TILTING MODULES, DOMINANT DIMENSION
AND EXACTNESS OF DUALITY FUNCTORS

By

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Dedicated to Professor Hiroyuki Tachikawa for his sixtieth birthday

Let R and S be rings and let $_RW_S$ be a bimodule. We shall denote both the functors $\mathrm{Hom}_R(\_, W)\colon R\text{-Mod}\to\text{Mod-}S$ and $\mathrm{Hom}_S(\_, W)\colon \text{Mod-}S\to R\text{-Mod}$ by Δ_W and the composition of the two, in either order, by Δ_W^2. Recall that (for fixed W) there is a natural transformation $\delta\colon 1_{R\text{-Mod}}\to\Delta_W^2$, defined via the usual evaluation maps $\delta_M\colon M\to\Delta_W^2(M)$. An R-module M is called W-reflexive (W-torsionless) in case δ_M is an isomorphism (a monomorphism). Then, an R-module M is W-torsionless if and only if it is isomorphic to a submodule of a direct product of copies of $_RW$. Also recall that $_RW_S$ is *balanced* in case $R\cong\mathrm{End}_S(W)$ and $S\cong\mathrm{End}_R(W)^{op}$ canonically, and that $_RW_S$ defines a *Morita Duality* if it is balanced and both $_RW$ and W_S are injective cogenerators (see [1], [3] or [10], for an account of Morita Duality).

We begin by studying exactness properties of the functor Δ_W^2. The case $W=R$ has been extensively studied in ([4], [5], [6] and [7]) and Theorem 1, Lemma 2, Proposition 3 and Proposition 4 are generalizations of results obtained there. A finite dimensional algebra R of positive dominant dimension possesses (what we consider to be) a canonical pair of tilting left and right modules $_RU$ and V_R. Associated with these are the endomorphism rings $S=\mathrm{End}_R(U)^{op}$ and $T=\mathrm{End}_R(V)$, and the bimodule $_TW_S=_T(V\otimes_R U)_S$. We relate exactness properties of the functors $\Delta_U, \Delta_V, \Delta_W$ and their squares to dominant dimension. For these canonically chosen tilting modules we show that

  1) if dom. dim. $R\geq 2$ then Δ_U^2 preserves monomorphisms both in Mod-S and in R-Mod;

  2) if dom. dim. $R\geq 3$ then Δ_U^2 is left exact on Mod-S and the functors Δ_W^2 preserve monomorphisms in Mod-S and in T-Mod. In this case, if $\Delta_W\colon T\text{-Mod}\leftrightarrow\text{Mod-}S\colon \Delta_W$ defines a Morita Duality, then R is QF (and conversely);

  3) if dom. dim. $R\geq 4$ then the functors Δ_W^2 are left exact on both Mod-S and on T-Mod.

We shall denote the injective envelope of a module M by $E(M)$ and, if M is an R-module, we denote the annihilator in M of a subset I of R by $\mathrm{Ann}_M(I)$.

THEOREM 1. *Let $_RW_S$ be a bimodule with $R=\mathrm{End}_S(W)$. Let I denote the ideal of all endomorphisms in R which factor through injective S-modules. The following are equivalent:*

1) *W_S cogenerates $E(W_S)$.*
2) *If $M\in\mathrm{Mod}\text{-}S$ is W-reflexive then $E(M_S)$ is W-torsionless.*
3) *$\Delta^2_W:\mathrm{Mod}\text{-}S\to\mathrm{Mod}\text{-}S$ preserves monomorphisms.*
4) *$\mathrm{Ann}_W(I)=0$.*

