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DERIVATIONS OF SPECIAL SUBRINGS OF MATRIX RINGS AND REGULAR GRAPHS

By

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Introduction.

Let R be a ring with identity and let $M_n(R)$ denote the ring of $n\times n$ matrices over R. We say that a subring P of $M_n(R)$ is a special with the relation ρ if P is of the form

$$P=\{A\in M_n(R)\,;\ A_{ij}=0 \text{ for } (i, j)\notin\rho\}\ ,$$

where ρ is a relation (reflexive and transitive) on the set $\{1, \cdots, n\}$.

It is well-known (see [3], p. 100) that if R is a field, then any R-derivation of $M_n(R)$ is inner. In this paper we give (Corollaries 4.4; 5.11) the following generalization of this result:

THEOREM. *If P is a special subring of $M_n(R)$ with the relation ρ then the following conditions are equivalent*

(1) *Any R-derivation of P is inner,*
(2) *ρ is regular over $Z(R)$,*
(3) $H^1(\tilde{\Gamma}, Z(R))=0$.

The definition of a regular relation over an abelian group G is given in Section 2 (some equivalent conditions are given in Section 5). The definition of the complex $\tilde{\Gamma}$ (dependent on the relation ρ only) is given in Section 5.

Moreover, if we assume that any derivation in R is inner, then the equivalent conditions (2), (3) of the above theorem imply that every derivation of P (non necessary R-derivation) is inner (Theorem 3.10, Corollary 5.11).

In Section 3 we give (Theorem 3.7) a description of all derivations of P. A similar result (without the uniqueness) was obtained by Burkow in [2] for generalized quasi-matrix algebras. See also [1], [6].

In Section 4 we study some properties of R-derivations of P. We show (Corollary 4.6) that if d_1, d_2 are R-derivations in P then the R-derivation $d_1d_2-d_2d_1$ is inner. At the end of Section 4 we give an example of such special

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ring P in which there exist outer R-derivations and the composition of any two R-derivations is an R-derivation too.

2. Preliminaries.

Throughout this paper, R is a ring with identity, n is a fixed natural number, and ρ is a reflexive and transitive relation on the set $I_n=\{1, \cdots, n\}$. We denote by $M_n(R)$ the ring of $n\times n$ matrices over R, and by $Z(R)$ the center of R. Moreover, we use the following conventions:

$F(R)$ =the set of mappings from R to R,

$\bar{\rho}$ =the relation on I_n defined by: $p\bar{\rho}q$ iff $p\rho q$ or $q\rho p$,

$\bar{\bar{\rho}}$ =the smallest equivalence relation on I_n containing ρ,

T_ρ =a fixed set of representatives of equivalence classes of $\bar{\bar{\rho}}$,

A_{ij} $=ij$-coefficient of a matrix A,

E^{ij} =the element of the standard basis of $M_n(R)$,

$\bar{r}$ =the diagonal matrix whose all coefficients on the diagonal are equal to $r\in R$,

$M_n(R)_\rho=\{A\in M_n(R)\,;\ A_{ij}=0 \text{ for } (i, j)\notin\rho\}$.

It is clear that $M_n(R)_\rho$ is a subring of $M_n(R)$. Conversely, if σ is a reflexive relation on I_n and $M_n(R)_\sigma$ is a subring of $M_n(R)$ then σ is transitive. We say that a subring P of $M_n(R)$ is special with the relation ρ iff $P\cong M_n(R)_\rho$.

If $\{U_{ij}\,;\ i, j\in I_n\}$ is a family of subsets of R, then we denote by $[U_{ij}]$ the set $\{A\in M_n(R)\,;\ A_{ij}\in U_{ij} \text{ for all } i, j\in I_n\}$.

