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SOME INEQUALITIES ON $|\nabla R|$, $|\nabla \mathrm{Ric}|$ AND $|dr|$ IN RIEMANNIAN MANIFOLDS

By

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Introduction. Let M^n be an $n(>1)$ dimensional Riemannian manifold. We denote by $g=(g_{ji})$, $R=(R_{kji}{}^h)$, $\mathrm{Ric}=(R_{ji})=(R_{rji}{}^r)$ and $r=(R_i{}^i)=(R_{ji}g^{ji})$ the metric tensor, the curvature tensor, the Ricci tensor and the scalar curvature respectively. ∇ means the operator of the covariant differential, and we put ${}^c\nabla R=(\nabla_r R_{kji}{}^r)$. The purpose of this paper is to give some inequalities which hold among the norms of ∇R, ${}^c\nabla R$ and dr. Though we do not know whether such inequalities are worthy to be studied or not, the cases when the equalities hold in our inequalities seem meaningful.

§1 will be devoted itself to preliminaries. Denoting the norm of a tensor T by $|T|$, we shall show in §2 two inequalities among $|\nabla R|$, $|{}^c\nabla R|$ and $|dr|$. In one of the inequalities the equality holds if and only if the manifold has harmonic Weyl tensor. An application will be given. In §3 an inequality which holds between $|\nabla \mathrm{Ric}|$ and $|dr|$ will be proved. In §4 we shall give among $|\nabla R|$, $|{}^c\nabla R|$ and $|dr|$ two inequalities which are different from those in §2. An inequality for the Codazzi tensor will be shown in §5, and in the last section $|\nabla \mathrm{Ric}|$ in Kaehlerian manifolds will be discussed.

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§1. Preliminaries. Let M^n be an n dimensional Riemannian manifold. We follow the notations in Introduction. Tensors are represented by their components with respect to the natural basis, unless otherwise stated, and the summation convention is assumed. ∇ denotes the operator of covariant differential. We have $\nabla R=(\nabla_l R_{kji}{}^h)$.

Let us put

$$ {}^c\nabla R=(S_{kji}), $$

where

$$ S_{kji}=\nabla_r R_{kji}{}^r . $$

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Then S_{kji} satisfies

(1.1) $$S_{kji}=-S_{jki},$$

(1.2) $$S_{kji}+S_{jik}+S_{ikj}=0,$$

(1.3) $$g^{lk}\nabla_l R_{kjih}=S_{hij},$$

(1.4) $$S_{kji}=\nabla_k R_{ji}-\nabla_j R_{ki}.$$

We also have

(1.5) $$S_{ki}{}^i=S_{kji}g^{ji}=\frac{1}{2}r_k,$$

because of $2\nabla_i R_k{}^i=r_k$, where $r_k=\nabla_k r$.

It is well known that ${}^C\nabla R=0$ is the condition of harmonic curvature.

The conformal curvature tensor (or Weyl tensor) $C=(C_{kji}{}^h)$ and the tensor $\hat{C}=(C_{kji})$ are defined as

$$\begin{aligned}C_{kjih}&=C_{kji}{}^l g_{lh}\\&=R_{kjih}-\frac{1}{n-2}(R_{ji}g_{kh}-R_{ki}g_{jh}+g_{ji}R_{kh}-g_{ki}R_{jh})\\&\quad+\frac{r}{(n-1)(n-2)}(g_{kh}g_{ji}-g_{jh}g_{ki}),\end{aligned}$$

$$C_{kji}=S_{kji}-\frac{1}{2(n-1)}(r_k g_{ji}-r_j g_{ki}).$$

M^n $(n>3)$ is conformally flat if and only if C vanishes. Because of the identity:

$$\nabla_r C_{kji}{}^r=\frac{n-3}{n-2}C_{kji},$$

$\hat{C}$ vanishes for a conformally flat M^n $(n>3)$.

$\hat{C}=0$ for $n>3$ is the condition for C to be harmonic (i.e. harmonic Weyl tensor in terminology of Besse [1]).

The norm of a tensor T, say $T=(T_{ji}{}^h)$, will be denoted by $|T|$ or $|T_{ji}{}^h|$, hence we have $|T|^2=T_{ji}{}^h T_{rs}{}^t g^{jr}g^{is}g_{ht}=T_{ji}{}^h T^{ji}{}_h$.

§2. Two inequalities on $|{}^C\nabla R|^2$.