PROOF. We modify the proof of [7, Theorem 1]. Let $E_S=E(W_S)$ and denote the injection of W_S into E_S by i. We first prove that 1) implies 4). Suppose that W_S cogenerates E_S. There is a sequence

$$W \xrightarrow{\ i\ } E \xrightarrow{\ j\ } W^X$$

in Mod-S where j is a monomorphism. For $x\in X$ let $p_x:W^X\to W$ be the canonical projection and let $b_x=p_x\circ j\circ i\in I$. Then if $K=\sum\{Rb_x:x\in X\}$, $K\subseteq I$ and note that $\mathrm{Ann}_W(K)=0$. Hence we also have $\mathrm{Ann}_W(I)=0$. Next, assume condition 4). Suppose $\alpha:M\to N$ is a monomorphism in Mod-S and consider the induced exact sequence

$$\Delta_W(N) \xrightarrow{\ \Delta_W(\alpha)\ } \Delta_W(M) \xrightarrow{\ \beta\ } \mathrm{Coker}\,\Delta_W(\alpha) \longrightarrow 0$$

in R-Mod. If $f\in\Delta_W(M)$ and $r\in I$ then $r\circ f$ factors through an injective so there exists $\bar f\in\mathrm{Hom}_S(N,W)$ such that $\bar f\circ\alpha=r\circ f$. That is $I\Delta_W(M)\subseteq\mathrm{Im}\,\Delta_W(\alpha)=\mathrm{Ker}\,\beta$. Thus we have

$$I\,\mathrm{Coker}\,(\Delta_W(\alpha))=I\beta(\Delta_W(M))=\beta(I\Delta_W(M))=0\,.$$

Now let $\phi\in\Delta_W(\mathrm{Coker}\,\Delta_W(\alpha))$. Since

$$I\phi(\mathrm{Coker}\,\Delta_W(\alpha))=\phi(I\,\mathrm{Coker}\,(\Delta_W(\alpha))=0$$

and $\mathrm{Ann}_W(I)=0$ by 4) we obtain that $\phi=0$. Thus $\Delta_W(\mathrm{Coker}\,\Delta_W(\alpha))=0$ so $\Delta^2_W(\alpha)$ is a monomorphism. This completes the proof that 4) implies 3). That 3) implies 2) follows from the observation that a non-zero kernel of $\delta_{E(M_S)}$ would have to intersect M non-trivially, and it is clear that 2) implies 1).

Straightforward modification of the proof of [4, Theorem 2] provides a proof of the following lemma.

LEMMA 2. *Let $_RW_S$ be a balanced bimodule and assume that the functor Δ^2_W preserves monomorphisms in Mod-S. Let*

$$0 \longrightarrow {}_RW \longrightarrow {}_RE_1 \longrightarrow {}_RE_2$$

be an injective copresentation of $_RW$. If $_RW$ cogenerates $_RE_1$ and $_RE_2$, then Δ_W^2 : Mod-$S \to$ Mod-S is left exact.

In case $W = R$, the equivalence of conditions 1) and 3) of the following result was observed in [6, Remark (d)].

PROPOSITION 3. *Let $_RW_S$ be a balanced bimodule. The following are equivalent.*

1) *W_S is injective.*
2) *If α is a monomorphism in Mod-S then $\Delta_W^3(\alpha)$ is an epimorphism in R-Mod.*
3) *Δ_W^2 : Mod-$S \to$ Mod-S preserves monomorphisms and Δ_W^2 : R-Mod$\to R$-Mod is right exact.*

In particular, if $_RW_S$ is a balanced bimodule, then both W_S and $_RW$ are injective if and only if both the Δ_W^2 functors are exact.

PROOF. If is clear that condition 1) implies condition 3). Assume condition 3) and let $\alpha : M \to N$ be a monomorphism in Mod-S. Since $\Delta_W^2(\alpha)$ is a monomorphism, we have $\Delta_W(\text{Coker}(\Delta_W(\alpha))=0$. Hence $\Delta_W^2(\text{Coker}(\Delta_W(\alpha))=0$ too so $\Delta_W^3(\alpha)$ is an epimorphism. Now assume condition 2). If $\alpha : M \to S$ is a monomorphism in Mod-S we obtain an exact sequence

$$\Delta_W(S) \xrightarrow{\ \Delta_W(\alpha)\ } \Delta_W(M) \longrightarrow \text{Coker}\, \Delta_W(\alpha) \longrightarrow 0$$

in R-Mod. Using 2), the W-reflexivity of $\Delta_W(S)=W$, and the fact that $\Delta_W(M)$ is W-torsionless, the commutativity and exact rows and columns of the diagram

$$
\begin{array}{ccc}
\Delta_W^3(S) & \longrightarrow \Delta_W{}^3(M) \longrightarrow 0 \\
\cong \Big\uparrow & \Big\uparrow \\
\Delta_W(S) & \longrightarrow \Delta_W(M) \longrightarrow \text{Coker}\, \Delta_W(\alpha) \longrightarrow 0 \\
& \Big\uparrow \\
& 0
\end{array}
$$

show that $\text{Coker}\, \Delta_W(\alpha)=0$ so $\Delta_W(\alpha)$ is an epimorphism. Thus 1) holds.