The following lemma describes all (two-sides) ideals of special subrings of $M_n(R)$

LEMMA 2.1. *Let P be a special subring of $M_n(R)$ with the relation ρ and let U be a subset of P. The following conditions are equivalent:*

(1) *U is an ideal of P,*

(2) *$U=[U_{ij}]$, where U_{ij} are ideals of R such that*

a) *$U_{ij}=0$, if $(i, j)\notin\rho$,*

b) *$U_{ij}+U_{jk}\subseteq U_{ik}$, if $i\rho j$ and $j\rho k$.*

PROOF. (1)$\Rightarrow$(2). Let U be an ideal of P. Put

$$U_{ij}=\{x\in R\,;\ x=A_{ij} \text{ for some } A\in U\}, \quad \text{for } i, j\in I_n.$$

If $x\in U_{ij}$, for some i, j, and $C=E^{ii}AE^{jj}$, where A is a matrix of U such that $A_{ij}=x$, then $C\in U$ and

$$C_{pq}=\begin{cases} x, & \text{if } (p,q)=(i,j) \\ 0, & \text{if } (p,q)\neq(i,j). \end{cases}$$

This implies that $U=[U_{ij}]$. Clearly, U_{ij} are ideals in R and $U_{ij}=0$, for $(i, j)\notin\rho$.

Now, let $i\rho j$, $j\rho k$, $x\in U_{ij}$, $y\in U_{jk}$, and let A, B are matrices of U such that $A_{ij}=x$, $B_{jk}=y$. Then the matrices $E^{ii}AE^{jk}$, $E^{ij}BE^{kk}$ belong to U and $(E^{ii}AE^{jk})_{ik}=x$, $(E^{ij}BE^{kk})_{ik}=y$. Therefore $U_{ij}+U_{jk}\subseteq U_{ik}$. The implication (2)⇒(1) is obvious.

A mapping f from ρ into an abelian group G will be called transitive iff $f(i, j)+f(j, k)=f(i, k)$, for $i\rho j$, $j\rho k$.

Let $f:\rho\to G$ be a transitive mapping. Then $f(i, i)=0$, for $i\in I_n$, and $f(i, j)=-f(j, i)$, if $i\rho j$, $j\rho i$.

We denote by $\bar{f}$ the mapping from $\bar{\rho}$ into G defined by

$$\bar{f}(i,j)=\begin{cases} f(i,j), & \text{if } i\rho j \\ -f(j,i), & \text{if } j\rho i, \end{cases}$$

and by $[f, -]$ (in the case $G=R$) the mapping from ρ to $F(R)$ defined by $[f, -](i, j)(r)=f(i, j)r-rf(i, j)$. Clearly, $[f, -]$ is transitive too.

We say that f is trivial if there exists a mapping $\sigma: I_n\to G$ such that $f(i, j)=\sigma(i)-\sigma(j)$, for $i\rho j$; and we say that f is quasi-trivial (in the case $G=R$) if $[f, -]$ is trivial.

Obviously every trivial transitive mapping from ρ to R is quasi-trivial. But the converse is not necessarily true.

For example, let

$$\rho_0=\begin{array}{ccc} 1 & \longrightarrow & 3 \\ \downarrow & & \uparrow \\ 4 & \longleftarrow & 2. \end{array}$$

Then the transitive mapping $f:\rho_0\to R$, $f(1, 3)=1$, $f(2, 3)=0$, $f(2, 4)=0$, $f(1, 4)=0$ is quasi-trivial, but it is not trivial.

DEFINITION 2.2. Let G be an abelian group. The relation ρ is called regular over G if every transitive mapping from ρ to G is trivial.

In Section 5 we give some necessary and sufficient conditions for ρ to be regular over G.

An additive mapping d of a ring S into itself is called a derivation if $d(ab)=ad(b)+d(a)b$. We say that d is an inner derivation of S if there exists an element $c\in S$ such that $d=[c, -]$ i.e. $d(x)=cx-xc$, for all $x\in S$. If d is a derivation of S then an ideal U in S is called a d-ideal if $d(U)\subseteq U$. A derivation

$d: P \to P$, where P is a special subring of $M_n(R)$, will be called an R-derivation if $d(\bar{r})=0$, for any $r \in R$.