We shall start from the following

PROPOSITION 1. $$\frac{4}{n}|{}^C\nabla R|^2\leqq|\nabla R|^2.$$

The equality holds if and only if M^n $(n>3)$ is locally symmetric.

PROOF. Let us put

$$B_{lkjih}=g_{lk}S_{hij}-g_{lj}S_{hik}+g_{lh}S_{kji}-g_{li}S_{kjh}. \tag{2.1}$$

The inequality required follows from

$$0\leqq\left|\nabla_l R_{kjih}-\frac{1}{n}B_{lkjih}\right|^2.$$

In fact, we have

$$|B_{lkjih}|^2=4n|S|^2, \qquad \nabla_l R_{kjih}B^{lkjih}=4|S|^2$$

by virtue of (1.1)∼(1.3), and

$$0\leqq|\nabla R|^2-\frac{2}{n}\nabla_l R_{kjih}B^{lkjih}+\frac{1}{n^2}|B_{lkjih}|^2$$

$$=|\nabla R|^2-\frac{8}{n}|S|^2+\frac{4n}{n^2}|S|^2=|\nabla R|^2-\frac{4}{n}|S|^2.$$

The equality holds if and only if $n\nabla_l R_{kjih}=B_{lkjih}$. By Bianchi's identity, we have

$$B_{lkjih}+B_{kjlih}+B_{jlkih}=0, \qquad \text{i.e.,}$$

$$g_{lh}S_{kji}-g_{li}S_{kjh}+g_{kh}S_{jli}-g_{ki}S_{jlh}+g_{jh}S_{lki}-g_{ji}S_{lkh}=0.$$

Transvecting the last equation with g^{lh} and then g^{ji} and taking account of (1.1) and (1.2), we obtain

$$(n-3)\ S_{kji}=g_{ki}S_{jr}{}^r-g_{ji}S_{kr}{}^r,$$

$$2(n-2)\ S_{kr}{}^r=0.$$

Thus $\nabla R=0$ holds for $n>3$. □

For a compact orientable M^n,

$$\int_M\{K-|^C\nabla R|^2\}d\sigma=-\frac{1}{2}\int_M|\nabla R|^2 d\sigma$$

has been known as Lichnerowicz' integral formula, where

$$K=R_{lm}R^{ljih}R^m{}_{jih}+\frac{1}{2}R^{lmpq}R_{lmih}R^{ih}{}_{pq}+2R^{limh}R_{lpmq}R^p{}_i{}^q{}_h.$$

Owing to Proposition 1 we have

$$\int_M K d\sigma=\int_M\left\{|^C\nabla R|^2-\frac{1}{2}|\nabla R|^2\right\}d\sigma\leqq\frac{n-2}{n}\int_M|^C\nabla R|^2 d\sigma.$$

The equality holds if and only if M^n $(n>3)$ is locally symmetric. Consequently we obtain

THEOREM 1. *Let M^n be an n (>3) dimensional compact Riemannian mani-*

fold. If $K \geqq \frac{n-2}{n} |{}^{C}\nabla R|^2$ *holds everywhere, then* M^n *is locally symmetric.*

The following proposition characterizes the case of harmonic Weyl tensor.

PROPOSITION 2. $$\frac{1}{2(n-1)} |dr|^2 \leqq |{}^{C}\nabla R|^2 .$$
The equality holds if and only if $\hat{C}=0$, *i.e.,*

$$S_{kji} = \frac{1}{2(n-1)} (r_k g_{ji} - r_j g_{ki}) .$$

PROOF. We have

$$0 \leqq |\hat{C}|^2 = |C_{kji}|^2 = \left| S_{kji} - \frac{1}{2(n-1)} (r_k g_{ji} - r_j g_{ki}) \right|^2$$
$$= |S|^2 - \frac{1}{2(n-1)} |dr|^2 ,$$

by taking account of (1.1), (1.5) and $|dr|^2 = r_i r^i$. □

§ 3. An inequality on $|\nabla \mathrm{Ric}|$

PROPOSITION 3. $$\frac{3n-2}{2(n-1)(n+2)} |dr|^2 \leqq |\nabla \mathrm{Ric}|^2 .$$
The equality holds if and only if

(3.1) $$\nabla_k R_{ji} = \alpha r_k g_{ji} + \beta (r_j g_{ki} + r_i g_{kj})$$

are satisfied, where

$$\alpha = \frac{n}{(n-1)(n+2)} , \qquad \beta = \frac{n-2}{2(n-1)(n+2)} .$$

PROOF. We have

$$0 \leqq |\nabla_k R_{ji} - \alpha r_k g_{ji} - \beta (r_j g_{ki} + r_k g_{ji})|^2$$
$$= |\nabla \mathrm{Ric}|^2 - \frac{3n-2}{2(n-1)(n+2)} |dr|^2 . \quad \square$$

It should be noticed that (3.1) implies $\hat{C}=0$.