We remark that if R and S are finite dimensional algebras, and $_RW$ and W_S are finitely generated, then Theorem 1, Lemma 2, and Proposition 3 remain true if we replace Mod-S and R-Mod by mod-S and R-mod, respectively.

Recall that $U \in R$-Mod has *dominant dimension* at least n (dom. dim. $_RU \geqq n$) if there is an exact sequence

$$0 \longrightarrow {}_RU \longrightarrow {}_RE_1 \longrightarrow \cdots \longrightarrow {}_RE_n$$

where each E_i is both projective and injective. If R is a finite dimensional algebra the dominant dimensions of $_RR$ and R_R are equal (see [8], [9], [10]) and this number is called the dominant dimension of the algebra R. Such algebras of dominant dimension greater than or equal to one are also known as QF-3 algebras. A ring R is a left QF-3 ring if it has a *minimal faithful* left module, i.e. a module which is isomorphic to a direct summand of every faithful module (see [10], for example). Of course, a minimal faithful module is both projective and injective and is isomorphic to a left ideal Re for some idempotent $e \in R$.

PROPOSITION 4. *Suppose R is a finite dimensional algebra over a field and that dom. dim. $R \geq 2$. Let $_RE$ be a minimal faithful left R-module with $S = \mathrm{End}_R(E)^{op}$. The following are equivalent:*

1) R *is QF.*
2) E_S *is injective.*
3) Δ_E^2 *is right exact on R-Mod.*

PROOF. Recall that $_RE_S$ is a balanced bimodule [10, Proposition 7.1]. Condition 2) implies condition 1) since E_S is a generator in Mod-S, hence if E_S is injective, S is QF and E_S is a progenerator, so R is Morita equivalent of S. Clearly condition 1) implies condition 3) since in this case $_RE$ is a progenerator so both $_RE$ and E_S are injective. Assume condition 3). By Proposition 3(3) and the remark following, to prove 2) it suffices to show that Δ_E^2 preserves mono-morphisms in mod-S. If M is a finitely generated (hence finitely presented) module in mod-S, then since S is E-reflexive and Δ_E^2 is right exact on Mod-S ($_RE$ is injective) it follows that M is E-reflexive. Thus Δ_E^2 is exact (hence preserves monomorphisms) on mod-S.

Recall that $U \in R$-Mod is a *tilting module* in case U has projective dimension at most one ($\mathrm{pd}_RU \leq 1$), $\mathrm{Ext}_R^1(U, U) = 0$, and there is an exact sequence $0 \to {}_RR \to {}_RU_1 \to {}_RU_2 \to 0$ where $U_1, U_2 \in \mathrm{add}$-$U$. We refer to [2] and the references given there for basic results concerning tilting modules. We next note that rings of positive dominant dimension have a canonical tilting module.

PROPOSITION 5. *Suppose R is a finite dimensional algebra over a field with dom. dim. $R \geq n$ where $n \geq 1$, and let $U = E \oplus E(_RR)/R$ where $_RE$ is a minimal faithful left R-module. Then $_RU$ is a tilting module and dom. dim. $_RU \geq n-1$.*

PROOF. Let $_RQ = E(_RR)$. Since $_RE$ is both projective and injective $\mathrm{Ext}_R^1(U, U) = 0$ will follow from $\mathrm{Ext}_R^1(Q/R, Q/R) = 0$. Since $_RQ$ is injective and $\mathrm{pd}_R(Q/R) \leq 1$, this is guaranteed by the exactness of the sequence

$$0 = \operatorname{Ext}^1(Q/R, Q) \longrightarrow \operatorname{Ext}^1(Q/R, Q/R) \longrightarrow \operatorname{Ext}^2(Q/R, R) = 0$$

which is induced by the exact sequence $0 \to {}_R R \to {}_R Q \to {}_R(Q/R) \to 0$. This latter exact sequence has both Q and Q/R in add-U and since Q is projective it is clear that $\operatorname{pd}_R U \leq 1$. Finally, since dom. dim. $R \geq n$ and Q is projective and injective, it is clear that dom. dim. ${}_R(Q/R) \geq n-1$ so dom. dim. ${}_R U \geq n-1$ as well.

