supp}\psi\subset B(0,1)=\{x\in R^n: |x|\leqq 1\}$ and $\hat{\psi}(0)=1$, we define $\psi_k(x)=k^n\psi(kx)$, $k=1, 2, \cdots$. From (2) it follows that

$$T*(\check{u}*\psi_k)=(-1)^n\check{\phi}*\psi_k,$$

and the convolutions $\check{u}*\psi_k$ and $(-1)^n\check{\phi}*\psi_k$ are in $\mathcal{K}_M$. Moreover, the sequence $\{\psi_k\}$ converges in $\mathcal{O}'_C(\mathcal{K}'_M\,;\mathcal{K}'_M)$ to δ, the Dirac measure as the origin. Hence $(-1)^n\check{\phi}*\psi\to(-1)^n\check{\phi}$ in $\mathcal{K}_M$ and $\check{u}*\psi_k\to\check{u}$ in $\mathcal{O}'_C(\mathcal{K}'_M\,;\mathcal{K}'_M)$. On the other hand, the sequence $\{\check{u}*\psi_k\}$ converges in $\mathcal{K}_M$, by the assumption that the inverse of $T*$ is continuous. The limit must be again $\check{u}$, and so $\check{u}$ is a function in $\mathcal{K}_M$.

(c)⇒(a). Let $\mathcal{F}$ be the space all functions $u\in C(R^n)$ such that

$$\sup_{x\in R^n} e^{M(kx)}|u(x)|<\infty,\qquad\text{for all } k$$

and $S*u\in\mathcal{K}_M$. We provide $\mathcal{F}$ with the topology defined by the seminorms

$$\|u\|_k=\sup_{x\in R^n} e^{M(kx)}|u(x)|+\nu_k(S*u), \qquad k=0, 1, 2, \cdots .$$

Then $\mathcal{F}$ becomes a Frechet space. Further, let $\mathcal{G}$ be the space of all functions $u\in C^1(R^n)$ such that

$$\|u\|=\sup_{x\in R^n\ |\alpha|\leqq 1} |D^\alpha u(x)|<\infty$$

with the norm $\| \ \|$, $\mathcal{G}$ is a Banach space.

By the fact $\mathcal{F}\subset\mathcal{O}'_C(\mathcal{K}'_M\,;\,\mathcal{K}'_M)$ and the assumption (c), each function $u\in\mathcal{F}$ is in $\mathcal{G}$. Also, the natural mapping $\mathcal{F}\to\mathcal{G}$ is closed and therefore continuous. Consequently there exist an integer $\mu>0$ and a constant C such that

$$\|u\|\leqq C\|u\|_\mu=C\{\sup_{x\in R^n} e^{M(\mu x)}|u(x)|+\nu_\mu(S*u)\}$$

for all $u\in\mathcal{F}$. Since the Fourier transformation is an isomorphism from $\mathcal{K}_M$ onto K_M, there exist another integer $\nu>0$ and a constant C_0 such that

$$\|u\|-C\sup_{x\in R^n} e^{M(\mu x)}|u(x)|\leqq C_0\omega_\nu(\hat{S}\cdot\hat{u}), \qquad \text{for all } u\in\mathcal{K}_M . \tag{3}$$

Suppose now that the condition (a) is not satisfied. Then there exists a sequence $\{\xi_j\}$ such that $|\xi_i|\to+\infty$ as $j\to\infty$ and

$$\sup_{\substack{z\in C^n\\ |z-\xi_j|\leqq j\Omega^{-1}(\log(2+|\xi_j|))}} |\hat{S}(z)|<\frac{1}{(1+|\xi_j|)^j}, \tag{4}$$

For each j, we define k_j to be the greatest integer equal or less than $\alpha_j=\Omega^{-1}(\log(2+|\xi_j|))$. Let $\phi\geqq 0$ in C_c^∞, $\operatorname{supp}\phi\subset B(0, 1)$ and $\hat{\phi}(0)=1$. We also define

$$\psi_j^1(x)=e^{i\langle\xi_j, x\rangle}(\phi_j*\cdots*\phi_j)(x),$$

and

$$\psi_j^2(x)=(\phi*(\phi_j*\cdots*\phi_j))(x)$$

where $\phi_j(x)=\alpha_j^n\phi(\alpha_j x)$ and the convolution product in the parenthesis is being taken k_j-times. Now we define

$$\psi_j(x)=(\phi*\psi_j^1)(x)$$

Since $\operatorname{supp}\psi_j\subset B(0, 2)$, clearly $\psi_j\in\mathcal{F}$.