We shall give an application of Proposition 3. Let us assume that $\hat{C}=0$, i.e.,

$$\nabla_k R_{ji} - \nabla_j R_{ki} = \frac{1}{2(n-1)} (r_k g_{ji} - r_j g_{ki}) .$$

Applying $\nabla^k = g^{kl} \nabla_l$ to this equation and making use of Ricci's identity, we can obtain

$$\nabla^k\nabla_k R_{ji}=R_{jr}R_i{}^r-R_{kji}{}^r R_r{}^k+\frac{n-2}{2(n-1)}\nabla_j r_i+\frac{1}{2(n-1)}(\nabla^t r_t)g_{ji}.$$

Hence it holds that

$$R^{ji}\nabla^k\nabla_k R_{ji}=H+\frac{n-2}{2(n-1)}R^{ji}\nabla_j r_i+\frac{1}{2(n-1)}r\nabla^t r_t, \tag{3.2}$$

where we have put

$$H=R_{jr}R_i{}^r R^{ji}-R_{kjih}R^{kh}R^{ji}. \tag{3.3}$$

On the other hand, we have

$$r\nabla^t r_t=\nabla^t(rr_t)-|dr|^2,$$

$$R^{ji}\nabla_j r_i=\nabla_j(R^{ji}r_i)-r_i\nabla_j R^{ji}$$

$$=\nabla_j(R^{ji}r_i)-\frac{1}{2}|dr|^2.$$

Substituting these equations into (3.2) we get

$$R^{ji}\nabla^k\nabla_k R_{ji}=H-\frac{n}{4(n-1)}|dr|^2$$

$$+\frac{1}{2(n-1)}\nabla^t(rr_t)+\frac{n-2}{2(n-1)}\nabla_j(R^{ji}r_i).$$

We integrate

$$\frac{1}{2}\nabla^k\nabla_k|\mathrm{Ric}|^2=|\nabla\mathrm{Ric}|^2+R^{ji}\nabla^k\nabla_k R_{ji}$$

over compact orientable M^n to obtain

$$\int_M\left\{H+|\nabla\mathrm{Ric}|^2-\frac{n}{4(n-1)}|dr|^2\right\}d\sigma=0,$$

and hence

$$\int_M H\,d\sigma=\int_M\left\{\frac{n}{4(n-1)}|dr|^2-|\nabla\mathrm{Ric}|^2\right\}d\sigma$$

$$\leqq\frac{(n-2)^2}{4(n-1)(n+2)}\int_M|dr|^2 d\sigma$$

by virtue of Proposition 3.

Consequently we have

THEOREM 2. *Suppose that an n (>1) dimensional compact Riemannian manifold M^n satisfies $\hat{C}=0$. If the scalar function H given by* (3.3) *satisfies*

$$H\geqq\frac{(n-2)^2}{4(n-1)(n+2)}|dr|^2$$

everywhere, then (3.1) *holds.*

Let M^n be conformally flat and satisfies (3.1). Then it is easy to see that ∇R of M^n is given by

$$(3.4)\qquad \nabla_l R_{kjih}=\frac{1}{2(n-1)(n+2)}(r_k A_{ljih}+r_j A_{klih}+r_i A_{kjlh}+r_h A_{kjil}+2r_l A_{kjih}),$$

where we have put

$$(3.5)\qquad A_{kjih}=g_{kh}g_{ji}-g_{jh}g_{ki}.$$

EXAMPLE. (3.1) is the same equation as in Besse [1, p. 433]:

$$2(n-1)(n+2)D[r-(2n-2)^{-1}sg]=(n-2)ds\circ g,$$

where r and s denote the Ricci tensor and scalar curvature respectively, and $ds\circ g$ is the symmetric product of ds and g. An example of the space satisfying (3.1) are given there as compact quotients of $(\boldsymbol{R}\times\overline{M},\ dt^2+f^{-2}(t)\bar{g})$, where $(\overline{M},\ \bar{g})$ is Einstein with scalar curvature $\bar{s}<0$ and f is a positive solution of

$$\frac{d^2f}{dt^2}-\frac{2\bar{s}f^3}{(n-1)(n-2)}=cf$$

with a constant $c>0$. If $\overline{M}$ is a space of constant curvature, the above example is conformally flat [3] and hence satisfies (3.4).