3. Derivations of special subrings of $M_n(R)$.

Throughout this section P will denote a special subring of $M_n(R)$ with the relation ρ.

LEMMA 3.1. *If $d: P \to P$ is a derivation, then*

(1) $d(E^{pq})_{ij}=0$, *for* $p\rho q$, $i \neq p$, $j \neq q$;
(2) $d(E^{pp})_{pp}=0$, *for* $p=1, \cdots, n$;
(3) $d(E^{pq})_{iq}=d(E^{pp})_{ip}$, *for* $p\rho q$, $p \neq i$;
(4) $d(E^{pq})_{pj}=d(E^{qq})_{qj}$, *for* $p\rho q$, $j \neq q$;
(5) $d(E^{pq})_{pq}+d(E^{qs})_{qs}=d(E^{ps})_{ps}$, *for* $p\rho q$, $q\rho s$;
(6) $d(E^{pi})_{pj}+d(E^{jq})_{iq}=0$, *for* $i \neq j$, $p\rho i$, $j\rho q$;
(7) $d(E^{pp})_{pq}+d(E^{qq})_{pq}=0$, *for all* $p, q \in I_n$

PROOF. We prove (3). Proofs of other conditions are similar.

$$\begin{aligned} d(E^{pq})_{iq} &= d(E^{pp}E^{pq})_{iq} \\ &= (E^{pp}d(E^{pq})+d(E^{pp})E^{pq})_{iq} \\ &= \sum_k E^{pp}_{ik} d(E^{pq})_{kq} + \sum_k d(E^{pp})_{ik} E^{pq}_{kq} \\ &= d(E^{pp})_{ip}. \end{aligned}$$

LEMMA 3.2. *If $d: P \to P$ is a derivation and $r \in R$, then*

(1) $d(\bar{r})_{ij}=d(E^{jj})_{ij}r-rd(E^{jj})_{ij}$
$=rd(E^{ii})_{ij}-d(E^{ii})_{ij}r$, *if* $i \neq j$,
(2) $d(\bar{r})_{ii}-d(\bar{r})_{jj}=d(E^{ij})_{ij}r-rd(E^{ij})_{ij}$, *for* $i\rho j$.

PROOF. (1) follows from the equality

$$0=d(\bar{r}E^{jj}-E^{jj}\bar{r})_{ij}$$

and from Lemma 3.1 (7).

(2) follows from the equality

$$0=d(\bar{r}E^{ij}-E^{ij}\bar{r})_{ij}.$$

For an arbitrary derivation $d: P \to P$ by f_d we denote the mapping of ρ into R defined by $f_d(i, j)=d(E^{ij})_{ij}$. By Lemmas 3.1(5), 3.2(2) we have

COROLLARY 3.3. *f_d is a quasi-trivial transitive mapping.*

Let $f:\rho\to R$ be a quasi-trivial transitive mapping. We show that there is a derivation d of P such that $f_d=f$.

LEMMA 3.4. *There exists a unique mapping $\tau: I_n\to F(R)$ such that*

(1) *$[f, -](i, j)=\tau(i)-\tau(j)$, for all $i\rho j$,*

(2) *$\tau(t)=0$, for $t\in T_\rho$.*

Moreover, $\tau(1), \cdots, \tau(n)$ are inner derivations of R.

PROOF. Let $\sigma: I_n\to F(R)$ be a mapping such that

$$[f, -](i, j)=\sigma(i)-\sigma(j), \quad \text{for } i\rho j.$$

If $i\in I_n$ and $i\bar{\rho}t$, where $t\in T_\rho$, then we put $\tau(i)=\sigma(i)-\sigma(t)$. Obviously, τ satisfies (1) and (2).