Substituting ψ_j's into the inequality (3), we will show that the left side of (3) goes to ∞ and the right to 0, as $j\to\infty$, which gives the desired contradiction.

To show this, we first estimate

(5)
$$\|\phi_j\| = \sup_{x\in R^n,\, |\alpha|\leqq 1} |D^\alpha \phi_j(x)|$$
$$\geqq \sup_{x\in R^n,\, |\alpha|=1} |\phi * \{e^{i\langle x,\xi_j\rangle} D^\alpha(\phi_j * \cdots * \phi_j)$$
$$+D^\alpha(i\langle x,\ \xi_j\rangle) e^{i\langle x,\xi_j\rangle}(\phi_j * \cdots * \phi_j)\}(x)|$$
$$\geqq \sup_{x\in R^n,\, |\alpha|=1} |D^\alpha(\langle x,\ \xi_j\rangle)(\phi*(\phi_j * \cdots * \phi_j)\}(x)|$$
$$\geqq \frac{|\xi_j|}{n} \sup_{x\in R^n} |\phi_j^2(x)|$$

and since $\operatorname{supp}\phi_j$, $\operatorname{supp}\phi_j^2 \subset B(0,\ 2)$,

(6)
$$\sup_{x\in R^n} e^{M(\mu x)}|\phi_j(x)| = \sup_{|x|\leqq 2} e^{M(\mu x)}|\phi_j^2(x)| \leqq C' \sup_{x\in R^n} |\phi_j^2(x)|,$$

where $C' = e^{nM(2\mu)}$.

Viewing

$$1 = \int_{R^n} \phi_j^2(x)dx \leqq C'' \sup_{x\in R^n} |\phi_j^2(x)|,$$

where C'' is the volume of $B(0,\ 2)$, we have

(7)
$$\sup_{x\in R^n} |\phi_j^2(x)| \geqq \frac{1}{C''}.$$

Substituting (5), (6) and (7) into (3), the left hand side of (3) behaves, as $j\to\infty$,

$$\lim_{j\to\infty} \{\|\phi_j\| - C \sup_{x\in R^n} e^{M(\mu x)}|\phi_j(x)|\}$$
$$\geqq \lim_{j\to\infty} \left\{\frac{|\xi_j|}{n} - CC'\right\}\frac{1}{C''} = \infty$$

On the other hand,

(8)
$$\omega_\nu(\hat{S}\cdot\hat{\phi}_j) = \sup_{\zeta\in C^n} (1+|\zeta|)^\nu e^{-\Omega(\eta/\nu)} |\hat{S}(\zeta)||\hat{\phi}_j(\zeta)|$$
$$\leqq \sup_{|\zeta-\xi_j|\leqq j\alpha_j} (1+|\zeta|)^\nu e^{-\Omega(\eta/\nu)} |\hat{S}(\zeta)|\,|\hat{\phi}_j(\zeta)|$$
$$+ \sup_{|\zeta-\xi_j|> j\alpha_j} (1+|\zeta|)^\nu e^{-\Omega(\eta/\nu)} |\hat{S}(\zeta)|\,|\hat{\phi}_j(\zeta)|,$$

where $\zeta = \xi + i\eta$.

It now sufficies to prove that both terms in the right side of (8) go to 0, as $j\to\infty$. We first observe that, by the Paley-Wiener theorem for ϕ as element of C_c^∞ with $\operatorname{supp}\phi\subset B(0,\ 1)$, there exist a $C_m \geqq 0$, $m=0, 1, 2, \cdots$, such tha1

(9)
$$|\hat{\phi}(\zeta)| \leqq C_m(1+|\zeta|)^{-m} e^{|\eta|}.$$

Also, we observe that

$$\hat{\phi}^1_j(\zeta)=[\hat{\phi}_j(\zeta-\xi_j)]^{k_j}=\left[\hat{\phi}\left(\frac{\zeta-\xi_j}{\alpha_j}\right)\right]^{k_j}$$

and, by (9),

$$|\hat{\phi}^1_j(\zeta)|\leqq\left[C_1\left(1+\left|\frac{\zeta-\xi_j}{\alpha_j}\right|\right)^{-1}e^{|\eta|/\alpha_j}\right]^{k_j}. \tag{10}$$

Also we observe that Ω grows faster than any linear function of $|\eta|$ as $|\eta|$ goes large and $\hat{\phi}_j(\zeta)=\hat{\phi}(\zeta)\cdot\hat{\phi}^1_j(\zeta)$.

