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A CHARACTERIZATION OF GORENSTEIN ORDERS

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

By

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Introduction.

In this paper, we give a characterization of Gorenstein orders over a commutative d-dimensional Gorenstein local ring (Theorem 1.1) and study some special classes of Gorenstein orders (Theorem 2.1). Auslander called an order Λ Gorenstein if $\Lambda^* \cong \Lambda$ as Λ-Λ-bimodules [2]. Our definition of Gorenstein orders is more general and there are other interesting orders between our sense of Gorenstein orders and that of Auslander's. These orders are studied in Theorem 2.1.

Let R be a commutative d-dimensional Gorenstein local ring with its maximal ideal $\mathfrak{m}$. Following Auslander [2], an R-algebra Λ is called an *R-order* if Λ is a finitely generated maximal Cohen-Macaulay R-module such that $\mathrm{Hom}_R(\Lambda, R)_{\mathfrak{p}}$ is a projective $\Lambda_{\mathfrak{p}}^{op}$-module for all nonmaximal prime ideals $\mathfrak{p}$ of R. We call an R-order Λ *Gorenstein* if $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ is a projective Λ^{op}-module. It is easily seen that the definition of Gorenstein orders is left-right symmetric and also that Λ^* is a progenerator for mod Λ when Λ is Gorenstein. In a classical case, that is, R being a discrete valuation ring, Gorenstein orders and their representation theories are widely studied (cf. [4, 7]). They are mostly the parallel results with those of QF algebras over a field. Gorenstein orders include both classical Gorenstein orders and QF algebras over a field. Thus many results for classical and algebra cases can be extended to our cases, and we give the most basic ones in sections 1, 2. In section 3, we give various examples concerning Gorenstein orders.

We define some more definitions and notation. We call M a Λ-lattice if it is a finitely generated Λ-module and a maximal Cohen-Macaulay R-module such that, $M_{\mathfrak{p}}$, respectively $\mathrm{Hom}_R(M, R)_{\mathfrak{p}}$ is a projective $\Lambda_{\mathfrak{p}}$, respectively $\Lambda_{\mathfrak{p}}^{op}$-module for all nonmaximal prime ideals $\mathfrak{p}$ of R. The category of all Λ-lattices is denoted by $\mathcal{L}(\Lambda)$. All modules are considered as right modules. Left modules

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are regarded as right modules over an opposite ring. We denote the category of all finitely generated Λ-modules by $\mathrm{mod}\,\Lambda$. We put $\mathrm{Ext}_\Lambda-\dim A=\sup\{j\geqq 0\,;\ \mathrm{Ext}_\Lambda^j(A, \Lambda)\neq 0\}$ for $A\in\mathrm{mod}\,\Lambda$[6]. In what follows, all the notation and definitions provided above are preserved.

1. A characterization of Gorenstein orders.

In this section, we give a characterization of Gorenstein orders and study the related topics. We see that Gorenstein orders are natural extension of both commutative Gorenstein rings and QF algebras over a field.

THEOREM 1.1. *The following conditions are equivalent for an order Λ over a d-dimensional Gorenstein local ring R.*

1) *Λ is a Gorenstein order.*
2) $\mathrm{inj\,dim}\,\Lambda=d$.
3) $\mathrm{Ext}_\Lambda-\dim S=d$ *for all simple Λ-modules S.*

Moreover, if R is complete and Λ is basic, then every condition above is equivalent to

4) $\Lambda^*=t\Lambda=\Lambda t$ *for* $t\in\Lambda^*$.

The proof of this theorem needs several lemmas. Firstly, we quote the definition and properties of injective lattices from [2]. A lattice I is called an injective lattice if every exact sequence $0\to I\to X\to Y\to 0$ in $\mathcal{L}(\Lambda)$ splits. It holds that I is an injective lattice if and only if there is a projective lattice P in $\mathcal{L}(\Lambda^{op})$ such that $I\cong\mathrm{Hom}_R(P, R)$ if and only if $\mathrm{Ext}_\Lambda^1(\ , I)|_{\mathcal{L}(\Lambda)}=0$([2, Ch I, Propositions 8.1, 8.2]).

Let $X\in\mathcal{L}(\Lambda)$ and let $0\to X\to I_0\xrightarrow{f_1} I_1\xrightarrow{f_2}\cdots$ be an injective resolution of X in $\mathcal{L}(\Lambda)$ ([2, Ch I, Proposition 8.2]). This means that the above sequence is exact and each I_j $(j=0, 1, \cdots)$ is an injective lattice. Note that $\mathrm{Ext}_\Lambda^j(A, X)\cong\mathrm{Ext}_\Lambda^1(A, \mathrm{Im}\,f_{j-1})$ for all $A\in\mathcal{L}(\Lambda)$ by [2, Ch I, Proposition 8.2 b)].