LEMMA 6. *If ${}_R U$ is a tilting module then Ker $\operatorname{Tor}^R_1(\_, U)$ is closed under taking submodules.*

PROOF. Since $\operatorname{pd}_R U \leq 1$, there is an exact sequence $0 \to P_2 \to P_1 \to U \to 0$ in R-Mod with P_i projective. Suppose $0 \to M \to N$ is exact in Mod-R and $\operatorname{Tor}^R_1(N, U) = 0$. These two sequences innduce the commutative diagram

$$
\begin{array}{ccccc}
& & 0 \longrightarrow N \otimes P_2 & \longrightarrow & N \otimes P_1 \\
& & \uparrow & & \uparrow \\
0 \longrightarrow \operatorname{Tor}^R_1(M, U) \longrightarrow & M \otimes P_2 & \longrightarrow & M \otimes P_1 \\
& & \uparrow & & \uparrow \\
& & 0 & & 0
\end{array}
$$

which, since P_1 and P_2 are projective, has exact rows and columns and from which $\operatorname{Tor}^R_1(M, U) = 0$ follows.

LEMMA 7. *Suppose ${}_R U$ and V_R are tilting left and right modules, respectively. Let $S = \operatorname{End}_R(U)^{op}$ and $T = \operatorname{End}_R(V)$. If V_R is a submodule of a flat module, $\operatorname{Ext}^1_T(V, V) = 0$, and ${}_T V_R$ is a balanced bimodule, then there are canonical isomorphisms $\operatorname{Hom}_T(V, V \otimes_R U) \cong U_S$ and $\operatorname{Hom}_T(V \otimes_R U, V \otimes_R U) \cong S$.*

PROOF. It suffices to establish the first isomorphism since, then, we have canonical isomorphisms

$$\operatorname{Hom}_T(V \otimes_R U, V \otimes_R U) \cong \operatorname{Hom}_R(U, \operatorname{Hom}_T(V, V \otimes_R U))$$
$$\cong \operatorname{Hom}_R(U, U)$$
$$\cong S.$$

Using the hypothesis on ${}_R V$ and Lemma 6, we have that $\operatorname{Tor}^R_1(V, U) = 0$ so by our hypothesis that $\operatorname{Ext}^1_T(V, V) = 0$ (hence $\operatorname{Ext}^1_T(V, V \otimes P_2) = 0$) an exact sequence

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow U \longrightarrow 0$$

in R-Mod with P_i projective induces an exact sequence

$$0 \longrightarrow \operatorname{Hom}_T(V, V \otimes P_2) \longrightarrow \operatorname{Hom}_T(V, V \otimes P_1) \longrightarrow \operatorname{Hom}_T(V, V \otimes U) \longrightarrow 0$$

so since $_RP_1$ and $_RP_2$ are finitely generated and projective and $_TV_R$ is balanced, the natural isomorphisms $\mathrm{Hom}_T(V, V\otimes P_i) \cong P_i$ induce the required isomorphism $\mathrm{Hom}_T(V, V\otimes U) \cong U$.