§ 4. Two inequalities on $|\nabla R|$.

Let us consider the following equation:

$$(4.1)\qquad \nabla_l R_{kjih}=\lambda B_{lkjih}+\mu r_l A_{kjih},$$

where

$$(4.2)\qquad \lambda=\frac{1}{n+2},\qquad \mu=\frac{1}{(n-1)(n+2)}.$$

As B_{lkjih} are defined by (2.1) and A_{kjih} by (3.5), the equation (4.1) is written explicitly as

$$(4.3)\quad \nabla_l R_{kjih}=\lambda(g_{lk}S_{hij}-g_{lj}S_{hik}+g_{lh}S_{kji}-g_{li}S_{kjh})+\mu r_l(g_{kh}g_{ji}-g_{jh}g_{ki}).$$

By transvection (4.3) with g^{lk} and taking account of (1.1)~(1.3), we have

$$(4.4)\qquad S_{hij}=\frac{1}{2(n-1)}(r_h g_{ij}-r_i g_{hj}),$$

i.e., $\hat{C}=0$. Then if we eliminate the tensor (S_{hij}) in (4.3) by virtue of (4.4), we can get (3.4), i.e.,

$$(4.5)\qquad \nabla_l R_{kjih}=\frac{1}{2(n-1)(n+2)}(r_k A_{ljih}+r_j A_{klih}+r_i A_{kjlh}+r_h A_{kjil}+2r_l A_{kjih}).$$

Conversely, let us suppose that (4.5) holds. As we have

$$g^{ji}A_{kjih}=(n-1)g_{kh},$$
$$r^{i}A_{klih}=r_{l}g_{kh}-r_{k}g_{lh}, \qquad r^{i}A_{kilh}=r_{l}g_{kh}-r_{h}g_{kl},$$

it follows from (4.5) that

$$\nabla_l R_{kh}=g^{ji}\nabla_l R_{kjih}=\alpha r_l g_{hk}+\beta(r_k g_{lh}+r_h g_{lk}),$$

which is nothing but (3.1). Thus it is not difficult to see that (4.3) holds too. Consequently we have

LEMMA. (4.1) *and* (4.5.) *are equivalent to each other. In this case,* (3.1) *holds.*

Next we shall prove

PROPOSITION 4.

$$\frac{2}{(n+2)^2}\left\{2(n+4)|{}^{C}\nabla R|^2+\frac{n}{n-1}|dr|^2\right\}\leqq|\nabla R|^2.$$

The equality holds if and only if (4.5) *is satisfied.*

PROOF. The inequality follows from

$$0\leqq|\nabla_l R_{kjih}-\lambda B_{lkjih}-\mu r_l A_{kjih}|^2, \tag{4.6}$$

where λ and μ are given by (4.2).

In fact, the right hand side of (4.6) is

$$|\nabla R|^2+\lambda^2 a_1+\mu^2|dr|^2 a_2-2\lambda a_3-2\mu a_4+2\lambda\mu a_5,$$

where

$$a_1=|B_{lkjih}|^2=4n|S|^2, \qquad a_2=|A_{kjih}|^2=2n(n-1),$$
$$a_3=\nabla_l R_{kjih}B^{lkjih}=4|S|^2, \qquad a_4=r^l\nabla_l R_{kjih}A^{kjih}=2|dr|^2,$$
$$a_5=r^l B_{lkjih}A^{kjih}=4|dr|^2.$$

Hence we have

$$\begin{aligned}0&\leqq|\nabla R|^2+4\lambda(n\lambda-2)|S|^2+2\{n(n-1)\mu-2+4\lambda\}\mu|dr|^2\\&=|\nabla R|^2-\frac{4(n+4)}{(n+2)^2}|S|^2-\frac{2n}{(n-1)(n+2)^2}|dr|^2. \qquad \square\end{aligned}$$

By Proposition 2 and Proposition 4, we have

PROPOSITION 5. $$\frac{4}{(n-1)(n+2)}|dr|^2\leqq|\nabla R|^2.$$

The equality holds if and only if (4.5) *is satisfied.*

It should be mentioned that Proposition 5 can be obtained from the following, too:

$$0 \leqq \left| \nabla_l R_{kjih} - \frac{1}{2(n-1)(n+2)} (r_k A_{ljih} + r_j A_{klih} \right.$$
$$\left. + r_i A_{kjlh} + r_h A_{kjil} + 2 r_l A_{kjih}) \right|^2 .$$

§5. **Codazzi tensor.** In this section we shall apply our method to the Codazzi tensor.