DEFINITION. Let $X\in\mathcal{L}(\Lambda)$. We call that X has an injective lattice dimension t, denoted by $\mathrm{inj}\,\mathcal{L}\dim X=t$, if $\mathrm{Ext}_\Lambda^{t+1}(\ , X)|_{\mathcal{L}(\Lambda)}=0$ and $\mathrm{Ext}_\Lambda^t(\ , X)|_{\mathcal{L}(\Lambda)}\neq 0$.

The following lemma whose proof is standard is very useful because it reduces the computations of injective dimensions to those of projective dimensions.

LEMMA 1.2. *Let $X\in\mathcal{L}(\Lambda)$. Then* $\mathrm{inj\,dim}\,X=\infty$ *if and only if* $\mathrm{inj}\,\mathcal{L}\dim X=\infty$, *and if* $\mathrm{inj\,dim}\,X$ *is finite then*

$$\mathrm{inj\,dim}\,X=\mathrm{inj}\,\mathcal{L}\dim X+d\,.$$

PROOF. If inj $\mathcal{L}$ dim $X=\infty$, then inj dim $X=\infty$. So it is sufficient to show that if inj $\mathcal{L}$ dim $X=t<\infty$ then inj dim $X=t+d$. Let S be an arbitrary simple Λ-module and $\Omega^i S$ the i-th syzygies of $S(i=0, 1, \cdots)$. It holds that $\Omega^i S\in\mathcal{L}(\Lambda)$ for $i\geqq d$ by [2, Ch I, Proposition 7.3]. So we have $\mathrm{Ext}_\Lambda^{t+i+1}(S, X)\cong\mathrm{Ext}_\Lambda^{t+1}(\Omega^i S, X)=0$ for all $i\geqq d$ and all simple Λ-modules S which implies inj dim $X\leqq d+t$ by [6, Proposition 2.7]. There is a lattice $A\in\mathcal{L}(\Lambda)$ such that $\mathrm{Ext}_\Lambda^t(A, X)\neq 0$. We have a maximal A-sequence $x_1, \cdots, x_d\in \boldsymbol{m}$. Applying $\mathrm{Ext}_\Lambda(\ , X)$ to an exact sequence $0\to A\xrightarrow{x_1}A\to A/Ax_1\to 0$ we get an exact sequence

$$\mathrm{Ext}_\Lambda^t(A, X)\xrightarrow{x_1}\mathrm{Ext}_\Lambda^t(A, X)\longrightarrow\mathrm{Ext}_\Lambda^{t+1}(A/Ax_1, X).$$

Thus $\mathrm{Ext}_\Lambda^{t+1}(A/Ax_1, X)\neq 0$ by Nakayama's Lemma. Repeating this procedure for $x_2, \cdots, x_d$ we get $\mathrm{Ext}_\Lambda^{t+d}(A/A(x_1, \cdots, x_d), X)\neq 0$. Hence we have inj dim $X=t+d$.

COROLLARY 1.3. *Let* $X\in\mathcal{L}(\Lambda)$ *and* proj dim X^* *finite with* $X^*=\mathrm{Hom}_R(X, R)$. *Then* inj dim $X=$proj dim X^*+d.

In Example 1 of section 3, we use Corollary 1.3 and compute self-injective dimension of the order which has finite self-injective dimension but is not Gorenstein.

We prove the following about the condition 4) of Theorem 1.1.

LEMMA 1.4. *Let* $\Lambda^*=t\Lambda=\Lambda t$. *Then* $\lambda t=0$ *or* $t\lambda=0$, *for* $\lambda\in\Lambda$, *implies* $\lambda=0$.

PROOF. We have $0=\lambda t(\mu)=t(\mu\lambda)=t\mu(\lambda)$ for all $\mu\in\Lambda$. Thus λ is in the kernel of the canonical homomorphism $\Lambda\to\Lambda^{**}$ which is an isomorphism. Hence $\lambda=0$.