LEMMA 8. *Suppose that $V_R \in \mathrm{mod}\text{-}R$ is a submodule of a flat module and that $_RU$ is a tilting module with dom. dim. $_RU \geqq 2$. Let $T = \mathrm{End}_R(V)$. Then $_TV$ and $_T(V\otimes_R U)$ finitely cogenerate each other.*

PROOF. By Lemma 6 $\mathrm{Tor}_1^R(V, U) = 0$ so an exact sequence $0 \to R \to U_1 \to U_2 \to 0$ in R-Mod with $U_i \in \mathrm{add}\text{-}U$ induces an exact sequence $0 \to {}_T(V\otimes R) \to {}_T(V\otimes U_1)$. Thus, since $U_1 \in \mathrm{add}\text{-}U$ there is an injection of $_TV$ into $_T(V\otimes U)^n$ for some n. Since dom. dim. $_RU \geqq 2$ there is an exact sequence $0 \to U \to E_1 \to E_2$ in R-Mod with E_i projective and injective. Hence, since E_2 is projective, by Lemma 6 we obtain an injection of $_T(V\otimes U)$ into $_T(V\otimes E_1)$, and, since E_1 is projective, one of $_T(V\otimes E_1)$ into $_TV^m$ for some m.

Suppose R is a finite dimensional algebra over a field. Then, if $_RU$ is a tilting module and $S = \mathrm{End}_R(U)^{op}$ then U_S is also a tilting module, $_RU_S$ is a balanced bimodule, and R and S have the same number of isomorphism classes of simple modules [2, Theorem 1.5] (our references to [2] do not require the standing hypothesis of algebraic closure made there).

THEOREM 9. *Suppose R is a finite dimensional algebra over a field and that dom. dim. $R \geqq 1$. Let F_R be a minimal faithful right module, let $_RU$ be a tilting left module, and $S = \mathrm{End}_R(U)^{op}$. Then $(F\otimes_R U)_S$ is an injective module. Consequently if dom. dim. $_RU \geqq 1$, then the functors Δ_U^0 preserve monomorphisms both in $\mathrm{Mod}\text{-}S$ and in R-Mod. Furthermore, if dom. dim. $_RU \geqq 2$ then Δ_U^2 is left exact on $\mathrm{Mod}\text{-}S$.*

PROOF. Let $H_S = E((F\otimes_R U)_S)$. The evaluation $\mathrm{Hom}_S(U, H)\otimes_R U_S \to H_S$ is an isomorphism by [2, Proposition 1.5a] and the evaluation $\mathrm{Hom}_S(U, F\otimes_R U)\otimes_R U_S \to F\otimes_R U_S$ is an isomorphism since F_R is finitely generated and projective and R is the endomorphism ring of U_S. Since the injective module $\mathrm{Hom}_S(U, F\otimes_R U)_R = F_R$ is a direct summand of $\mathrm{Hom}_S(U, H)$, $(F\otimes_R U)_S$ is a direct summand of the injective module $\mathrm{Hom}_S(U, H)\otimes_R U_S = H_S$. Thus $(F\otimes_R U)_S$ is injective. In order to prove the remaining assertions, identify F_R with a right ideal fR and let Re be a minimal faithful left R-module where f and e are idempotents of R. Considering $R = \mathrm{End}_S(U)$, f is the canonical projection of U_S onto $F\otimes_R U_S$ so we have $fR \subseteq I$ where I is the ideal of Theorem 1. Suppose dom. dim. $_RU \geqq 1$. Since fR_R is faithful and $_RU$ is R-torsionless, $\mathrm{Ann}_U(fR) = 0$ so $\mathrm{Ann}_U(I) = 0$ also. Thus

Δ_U^2 preserves monomorphisms in Mod-S by Theorem 1(4). Also, since dom. dom. $R \geq 1$, $_R Re$ cogenerates $_R R$ hence also $_R U$ (since $_R U$ is a snbmodule of a projective). Thus Δ_U^2 preserves monomorphisms in R-Mod by Theorem 1(1) and [2, Corollary to Theorem 2.1]. The final assertion follows from Lemma 2 since the minimal faithful left R-module cogenerates any projective and is a direct summand of $_R U$.