A symmetric tensor $H=(H_{ji})$ is called a Codazzi tensor if it satisfies

$$\nabla_k H_{ji} = \nabla_j H_{ki} .$$

Let us put

$$h = H_i{}^i , \qquad h_j = \nabla_j h .$$

Then we have

$$h_j = \nabla_i H_j{}^i$$

for a Codazzi tensor H.

PROPOSITION 6. *For a Codazzi tensor H, we have*

$$\frac{3}{n+2} |dh|^2 \leqq |\nabla H|^2 .$$

The equality holds if and only if

$$\nabla_k H_{ji} = \frac{1}{n+2} (h_k g_{ji} + h_j g_{ki} + h_i g_{kj})$$

are satisfied.

PROOF. Let H be any Codazzi tensor, and the inequality follows from

$$0 \leqq |(n+2)\nabla_k H_{ji} - (h_k g_{ji} + h_j g_{ki} + h_i g_{kj})|^2$$
$$= (n+2)\{(n+2)|\nabla H|^2 - 3|dh|^2\} . \quad \square$$

§6. **An inequality in Kaehlerian manifolds.** A Kaehlerian manifold M^n of real dimension n $(=2m)$ is a Riemannian manifold admitting a parallel tensor field $F=(F_i{}^h)$ such that

$$F_i{}^r F_r{}^h = -\delta_i{}^h , \qquad g_{ji} F_k{}^j F_l{}^i = g_{kl} .$$

If we put $F_{ji} = F_j{}^h g_{hi}$, $F_{ji} = -F_{ij}$ hold. It is well known that the Ricci tensor $\mathrm{Ric}=(R_{ji})$ satisfies

$$R_{ji} F_k{}^j F_l{}^i = R_{kl} .$$

As the set of Kaehlerian manifolds constitutes a special class of the set of Riemannian manifolds, we may expect better inequalities than those in Proposition 2~6.

We are especially interested in Proposition 3 and prove

PROPOSITION 7. $$\frac{1}{m+1}|dr|^2 \leqq |\nabla \mathrm{Ric}|^2.$$

The equality holds if and only if

$$\nabla_k R_{ji} = \frac{1}{4(m+1)}(2r_k g_{ji} + r_j g_{ki} + r_i g_{kj} + \tilde{r}_j F_{ik} + \tilde{r}_i F_{jk}) \tag{6.1}$$

are satisfied, where $\tilde{r}_j = F_j{}^h r_h$.

PROOF. Let us put $\rho = \dfrac{1}{4(m+1)}$ for simplicity. Then we have

$$\begin{aligned} 0 &\leqq |\nabla_k R_{ji} - \rho(2r_k g_{ji} + r_j g_{ki} + r_i g_{kj} + \tilde{r}_j F_{ik} + \tilde{r}_i F_{jk})|^2 \\ &= |\nabla \mathrm{Ric}|^2 - 8\rho |dr|^2 + 16(m+1)\rho^2 |dr|^2 \\ &= |\nabla \mathrm{Ric}|^2 - 8\rho\{1 - 2(m+1)\rho\}|dr|^2 \\ &= |\nabla \mathrm{Ric}|^2 - \frac{2}{m+1}|dr|^2. \quad \square \end{aligned}$$

Let $K = (K_{kji}{}^h)$ be the Bochner curvature tensor [4] and $\hat{K} = (K_{kji})$ be the tensor given by

$$\begin{aligned} K_{kji} &= \nabla_k R_{ji} - \nabla_j R_{ki} \\ &+ \frac{1}{4(m+1)}(g_{ki}\delta_j^h - g_{ji}\delta_k^h + F_{ki}F_j{}^h - F_{ji}F_k{}^h + 2F_{kj}F_i{}^h) r_h. \end{aligned}$$

Then it is known that

$$\nabla_h K_{kji}{}^h = \frac{m}{m+2} K_{kji}$$

hold.

We remark that (6.1) implies $\hat{K} = 0$.

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