PROOF OF THEOREM 1.1. 1)$\Leftrightarrow$2): Λ is Gorenstein.$\Leftrightarrow$proj dim ${}_{\Lambda^{op}}\Lambda^*=0\Leftrightarrow$ inj dim $\Lambda=d$ by Corollary 1.3. By [6, Corollary 2.8] we have 3)$\Rightarrow$2). We shall prove 1)$\Rightarrow$3). Let $0\to R\to I_0\to I_1\to\cdots\to I_d\to 0$ be a minimal injective resolution of R. Since R is Gorenstein and Λ is a maximal Cohen-Macaulay R-module, we have $\mathrm{Ext}_R^i(\Lambda, R)=0$ $(i\geqq 1)$, so that (*) $0\to\mathrm{Hom}_R(\Lambda, R)\to\mathrm{Hom}_R(\Lambda, I_0)\to\cdots\to\mathrm{Hom}_R(\Lambda, I_d)\to 0$ is exact and $\mathrm{Hom}_R(\Lambda, I_i)$ $(0\leqq i\leqq d)$ are injective Λ-modules. Let S be a simple Λ-module. Applying $\mathrm{Hom}_R(S,\ )$ to the sequence (*) we get the complex $0\to\mathrm{Hom}_\Lambda(S, \mathrm{Hom}_R(\Lambda, I_0))\to\cdots\to\mathrm{Hom}_\Lambda(S, \mathrm{Hom}_R(\Lambda, I_d))\to 0$. Since S is an R-module of finite length, we have $\mathrm{Hom}_\Lambda(S, \mathrm{Hom}_R(\Lambda, I_j))\cong\mathrm{Hom}_R(S, I_j)=0$ for $j=0, \cdots, d-1$. Thus $\mathrm{Ext}_\Lambda^d(S, \Lambda^*)\cong\mathrm{Hom}_\Lambda(S, \mathrm{Hom}_R(\Lambda, I_d))\cong\mathrm{Hom}_R(S, I_d)\neq 0$. By 1) Λ^* is a progenerator, so that $\mathrm{Ext}_\Lambda^d(S, \Lambda)\neq 0$. Since inj dim $\Lambda=\sup\{\mathrm{Ext}_\Lambda-$

dim S; S is simple} by [6, Corollary 2.8], we have Ext_{Λ}-dim $S=d$ for all simple Λ-modules S. This proves 1)$\Rightarrow$3). 4)$\Rightarrow$1) is an easy consequence of Lemma 1.4. 1)$\Rightarrow$4): Suppose that R is complete and that Λ is basic. Let $\{e_1, \cdots, e_n\}$ be the complete set of primitive idempotents of Λ. Since Λ is Gorenstein and basic, there exists a permutation π of the set $\{1, \cdots, n\}$ such that we have Λ^{op}-isomorphisms $u_i: \Lambda e_i \xrightarrow{\sim} \mathrm{Hom}_R(e_{\pi(i)}\Lambda, R)$ for $i=1, \cdots, n$. Let $v_{\pi(i)}$ be the composition of a canonical isomorphism $\phi_{\pi(i)}: e_{\pi(i)}\Lambda \xrightarrow{\sim} (e_{\pi(i)}\Lambda)^{**}$ and the induced isomorphism $\mathrm{Hom}_R(u_i, R): (e_{\pi(i)}\Lambda)^{**} \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda e_i, R)$ $(1 \leqq i \leqq n)$. Put $t_i=u_i(e_i)$ and $s_j=v_j(e_j)$ $(1\leqq i, j\leqq n)$. Letting the value of t_i on $e_j\Lambda$ be zero if $j\neq\pi(i)$, we consider $t_i\in\mathrm{Hom}_R(\Lambda, R)$. Similary, we consider $s_i\in\mathrm{Hom}_R(\Lambda, R)$. It holds that $e_it_j=\delta_{ij}t_j$, $s_ie_j=\delta_{ij}s_i$ $(1\leqq i, j\leqq n)$. For all $\lambda\in\Lambda$, $s_{\pi(i)}(\lambda e_i)=\mathrm{Hom}_R(u_i, R)(\phi_{\pi(i)}(e_{\pi(i)}))(\lambda e_i)=\phi_{\pi(i)}(e_{\pi(i)})(\lambda t_i)=\lambda t_i(e_{\pi(i)})=t_i(e_{\pi(i)}\lambda)$. Thus, for all $\lambda\in\Lambda$, $t_i(\lambda)=t_i(e_{\pi(i)}\lambda)=s_{\pi(i)}(\lambda e_i)=s_{\pi(i)}(\lambda)$. Let $t=\sum_{i=1}^n t_i=\sum_{i=1}^n s_{\pi(i)}\in\mathrm{Hom}_R(\Lambda, R)$. Since $\mathrm{Hom}_R(\Lambda, R)=\bigoplus\mathrm{Hom}_R(e_i\Lambda, R)=\bigoplus\Lambda t_i$, $f=\sum\lambda_it_i(\lambda_i\in\Lambda)$ for every $f\in\mathrm{Hom}_R(\Lambda, R)$. Thus $f=(\sum\lambda_ie_i)(\sum t_i)=\lambda t$ by the above computation, where $\lambda=\sum\lambda_ie_i$. Hence $\mathrm{Hom}_R(\Lambda, R)=\Lambda t$. Similary, using $t=\sum s_i$ we can prove $\Lambda^*=t\Lambda$. This completes the proof of the theorem.