From these observations, the first term of the last estimate in (8) is bounded by

$$\begin{aligned}
&\sup_{|\zeta-\xi_j|\leqq j\alpha_j}(1+|\zeta|)^{\nu}e^{\Omega(\eta/\nu)}|\hat{S}(\zeta)|(C_1(1+|\zeta|)^{-1}e^{|\eta|})\\
&\qquad\times\left(C_1\left(1+\left|\frac{\zeta-\xi_j}{\alpha_j}\right|\right)^{-1}e^{|\eta|/\alpha_j}\right)^{k_j}\\
&\leqq C_2\sup_{|\zeta-\xi_j|\leqq j\alpha_j}(1+|\zeta-\xi_j|)^{\nu-1}(1+|\xi_j|)^{\nu-1}C_1^{k_j}\\
&\qquad\times\left(1+\left|\frac{\zeta-\xi_j}{\alpha_j}\right|\right)^{-k_j}|\hat{S}(\zeta)|\\
&\leqq C_2'\sup_{|\zeta-\xi_j|\leqq j\alpha_j}(1+|\xi_j|)^{2\nu-2+d}\left(1+\left|\frac{\zeta-\xi_j}{\alpha_j}\right|\right)^{\nu-1-k_j}|\hat{S}(\zeta)|\\
&\leqq C_2''(1+|\xi_j|)^{2\nu-2+d-j},
\end{aligned}$$

where we used that $e^{-\Omega(\eta/\nu)+|\eta|+(k_j/\alpha_j)|\eta|}$ is bounded in R^n and $d=\log C_1$.

Therefore the first term of the last part in (8) approaches to 0 as $j\to\infty$.

From the lemma for S as element of $\mathcal{O}'_C(\mathcal{K}'_M\,;\,\mathcal{K}'_M)$ and (9), (10), the second term of the last estimate in (8) is bounded by

$$\begin{aligned}
&\sup_{|\zeta-\xi_j|>j\alpha_j}C_{S,2\nu+N}(1+|\zeta|)^{\nu}e^{-\Omega(\eta/\nu)}(1+|\zeta|)^{N}e^{\Omega(\eta/2\nu)}\\
&\qquad\times(1+|\zeta|)^{-(2\nu+N)}e^{|\eta|}\left(C_1\left(1+\frac{|\zeta-\xi_j|}{\alpha_j}\right)^{-1}e^{|\eta|/\alpha_j}\right)^{k_j}\\
&\leqq\sup_{|\zeta-\xi_j|>j\alpha_j}C'_{S,2\nu+N}(1+|\zeta|)^{\nu+N-(2\nu+N)}C_1^{k_j}\left(1+\frac{|\zeta-\xi_j|}{\alpha_j}\right)^{-k_j}\\
&\leqq\sup_{|\zeta-\xi_j|>j\alpha_j}C'_{S,2\nu+N}C_1^{k_j}\left(1+\frac{|\zeta-\xi_j|}{\alpha_j}\right)^{-k_j}\\
&\leqq C'_{S,2\nu+N}\left(\frac{1+j}{C_1}\right)^{-k_j},
\end{aligned}$$

where we used that $e^{-\Omega(\eta/\nu)+\Omega(\eta/2\nu)+|\eta|+(k_j/\alpha_j)|\eta|}$ is bounded in R^n. Here $C_{S,2\nu+N}$ and $C'_{S,2\nu+N}$ are constants which depend on S, ϕ, ν and N only.

Hence the second term of the last part in (8) approaches to 0 as $j\to\infty$.

Combining both estimates we have

$$\lim_{j\to\infty} \omega_\nu(\hat{S}\cdot\hat{\phi}_j)=0\,,$$

which gives the desired contradiction.

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