PROPOSITION 10. *Suppose R is a finite dimensional algebra over a field. Assume that $_R U$ and V_R are tilting left and right modules respectively, that V_R is torsionless and that dom. dim. $_R U \geq 2$ The following are equivalent:*

1) *$_T V$ is injective.*
2) *$_T V$ is a cogenerator.*
3) *$_T(V \otimes U)$ is a cogenerator.*
4) *R is QF.*

PROOF. Suppose T has n simple modules. By [2, Theorem 2.1], $_T V$ has n isomorphism classes of indecomposable direct summands. Hence, if $_T V$ is injective, every indecomposable injective is a direct summand of $_T V$ so $_T V$ is a cogenerator. Similarly, if $_T V$ is a cogenerator, $_T V$ is injective since it has n isomorphism classes of indecomposable injective direct summands. Thus conditions 1) and 2) are equivalent and their equivalence with 3) follows from Lemma 8. Now $_T V_R$ is a balanced bimodule so if 1) and 2) hold then $_T V_R$ defines a Morita Duality. Hence V_R is injective. Again by [2, Theorem 2.1], V_R has exactly n isomorphism classes of indecomposable direct summands and this is the number of simple R-modules. Thus R is QF. Finally, condition 4) implies conditions 1) and 2) by [2, Corollary to Theorem 2.1].

THEOREM 11. *Suppose R is a finite dimensional algebra over a field and that dom. dim. $_R R \geq 1$. Suppose $_R U$ and V_R are tilting left and right R-modules, respectively, each having dominant dimension at least 1. Let $S = \mathrm{End}_R(U)^{op}$, $T = \mathrm{End}_R(V)$, and $_T W_S = {}_T(V \otimes_R U)_S$. Then $_T W_S$ is a balanced bimodule, $\Delta_W(_T V) = U_S$, and $\Delta_W(U_S) = {}_T V$. Furthermore,*

1) *If dom. dom. $_R U \geq 2$ and dom. dim. $V_R \geq 2$, then the functors Δ_W^2 preserve monomorphisms both in Mod-S and in T-Mod,*

2) *If dom. dom. $_R U \geq 3$ and dom. dim. $V_R \geq 3$, then the functors Δ_W^2 are left exact both in Mod-S and T-Mod, and*

3) *If dom. dom. $_R U \geq 2$ and dom. dim. $V_R \geq 2$ then R is QF if and only if Δ_W: T-Mod$\leftrightarrow$Mod-S: Δ_W defines a Morita Duality.*

Proof. The first assertion follows from Lemma 7. Suppose dom. dom. $_RU$ $\geqq 2$ and dom. dim. $V_R \geqq 2$. By Theorem 9 Δ_V^2 is left exact on T-Mod and Δ_U^2 is left exact on Mod-S. Hence by Theorem 1, U_S cogenerates $E(U_S)$ and $_TV$ cogenerates $E(_TV)$. By Lemma 8 $E(_TV)$ cogenerates $E(_TW)$ so $_TV$ cogenerates $E(_TW)$, but then, since $_TW$ cogenerates $_TV$, $_TW$ cogenerates $E(_TW)$. Thus Δ_W^2 preserves monomorphisms in Mod-S by Theorem 1. Similarly, Δ_W^2 preserves monomorphisms in T-Mod. Next assume that dom. dim. $_RU \geqq 3$ and dom. dim. $V_R \geqq 3$. Let F_R be a minimal faithful right R-module. There is an exact sequence

$$0 \longrightarrow V_R \longrightarrow F_R^1 \longrightarrow F_R^2 \longrightarrow F_R^3$$

where $F_R^i \in \text{add-}F_R$. Applying Lemma 6 twice, we conclude that the induced sequence

$$0 \longrightarrow W_S \longrightarrow (F^1 \otimes_R U)_S \longrightarrow (F^2 \otimes_R U)_S$$

is exact. Since $(F \otimes_R U)_S$ is injective by Theorem 9 and since F_R is isomorphic to a direct summand of V_R by [2, corollary to Theorem 2.1] we conclude that Δ_W^2 is left exact on T-Mod by Lemma 2. Similarly, Δ_W^2 is left exact on Mod-S. Statement 3) follows from Proposition 10.

Example. Let A be the algebra of 3×3 lower triangular matrices over an field and let $R = A/J^2$ where J is the radical of A. Then R has dominant dimension 2 (and is not QF). Computation shows that, with notation as in Theorem 11 and $_RU$, V_R chosen as in Proposition 5 (and having dominant dimension 1), $\Delta_W : T\text{-Mod} \leftrightarrow \text{Mod-}S : \Delta_W$ does define a Morita Duality.

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