REMARK 1.5. a) In the case of dim $R=1$, the equivalence of 1) and 2) of Theorem 1.1 was shown in [4, Proposition 6.1].

b) If Λ is quasi-local, i.e., $\Lambda/\mathrm{rad}\,\Lambda$ is a simple ring, then Λ is Gorenstein if and only if inj dim $\Lambda<\infty$ (see Corollary 1.8 below). However, there exists an order having the finite global dimension greater than d. Then inj dim Λ=gl dim Λ is finite, but Λ is not Gorenstein. Moreover, we will give the order Λ which has infinite global dimension, finite self-injective dimension, and is not Gorenstein in Example 1 of section 3.

c) Let $\phi: \Lambda\times\Lambda\to R$ be $\phi(\lambda, \mu)=t(\lambda\mu)$. Then ϕ is a nondegenerate associative bilinear form by Lemma 1.4. However, differing from an algebra case the existence of such ϕ doesn't necessarily imply the condition 4). Because if we define $t\in\Lambda^*$ by $t(\lambda)=\phi(\lambda, 1)$, then $\Lambda\cong\Lambda t\subset\Lambda^*$. Thus $\mathrm{rank}_R\Lambda=\mathrm{rank}_R\Lambda^*=\mathrm{rank}_R\Lambda t$. But we can't conclude $\Lambda t=\Lambda^*$ by this.

We generalize [6, Proposition 2.14] in the following.

PROPOSITION 1.6. *Let Λ be a Gorenstein order and E a finitely generated Λ-module. Then we have*

$$\mathrm{Ext}_{\Lambda}\text{-dim } E+\mathrm{depth}_R E=d\,.$$

PROOF. Using Theorem 1.1, the proof of [6, Proposition 2.14] works as it is.

COROLLARY 1.7. *If Λ is a Gorenstein order, then $M\in\mathcal{L}(\Lambda)$ if and only if* $\mathrm{Ext}^i_\Lambda(M, \Lambda)=0$ $(i\geqq 1)$ *and $M_{\mathfrak{p}}$ is $\Lambda_{\mathfrak{p}}$-projective for all nonmaximal prime ideals $\mathfrak{p}$ of R. Therefore,* $\mathrm{Hom}_\Lambda(\ ,\Lambda)$ *induces a duality between $\mathcal{L}(\Lambda)$ and $\mathcal{L}(\Lambda^{op})$.*

Combining Theorem 1.1 with [6, Proposition 2.14] we get the following.

COROLLARY 1.8. *Let Λ be quasi-local. Then Λ is Gorenstein if and only if* $\operatorname{inj\,dim}\Lambda<\infty$.

The above homological properties generalize those of commutative cases. The condition 4) provides another properties. Let $\Lambda^*=t\Lambda=\Lambda t$ and $\sigma:\Lambda\to\Lambda$ be defined by $t\lambda=\lambda^\sigma t$ $(\lambda\in\Lambda)$. Then σ is an automorphism of Λ by Lemma 1.4. Denote this σ by σ_t. If there exists another $s\in\Lambda^*$ with $\Lambda^*=s\Lambda=\Lambda s$, then there exists a unit λ of Λ such that $\sigma_s=i_\lambda\sigma_t$, where i_λ is an inner automorphism of Λ defined by $\mu^{i_\lambda}=\lambda\mu\lambda^{-1}$ $(\mu\in\Lambda)$. We generalize the computation in [3, Section 3] to our situation and describe the Nakayama functor $\mathcal{N}=\mathrm{Hom}_R(\mathrm{Hom}_\Lambda(\ ,\Lambda), R)$ using the above automorphism $\sigma=\sigma_t$. For $M\in\operatorname{mod}\Lambda$, let M^σ be the same additive group as M. The action of $\lambda\in\Lambda$ to M^σ is defined by $m\circ\lambda=m\lambda^\sigma$ $(m\in M)$. Then M^σ is a Λ-module.