Now suppose that τ_1 also satisfies (1) and (2). Then, for $i\in I_n$ and for $i_1, \cdots, i_s\in I_n$ such that $i\bar{\rho}i_1, i_1\bar{\rho}i_2, \cdots, i_s\bar{\rho}t$ we have

$$\begin{aligned}\tau_1(i)&=(\tau_1(i)-\tau_1(i_1))+(\tau_1(i_1)-\tau_1(i_2))+\cdots+(\tau_1(i_s)-\tau_1(t))\\&=(\tau(i)-\tau(i_1))+(\tau(i_1)-\tau(i_2))+\cdots+(\tau(i_s)-\tau(t))\\&=\tau(i),\end{aligned}$$

i.e. $\tau_1=\tau$.

Moreover, $\tau(i)=[\bar{f}(i, i_1), -]+\cdots+[\bar{f}(i_s, t), -]$, so $\tau(i)$ is an inner derivation of R.

Denote by Δ_f the mapping from P into P defined by

$$\Delta_f(B)_{pq}=B_{pq}f(p, q)+\tau_f(p)(B_{pq}),$$

for $B\in P$, $p\rho q$, where τ_f is the mapping τ from Lemma 3.4.

PROPOSITION 3.5.

(1) *Δ_f is a derivation of P.*

(2) *$f_{\Delta_f}=f$.*

(3) *Δ_f is inner if and only if f is trivial.*

PROOF. (1) Let $B, C\in P$, $p\rho q$ and denote $\Delta=\Delta_f$, $\tau=\tau_f$. Then

$$(B\Delta(C)+\Delta(B)C)_{pq}=\sum_{p\rho s\rho q} M_s,$$

where

$$M_s=B_{ps}(C_{sq}f(s, q)+\tau(s)(C_{sq}))+(B_{ps}f(p, s)+\tau(p)(B_{ps}))C_{sq}.$$

By Lemma 3.4 we have

$$\begin{aligned} M_s &= B_{ps}C_{sq}f(s, q)+B_{ps}\tau(s)(C_{sq})+B_{ps}(f(p, s)C_{sq}-C_{sq}f(p, s)) \\ &\quad +B_{ps}C_{sq}f(p, s)+\tau(p)(B_{ps})C_{sq} \\ &= B_{ps}C_{sq}(f(p, s)+f(s, q))+B_{ps}\tau(p)(C_{sq})+\tau(p)(B_{ps})C_{sq} \\ &= B_{ps}C_{sq}f(p, q)+\tau(p)(B_{ps}C_{sq}), \end{aligned}$$

Therefore

$$\begin{aligned} (B\Delta(C)+\Delta(B)C)_{pq} &= (\sum_s B_{ps}C_{sq})f(p, q)+\tau(p)(\sum_s B_{ps}C_{sq}) \\ &= (BC)_{pq}f(p, q)+\tau(p)((BC)_{pq}) \\ &= \Delta(BC)_{pq}, \end{aligned}$$

i.e. Δ is a derivation of P.

(2) is obvious.

(3) Let $\Delta=\Delta_f$. If $\Delta=[A, -]$, where $A\in P$, then, by (2), we have

$$\begin{aligned} f(p, q) &= f_\Delta(p, q)=\Delta(E^{pq})_{pq} \\ &= (AE^{pq})_{pq}-(E^{pq}A)_{pq} \\ &= A_{pp}-A_{qq}, \end{aligned}$$

and hence f is trivial.

Assume now that f is trivial and set that $f(p, q)=\sigma(p)-\sigma(q)$, for all $p\rho q$, where $\sigma: I_n\to R$ is a mapping such that $\sigma(t)=0$ for $t\in T_\rho$ (see the proof of Lemma 3.4). Then $\tau_f(p)=[\sigma(p), -]$ and we see that

$$\Delta(B)_{pq}=\sigma(p)B_{pq}-B_{pq}\sigma(q), \quad \text{for } B\in P.$$

Thus, we obtain $\Delta=[A, -]$, where A is the matrix in P defined by

$$A_{pq}=\begin{cases} 0, & \text{for } p\neq q \\ \sigma(p), & \text{for } p=q. \end{cases}$$

This completes the proof.

