

## CARDINAL FUNCTIONS OF SPACES WITH ORTHO-BASES

By

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### § 1. Introduction.

Throughout this paper, "space" will mean  $T_1$ -space. Let  $\mathcal{B}$  be a base of a space  $X$ .  $\mathcal{B}$  is said to be an ortho-base if for every  $\mathcal{B}' \subset \mathcal{B}$ ,  $\cap \mathcal{B}'$  is open or  $\mathcal{B}'$  is a neighborhood base of some point.  $\mathcal{B}$  is said to have subinfinite rank if for every  $\mathcal{B}' \subset \mathcal{B}$  such that  $\cap \mathcal{B}' \neq \phi$  and  $\mathcal{B}'$  is infinite, at least two elements of  $\mathcal{B}'$  are related by set inclusion. Spaces having an ortho-base, and spaces having a base of subinfinite rank were introduced by Nyikos as natural generalizations of non-archimedean spaces [4] [5].

Concerning cardinal functions of spaces with special bases, Gruenhage showed that for each regular space  $X$  having a base of subinfinite rank,  $d(X) = hd(X) \geq hl(X) = s(X)$  holds [3].  $d(X)$  is the density of  $X$ ,  $hd(X)$  is the hereditary density,  $hl(X)$  is the hereditary Lindelöf degree, and  $s(X)$  is the spread (i. e., the supremum of the discrete subspaces of  $X$ ). In this paper we investigate cardinal functions of spaces having ortho-bases. We shall show that  $hd(X) \geq hl(X) = s(X)$  holds for each space  $X$  having an ortho-base.

### § 2. Main result.

We need two lemmas. For convenience, for a cardinal  $\tau$ , we say a space  $X$  to be  $\tau$ -developable if there exist  $\tau$  open covers  $\{\mathcal{H}_\alpha\}_{\alpha < \tau}$  such that for each  $x \in X$   $\{\text{St}(x, \mathcal{H}_\alpha)\}_{\alpha < \tau}$  is a neighborhood base of  $x$ .

LEMMA 1. *Let  $X$  be a space having an ortho-base  $\mathcal{B}$  and  $D$  be the set of isolated points of  $X$ . If  $D$  is dense in  $X$ , then  $X$  is  $|D|$ -developable.*

PROOF. Set  $D = \{d_\alpha \mid \alpha < \tau\}$ , where  $\tau$  is a cardinal. For each  $x \in X - D$  and  $\alpha < \tau$ , we take  $B_\alpha(x) \in \mathcal{B}$  such that  $x \in B_\alpha(x)$  and  $d_\alpha \notin B_\alpha(x)$ . Put  $\mathcal{H}_\alpha = \{\{d_\alpha\} \mid \alpha < \tau\} \cup \{B_\alpha(x) \mid x \in X - D\}$ .  $\mathcal{H}_\alpha$  is obviously an open cover of  $X$ . Let  $x$  be a point of  $X$  and  $W$  be a neighborhood of  $x$ . If  $x \in D$ , then  $\text{St}(x, \mathcal{H}_\alpha) = \{x\} \subset W$  for some  $\alpha$ . So, we assume  $x \in X - D$ . Suppose that  $\text{St}(x, \mathcal{H}_\alpha) \not\subset W$  for any  $\alpha < \tau$ . Then for each  $\alpha$ , we can take  $H_\alpha \in \mathcal{H}_\alpha$  such that  $x \in H_\alpha$  and  $H_\alpha \not\subset W$ . Since  $\{H_\alpha\}_{\alpha < \tau}$  can not be a neigh-

neighborhood base of  $x$ ,  $H = \bigcap_{\alpha < \tau} H_\alpha$  must be open. But  $H \cap D = \emptyset$ , because  $H_\alpha \not\supset d_\alpha$ . Since  $D$  is dense in  $X$ , this is a contradiction.

The following lemma is well known in the countable case and can be easily carried over to the general case. So we omit the proof.