PROPOSITION 1.9. *Let Λ be a Gorenstein order. Then $\mathcal{N}(M)\cong M\otimes_\Lambda\mathrm{Hom}_R(\Lambda, R)$ $\cong M^\sigma$ for $M\in\mathcal{L}(\Lambda)$.*

PROOF. Note that $\mathrm{Hom}_R(M\otimes_\Lambda\Lambda^*, R)\cong\mathrm{Hom}_\Lambda(M,\mathrm{Hom}_R(\Lambda^*, R))\cong\mathrm{Hom}_\Lambda(M,\Lambda)$. Since Λ is Gorenstein, $M\otimes_\Lambda\Lambda^*$ is a Λ-lattice, so that $\mathcal{N}(M)\cong M\otimes_\Lambda\Lambda^*$. Since $\Lambda^*=\Lambda t$, each element of $M\otimes_\Lambda\Lambda^*$ is of the form $m\otimes t$ where m is in M. For $\lambda\in\Lambda$ and $m\otimes t\in M\otimes_\Lambda\Lambda^*$, $(m\otimes t)\lambda=m\otimes t\lambda=m\lambda^\sigma\otimes t$. Hence the map $M\otimes_\Lambda\Lambda^*\to M^\sigma$, $m\otimes t\mapsto m$, is an isomorphism of Λ-lattices.

2. Gorenstein orders in the sence of Auslander.

In [2], Auslander called an order Λ Gorenstein if $\Lambda^*\cong\Lambda$ as Λ-Λ-bimodules. This class of orders occupies the position of "symmetric orders" in the class of Gorenstein orders in our sense. Thus we investigate these orders and related ones in Theorem 2.1.

Let Λ be a Gorenstein order. Then, for any indecomposable projective Λ-module P, there exists a unique simple Λ-module $S(P)$ such that $\mathrm{Ext}^d_\Lambda(S(P), P)$ $\neq 0$ [2, Ch III, Proposition 3.3]. The notation provided in the end of the previous section is preserved in this section.

THEOREM 2.1. *Consider the following conditions for an order* Λ.

1) $\Lambda \cong \Lambda^*$ *as* Λ-Λ-*bimodules.*

2) *There exists* $t \in \Lambda^*$ *such that* $\Lambda^* = t\Lambda$ *and* $t\lambda = \lambda t$ *for all* $\lambda \in \Lambda$.

3) Λ *is Gorenstein and there exists* $s \in \Lambda^*$ *with* $\Lambda^* = s\Lambda = \Lambda s$ *such that* $\sigma_s = i_\lambda$ *for a unit* λ *of* Λ.

4) Λ *is Gorenstein and all* $s \in \Lambda^*$ *with* $\Lambda^* = s\Lambda = \Lambda s$ *satisfy* $\sigma_s = i_\lambda$ *for a unit* λ *of* Λ.

5) $\mathcal{N} = \mathrm{id}$ *on* $\mathcal{L}(\Lambda)$, *where* $\mathcal{N} = \mathrm{Hom}_R(\mathrm{Hom}_\Lambda(\ ,\Lambda), R)$.

5. a) $\mathcal{N} = \mathrm{id}$ *on* $\mathrm{pr}(\Lambda)$, *where* $\mathrm{pr}(\Lambda)$ *is a full snbcategory of* $\mathcal{L}(\Lambda)$ *consisting of all projective* Λ-*modules.*

5. b) $\mathcal{N} = \mathrm{id}$ *on* $\mathcal{L}(\Lambda) - \mathrm{pr}(\Lambda)$.

6) Λ *is Gorenstein and* $S(P) \cong P/\mathrm{rad}\, P$ *for all indecomposable projective* Λ-*modules* P.

Then we have 1)$\Leftrightarrow$2)$\Leftrightarrow$3)$\Leftrightarrow$4)$\Rightarrow$5)$\Rightarrow$5. a)$\Leftrightarrow$6) *and* 5)$\Rightarrow$5. b). *Moreover, if* $\dim R = 2$, *then* 5. b) *implies that* Λ *is Gorenstein.*