Now we present another class of derivations of P. Let $\delta=\{\delta_t; t\in T_\rho\}$ be a set of derivations of R. Denote by Θ_δ the mapping from P to P defined by

$$\Theta_\delta(B)_{pq}=\delta_t(B_{pq})$$

for $B\in P$, $p\rho q$, where $t\in T_\rho$ such that $p\bar{\bar{\rho}}t$.

It is easy to prove the following

PROPOSITION 3.6. *If $\Theta=\Theta_\delta$, then*

(1) Θ *is a derivation of* P.

(2) $f_\Theta=0$.

(3) Θ *is inner if and only if* δ_t *is inner, for any* $t\in T_\rho$.

The following theorem describes all derivations of P.

THEOREM 3.7. *Let* P *be a special subring of* $M_n(R)$ *with the relation* ρ. *Every derivation* d *of* P *has a unique representation:*

$$d=[A, -]+\Delta_f+\Theta_\delta,$$

where (1) A *is a matrix in* P *such that* $A_{pp}=0$, *for* $p=1, \cdots, n$,

(2) $\delta=\{\delta_t ; t\in T_\rho\}$ *is a set of derivations of* R.

(3) $f:\rho\rightarrow R$ *is a quassi-trivial transitive mapping.*

PROOF. Let $d:P\rightarrow P$ be a derivation. We define the matrix A by $A_{pq}=d(E^{qq})_{pq}$. Then $A\in P$ and, by Lemma 3.1(2), $A_{pp}=0$, for $p=1, \cdots, n$. For $t\in T_\rho$ we define the derivation δ_t by $\delta_t(r)=d(\bar{r})_{tt}$, where $r\in R$. Set $\delta=\{\delta_t ; t\in T_\rho\}$. Moreover, let $f(p, q)=d(E^{pq})_{pq}$, for $p\rho q$. Then, by Corollary 3.3, f is a quasi-trivial transitive mapping.

Now, from Lemma 3.2(2) and by the proof of Lemma 3.4 we have

$$\begin{aligned}\tau_f(i)(r)&=d(\bar{r})_{ii}-d(\bar{r})_{tt}\\&=d(\bar{r})_{ii}-\delta_t(r),\end{aligned}$$

where t is the only element of T_ρ such that $i\bar{\rho}t$. Let $D=[A, -]+\Delta_f+\Theta_\delta$. For proving $D=d$ it suffices to verify that $D(\bar{r})_{ij}=d(\bar{r})_{ij}$, $D(E^{pq})_{ij}=d(E^{pq})_{ij}$, for $r\in R$, $i\rho j$, $p\rho q$, but these equalities we get easily from Lemmas 3.1, 3.2. Finally, we prove that the representation of d is unique. Let A', δ', f' satisfy the conditions (1), (2), (3), and let $d=D'$, where $D'=[A', -]+\Delta_{f'}+\Theta_{\delta'}$. Then

$$A'_{pq}=D'(E^{qq})_{pq}=D(E^{qq})_{pq}=A_{pq}$$

$$f'(p, q)=D'(E^{pq})_{pq}=D(E^{pq})_{pq}=f(p, q), \quad \text{for } p\rho q,$$

and

$$\delta'_t(r)=D'(\bar{r})_{tt}=D(\bar{r})_{tt}=\delta_t(r), \quad \text{for } t\in T_\rho,\ r\in R.$$

Therefore $A'=A$, $f'=f$, $\delta'=\delta$. This completes the proof.