LEMMA 2. *Let  $X$  be  $\tau$ -developable. If the cardinality of each closed discrete subspace is at most  $\tau$ , then  $X$  is  $\tau$ -Lindelöf (i.e., every open cover has a subcover of the cardinality  $\tau$ ).*

THEOREM 3. *Let  $X$  be a space having an ortho-base  $\mathcal{B}$ . Then  $hd(X) \geq s(X) = hl(X)$  holds.*

PROOF. Since  $hd(X) \geq s(X)$  and  $hl(X) \geq s(X)$  are obvious, we show  $s(X) \geq hl(X)$ . Let  $s(X) = \tau$ . Since for each subspace  $Y$  of  $X$ ,  $s(Y) \leq \tau$  and  $Y$  has an ortho-base, the proof is complete if we show that  $X$  is  $\tau$ -Lindelöf. Suppose that there exists an open cover  $\mathcal{Q}$  of  $X$  which has not a subcover of the cardinality  $\tau$ . Firstly we take  $x_0 \in X$  and  $U_0 \in \mathcal{Q}$  such that  $x_0 \in U_0$ . Put  $V_0 = U_0$ . Let  $\gamma < \tau^+$ . We assume that for each  $\beta < \gamma$  we could take  $x_\beta \in X$  and an open set  $V_\beta$  such that the following (\*) is satisfied.

$$(*) \quad \begin{cases} V_\beta \cap \{x_\alpha | \alpha < \gamma\} = \{x_\beta\} & \text{for each } \beta < \gamma. \\ \text{There exists } U_\beta \in \mathcal{Q} \text{ such that } V_\beta \subset U_\beta & \text{for each } \beta < \gamma. \end{cases}$$

Then, if we set  $A = \{x_\alpha | \alpha < \gamma\}$ , since  $|A| \leq \tau$ ,  $\text{Cl} A$  is  $\tau$ -Lindelöf by Lemma 1 and Lemma 2. Thus  $\text{Cl} A \cup (\bigcup_{\beta < \gamma} V_\beta)$  is covered by  $\tau$  elements of  $\mathcal{Q}$ . So we can take  $x_\gamma \in X - \text{Cl} A \cup (\bigcup_{\beta < \gamma} V_\beta)$ . We take  $U_\gamma \in \mathcal{Q}$  and an open set  $V_\gamma$  such that  $x_\gamma \in V_\gamma \subset U_\gamma$  and  $V_\gamma \cap A = \emptyset$ . Now by the induction we get the discrete space  $\{x_\alpha | \alpha < \tau^+\}$ . This is a contradiction to  $s(X) = \tau$ .

There exists a space having an ortho-base such that  $hd(X) \neq d(X)$ . In fact the space in [6, 3.6.I] is such a space.

Concerning SH (Souslin's hypothesis), we note the following theorem.

THEOREM 4. *The following (a), (b) and (c) are equivalent.*

(a) *SH is false.*

(b) *There exists a non-metrizable non-archimedean space such that  $s(X)$  is countable.*

(c) *There exists a non-metrizable regular space having an ortho-base such that  $s(X)$  is countable.*

PROOF. The equivalence of (a) and (b) is due to [1]. Also, refer [5, Theorem 1.7]. (b) $\rightarrow$ (c) is trivial. We show (c) $\rightarrow$ (b). Let  $X$  be a space of (c). Since by Theorem 3  $X$  is regular Lindelöf, it is paracompact. Therefore  $X$  is a proto-metrizable space (i. e., paracompact space having an ortho-base). It follows from Fuller's result [2, Theorem 6] that  $X$  is the perfect irreducible image of a non-archimedean space  $Y$ . Since metrizable is an invariant of perfect maps,  $Y$  is not metrizable. Since the spread of a non-archimedean space is equal to the cellularity, by the irreducibility of the map,  $s(Y)$  must be countable. Thus  $Y$  is the desired space.

COROLLARY 5. *The following (a) and (b) are equivalent.*

(a) *SH.*

(b) *Each regular space having an ortho-base is metrizable if the spread is countable.*

#### References

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