PROOF. Using Lemma 1.4 1)$\Leftrightarrow$2) is easily obtained. 2)$\Rightarrow$4): There exists a unit λ of Λ such that $\sigma_s = i_\lambda \sigma_t = i_\lambda$ by 2). 4)$\Rightarrow$3): This is trivial. 3)$\Rightarrow$2): By assumption there exists a unit λ of Λ with $\mu^{\sigma_s} = \lambda\mu\lambda^{-1}$, $\mu \in \Lambda$. Put $t = \lambda^{-1}s \in \Lambda^*$. Then $t\mu = \lambda^{-1}s\mu = \lambda^{-1}\mu^{\sigma_s}s = \mu t$ for all $\mu \in \Lambda$. 1)$\Rightarrow$5): We have $\mathrm{Hom}_\Lambda(X, \Lambda) \cong \mathrm{Hom}_R(X, R)$ for all $X \in \mathcal{L}(\Lambda)$ by assumption. Thus $\mathcal{N} = \mathrm{id}$ on $\mathcal{L}(\Lambda)$. 5)$\Rightarrow$5. a), 5. b): They are trivial. 5. a)$\Leftrightarrow$6): We prove this under the assumption that Λ is Gorenstein. By [2, Ch III, Proposition 3.3], for a simple Λ-module S and a projective Λ-module P, we have $\mathrm{Ext}_\Lambda^d(S, P) \cong \mathrm{Hom}_R(S \otimes_\Lambda \mathrm{Hom}_R(P, R), I_d)$, where $0 \to R \to I_0 \to \cdots \to I_d \to 0$ is a minimal injective resolution of R. Put $P' = \mathrm{Hom}_\Lambda(\mathrm{Hom}_R(P, R), \Lambda)$. Then $\mathrm{Hom}_\Lambda(P', \Lambda) \cong \mathrm{Hom}_R(P, R)$ and $\mathrm{Ext}_\Lambda^d(S, P) \cong \mathrm{Hom}_R(\mathrm{Hom}_\Lambda(P', S), I_d)$. Thus we have $\mathrm{Ext}_\Lambda^d(S, P) \neq 0 \Leftrightarrow P'/\mathrm{rad}\, P' \cong S$. This implies $S(P) \cong P/\mathrm{rad}\, P \Leftrightarrow P \cong P' \Leftrightarrow \mathrm{Hom}_R(P, R) \cong \mathrm{Hom}_\Lambda(P, \Lambda) \Leftrightarrow \mathcal{N}(P) \cong P$. This proves 5. a)$\Leftrightarrow$6). Finally we assume $\dim R = 2$ and 5. b) and prove Λ to be Gorenstein. It suffices to show that $Q^* \in \mathrm{pr}(\Lambda)$ for all $Q \in \mathrm{pr}(\Lambda^{op})$. If it doesn't hold, then there exists $Q \in \mathrm{pr}(\Lambda^{op})$ with Q^* not projective. By assumption $Q^* \cong \mathcal{N}(Q^*)$. Since $\dim R = 2$, $\mathrm{Hom}_\Lambda(Q^*, \Lambda)$ is a maximal Cohen-Macaulay R-module. Thus $Q \cong \mathrm{Hom}_\Lambda(Q^*, \Lambda)$. By [2, Ch I, Lemma 7.8] we have $Q^* \cong \mathrm{Hom}_{\Lambda^{op}}(\mathrm{Hom}_\Lambda(Q^*, \Lambda), \Lambda^{op})$ which is a projective Λ-module, a contradiction. This completes the proof.

REMARK 2.2. a) We explain that some implications in Theorem 2.1 are proper. The examples are given in section 3.

i) There exists an order which satisfies 5) but not 1) (Example 2).

ii) There exists an order which satisfies 5. b) but not 5. a) (Example 3).

iii) Since a local Gorenstein order always satisfies 5. a), the orders of type (IV), (V) in [1] satisfy 5. a) but not 5. b) (see [3, Section 3]).

iv) There exists a Gorenstein order which satisfies neither 5. a) nor 5. b) (Example 4).

v) When $\dim R=1$ there exists an order which satisfies 5. b) but is not Gorenstein (Example 5).

b) Whether 5. b)$\Rightarrow \Lambda$ Gorenstein holds, or not for $\dim R\geqq 3$ is an open question. It holds that $\mathrm{Hom}_{\Lambda^{op}}(\mathrm{Hom}_{\Lambda}(M, \Lambda), \Lambda^{op})\cong M$ for all $M\in\mathcal{L}(\Lambda)$ when $\dim R\geqq 2$ by [2, Ch I, Lemma 7.8]. However, we can't prove that $\mathrm{Hom}_{\Lambda}(M, \Lambda)$ is a maximal Cohen-Macaulay R-module for $M\in\mathcal{L}(\Lambda)$ when $\dim R\geqq 3$.

The results in [2, Ch III, Section 1] for Gorenstein orders in the sense of Auslander also hold for those in our sense after a slight modification using the Nakayama functor. In particular, for orders satisfying 5. b), i. e., $\mathcal{N}=\mathrm{id}$ on $\mathcal{L}(\Lambda)-\mathrm{pr}(\Lambda)$, they hold without any change. Here we only state the results about Auslander-Reiten translation $\tau=DTr_L$ in the following.

PROPOSITION 2.3. ([2, Ch III, Proposition 1.8]) *Let R be a complete Gorenstein local ring with $\dim R=d$ and Λ a Gorenstein R-order. Then*

1) $\tau(A)=\mathcal{N}\Omega^{2-d}(A)$ *for any nonprojective Λ-lattice A.*

2) $\tau^{-1}(A)=\Omega^{d-2}\mathcal{N}^{-}(A)$ *for any noninjective Λ-lattice A, where* $\mathcal{N}^{-}=\mathrm{Hom}_{\Lambda^{op}}(\mathrm{Hom}_R(\ , R), \Lambda^{op})$.