By the last theorem and by Lemma 2.1 we obtain

COROLLARY 3.8. *Let* $d:P\rightarrow P$ *be a derivation and let* $U=[U_{ij}]$ *be an ideal of* P. *If* $i\rho j$, *then we denote by* d_{ij} *the derivation of* R *defined by* $d_{ij}(r)=d(\bar{r})_{ij}$, *where* $r\in R$. *The following conditions are equivalent*

(1) *U is a d-ideal,*

(2) $d_{ii}(U_{ij}) \subseteq U_{ij}$, *for all* $i\rho j$,

(3) $d_{jj}(U_{ij}) \subseteq U_{ij}$, *for all* $i\rho j$,

(4) $d_{tt}(U_{ij}) \subseteq U_{ij}$, *for all* $i\rho j$, *and* $t \in T_\rho$ *such that* $i\bar{\rho}t$.

THEOREM 3.9. *Let d be a derivation of a special subring P (of $M_n(R)$) with the relation ρ. The following conditions are equivalent*

(1) *d is inner,*

(2) *f_d is trivial and all d_{ij} are inner,*

(3) *f_d is trivial and $d_{11}, \cdots, d_{nn}$ are inner,*

(4) *f_d is trivial and d_{tt} is inner for all $t \in T_\rho$.*

PROOF. (1)$\Rightarrow$(2). Let $d=[B, -]$, where $B \in P$. Then

$$f_d(p, q)=(BE^{pq}-E^{pq}B)_{pq}=B_{pp}-B_{qq},$$

$$d_{pq}(r)=(B\bar{r}-\bar{r}B)_{pq}=B_{pq}r-rB_{pq},$$

so f_d is trivial and d_{pq} is inner.

(4)$\Rightarrow$(1). By Theorem 3.7, we have $d=[A, -]+\Delta_f+\Theta_\delta$, where $A \in P$ and and $f=f_d$, $\delta=\{d_{tt}, t \in T_\rho\}$ (by the proof of Theorem 3.7). Now, from Proposition 3.5 we know that Δ_f is inner, and, by Proposition 3.6, Θ_δ is inner too. Finally d is inner, because it is a sum of three inner derivations. The implications (2)$\Rightarrow$(3), (3)$\Rightarrow$(4) are obvious.

We end this section with the following

THEOREM 3.10. *Let R be a ring in which any derivation is inner and let P be a special subring of $M_n(R)$ with the relation ρ. The following conditions are equivalent*

(1) *Any derivation of P is inner,*

(2) *The relation ρ is regular over $Z(R)$.*

PROOF. (1)$\Rightarrow$(2). Notice, that any transitive mapping from ρ to $Z(R)$ is quasi-trivial. Therefore, if $f: \rho \to Z(R)$ is a transitive mapping then we have the derivation Δ_f (Proposition 3.5) which is inner (by (1)) so, by Proposition 3.9, f is trivial. The proof of the implication (2)$\Rightarrow$(1) we give in Section 5 (see Remark 5.6).

4. R-derivations.

Let P be a special subring of $M_n(R)$ with the relation ρ. In this section we shall use results of Section 3 to establish some properties of R-derivations of P.

Notice at first, that from Lemmas 3.2; 3.1, we have the following

LEMMA 4.1. *If d is an R-derivation of P then $d(E^{pq})_{ij}$ belongs to $\boldsymbol{Z}(R)$, for all $i, j, p, q \in I_n$ such that $p \rho q$.*

The above lemma and Corollary 3.3 imply that, if d is an R-derivation of P, then f_d is a transitive mapping from ρ to $\boldsymbol{Z}(R)$. Conversely, if $f: \rho \rightarrow \boldsymbol{Z}(R)$ is a transitive mapping then f is quasi-trivial and, by Proposition 3.5 (since $\Delta_f(B)_{pq}=f(p, q)B_{pq}$) the mapping Δ_f is an R-derivation of P.