3. Examples.

EXAMPLE 1. (Fujita) Let R be a discrete valuation ring with prime element π. Then $\dim R=1$. We provide an R-order Λ with $\mathrm{gl\,dim}\,\Lambda=\infty$ and $\mathrm{inj\,dim}\,\Lambda=2$.

Let $\Lambda=\begin{pmatrix} R & \pi R & \pi R & \pi R & \pi R \\ R & R & \pi R & R & \pi R \\ R & \pi R & R & R & R \\ \pi R & \pi R & \pi R & R & \pi R \\ \pi R & \pi R & \pi R & R & R \end{pmatrix}$. Put $e_i=\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 \cdots\cdots & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}(i$ and $P_i=e_i\Lambda$ $(1\leqq i\leqq 5)$. Then $P_1^*\cong\Lambda e_4$, $P_2^*\cong\Lambda e_5$, $P_3^*\cong\Lambda e_2$, $P_5^*\cong\Lambda e_1$ are projective Λ^{op}-modules and $P_4^*\cong(R\ R\ R\ \pi R\ R)^t$ is not projective with its projective resolution $0\to\Lambda e_3\to\Lambda e_1\oplus\Lambda e_5\to P_4^*\to 0$. Thus $\mathrm{proj\,dim}\,P_4^*=1$, and so $\mathrm{proj\,dim}\,\Lambda^*=1$. By Corollary 1.3, $\mathrm{inj\,dim}\,\Lambda=2$. On the other hand, we have $\mathrm{gl\,dim}\,\Lambda=\infty$ by [5, Example 3.4].

Next example is due to Artin ([1], see also [3]).

EXAMPLE 2. The order of type (II_k) in [1]. Let R be the power series ring $\boldsymbol{k}[[u, v]]$, where $\boldsymbol{k}$ is an algebraically closed field with char $\boldsymbol{k}=0$. Λ is an R-order generated by x, y with the relations

$$x^2=u, \quad y^2=u^k v, \quad xy+yx=2v.$$

By [3] we have $\mathcal{N}=\mathrm{id}$. Thus Λ satisfies 5) of Theorem 2.1. It is also noted in [3] that Λ doesn't satisfy 1). However, we provide here some more computation in order to see how Theorem 2.1 can be applied to this case. We use the same notation as in [3]. We have $\Lambda^*=f_{xy}\Lambda=\Lambda f_{xy}$. For $\phi\in\Lambda^*$, $\phi=((\phi(xy)-2v\phi(1))-\phi(y)x+\phi(x)y+\phi(1)xy)f_{xy}=f_{xy}((\phi(xy)-2v\phi(1))+\phi(y)x-\phi(x)y+\phi(1)xy)$. Thus $\sigma=\sigma_{f_{xy}}$ is given by $\sigma(r_1+r_2x+r_3y+r_4xy)=r_1-r_2x-r_3y+r_4xy$. If Λ satisfies 1), then there exists a unit λ of Λ with $\mu^\sigma=\lambda\mu\lambda^{-1}$ for all $\mu\in\Lambda$ by Theorem 2.1. Since $\sigma^2=\mathrm{id}$, λ^2 is in the center of Λ, i.e., $\lambda^2\in R$. Put $\lambda=r_1+r_2x+r_3y+r_4xy$. Then $\lambda^2=r_1^2+r_2^2u+r_3^2u^kv-r_4^2u^{k+1}v+2r_2r_3v+2r_2(r_1+r_4v)x+2r_3(r_1+r_4v)y+2r_4(r_1+r_4v)xy$. Thus $r_2=r_3=r_4=0$ or $r_1+r_4v=0$. In the former case, we have $\lambda\in R$, and then $\sigma=\mathrm{id}$, a contradiction. In the latter case, a unit $\lambda^2\in\mathrm{rad}\,R$, a contradiction. Thus Λ doesn't satisfy 1).

In the following examples 3, 4, 5, R is the same as in Example 1.

EXAMPLE 3. (Roggenkamp [7]) An order Λ such that $\mathcal{N}=\mathrm{id}$ on $\mathcal{L}(\Lambda)-\mathrm{pr}(\Lambda)$, but $\mathcal{N}\neq\mathrm{id}$. Let

$$\Lambda=\begin{pmatrix} R & \pi R & R & R \\ \pi R & R & R & R \\ \pi R & \pi R & R & R \\ \pi R & \pi R & \pi R & R \end{pmatrix}, \quad e_1=\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad e_2=\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad e_3=\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where $R=R$ means the set $\{(x+\pi y, x);\ x, y\in R\}$. Put $P_i=e_i\Lambda$ $(i=1, 2, 3)$. The Auslander-Reiten quiver of $\mathcal{L}(\Lambda)$ is

$$\begin{tikzcd}[column sep=small]
 & (R\ \pi R\ R\ R)=P_1 \arrow[dr] & & & \\
(\pi R\ \pi R\ R\ R) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & (R\ R\ R\ R) \arrow[dr] \arrow[rr, dotted, no head] & & (\pi R\ \pi R\ R\ R) \\
 & (\pi R\ R\ R\ R)=P_2 \arrow[ur] & & \begin{pmatrix} \pi R & \pi R & R & R \\ \pi R & \pi R & \pi R & R \end{pmatrix}=P_3 \arrow[ur] \arrow[dr] & \\
 & & (\pi R\ \pi R\ \pi R\ R) \arrow[ur] \arrow[rr, dotted, no head] & & (\pi R\ \pi R\ \pi R\ R)
\end{tikzcd}$$

where dotted lines represents τ-orbits and isomorphic lattices are identified. Using Proposition 2.3 we can show $\mathcal{N}=\mathrm{id}$ on $\mathcal{L}(\Lambda)-\mathrm{pr}(\Lambda)$. On the other hand, $\mathcal{N}(P_1)=P_2$, $\mathcal{N}(P_2)=P_1$, $\mathcal{N}(P_3)=P_3$ hold.

EXAMPLE 4. A Gorenstein order which satisfies neither 5. a) nor 5. b). Let $\Lambda=\begin{pmatrix} R & \pi^2R & \pi^2R \\ R & R & \pi^2R \\ R & R & R \end{pmatrix}$. Put $e_1=\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$, $e_2=\begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$, $e_3=\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$ and $P_i=e_i\Lambda$, $Y_i=\mathrm{rad}\,P_i$ $(i=1, 2, 3)$. Then we have $P_1^*=\Lambda e_2$, $P_2^*=\Lambda e_3$, $P_3^*=\Lambda e_1$ and $\mathcal{N}(P_1)=P_2$, $\mathcal{N}(P_2)=P_3$, $\mathcal{N}(P_3)=P_1$, $\mathcal{N}(Y_1)=Y_3$, $\mathcal{N}(Y_2)=Y_1$, $\mathcal{N}(Y_3)=Y_2$ by direct computations. Thus Λ satisfies our requirement.

EXAMPLE 5. Let $\Lambda=\begin{pmatrix} R & \pi R & \pi R \\ \pi R & R & \pi R \\ R & R & R \end{pmatrix}$ and e_i, P_i be the same as in Example 4 $(i=1, 2, 3)$. We have $P_1^*=\Lambda e_2$, $P_2^*=\Lambda e_1$, and P_3^* is not Λ^{op}-projective. Thus Λ is not Gorenstein. Put $Y=(R\ R\ \pi R)=\mathrm{rad}\,P_3$. Then $\mathcal{N}(Y)=Y$, $\mathcal{N}(P_1)=P_2$, $\mathcal{N}(P_2)=P_1$, $\mathcal{N}(P_3)=Y$. Since P_3 is a noninjective lattice and $P_3\cong\mathrm{rad}\,P_2$, we have an almost split sequence $0\to P_3\to P_1\oplus P_2\to Y\to 0$. Since $Y\cong(\Lambda e_3)^*$ is an injective lattice, the following is a connected component of the Auslander-Reiten quiver of $\mathcal{L}(\Lambda)$.

$$\begin{tikzcd}
 & P_1 \arrow[dr] & \\
P_3 \arrow[ur] \arrow[dr] & & Y \arrow[ll] \\
 & P_2 \arrow[ur] &
\end{tikzcd}.$$

Thus it coincides with the Auslander-Reiten quiver of $\mathcal{L}(\Lambda)$. Therefore, $\mathcal{N}=\mathrm{id}$ on $\mathcal{L}(\Lambda)-\mathrm{pr}(\Lambda)$, but Λ is not Gorenstein. We note that $\mathrm{Hom}_{\Lambda^{op}}(\mathrm{Hom}_\Lambda(Y, \Lambda), \Lambda^{op})\cong(R\ R\ R)\neq Y$ holds in this case.

Note added in proof. Vasconcelos obtained Proposition 1.6 in the more general context (On quasi-local regular algebras, Sympos. Math. XI, Academic Press, 1973, 11-22). He called an algebra which satisfies the condition 3) of Theorem 1.1 moderated Gorenstein algebra and showed the equality of Proposition 1.6 (section 3, Remark of the above cited paper).

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