Combining these remarks with results of Section 3 we obtain the following four corollaries

COROLLARY 4.2. *Any R-derivation d of P has a unique representation*

$$d=[A, -]+\Delta_f,$$

where (1) *A is a matrix in P such that $A_{ij} \in Z(R)$, for $i, j=1, \cdots, n$, and $A_{ii}=0$, for $i=1, \cdots, n$.*

(2) *$f: \rho \rightarrow \boldsymbol{Z}(R)$ is a transitive mapping.*

COROLLARY 4.3. *If $d: P \rightarrow P$ is an R-derivation, then the following conditions are equivalent*

(1) *d is inner,*

(2) *f_d is trivial,*

COROLLARY 4.4. *The following conditions are equivalent*

(1) *Any R-derivation of P is inner,*

(2) *The relation ρ is regular over $\boldsymbol{Z}(R)$.*

COROLLARY 4.5. *If d is an R-derivation of P, then every ideal of P is a d-ideal.*

The following two properties of R-derivations of P seem to be useful.

COROLLARY 4.6. *If d', d'' are R-derivations of P, then the R-derivation $d'd''-d''d'$ is inner.*

PROOF. Let $d=d'd''-d''d'$. Then we have

$$d'=[A', -]+\Delta_{f'}$$

$$d''=[A'', -]+\Delta_{f''}$$

$$d=[A, -]+\Delta_f,$$

where A', A'', A; f', f'', f satisfy the conditions (1) and (2) of Corollary 4.2. It is easy to verify that

$$f_d(p, q)=[A', A'']_{pp}-[A', A'']_{qq}.$$

So f_d is trivial and therefore, by Corollary 4.3, d is inner.

COROLLARY 4.7. *If d is an R-derivation of P, then $d(\boldsymbol{Z}(P))=0$.*

PROOF. Let $d=[A, -]+\Delta_f$ as in Corollary 4.2, and let $B\in\boldsymbol{Z}(P)$. Since $B_{ij}=0$, for $i\neq j$, and $B_{ii}\in\boldsymbol{Z}(R)$, for $i=1, \cdots, n$, then we have $\Delta_f(B)=0=[A, B]$ i.e. $d(B)=0$.

For $n\leqq 3$ every relation ρ (reflxexive and transitive) on I_n is regular over any group. Then (by Corollary 4.4) in this case, any special subring of $\boldsymbol{M}_n(R)$ has only inner R-derivations.

For $n=4$, it is not true. There exists the unique (with respect to an isomorphism) relation ρ_0 on I_4 which isn't regular over any ring R. The relation ρ_0 is defined by the graph

$$\begin{array}{ccc} 1 & \longrightarrow & 3 \\ \downarrow & & \uparrow \\ 4 & \longleftarrow & 2 \end{array}$$

(The mapping $f:\rho_0\to R$, $f(1, 3)=1$, $f(2, 3)=f(2, 4)=f(1, 4)=0$ is transitive but it isn't trivial).

Denote by $S_4(R)$ the special subring of $\boldsymbol{M}_4(R)$ with the relation ρ_0. Then we have

$$S_4(R)=\begin{bmatrix} R & 0 & R & R \\ 0 & R & R & R \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}.$$

We write all R-derivations of $S_4(R)$. Since any mapping $f:\rho_0\to\boldsymbol{Z}(R)$ is transitive then by Corollary 4.2 we get

COROLLARY 4.8. *If d is an R-derivation of $S_4(R)$ then there exists the unique system a_{13}, a_{14}, a_{23}, a_{24}; p_{13}, p_{14}, p_{23}, p_{24} of elements of $\boldsymbol{Z}(R)$ such that*

$$(*)\qquad d\left(\begin{bmatrix} x_{11} & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{bmatrix}\right)$$

$$=\begin{bmatrix} 0 & 0 & a_{12}(x_{33}-x_{11})+p_{13}x_{13} & a_{14}(x_{44}-x_{11})+p_{14}x_{14} \\ 0 & 0 & a_{23}(x_{33}-x_{22})+p_{23}x_{23} & a_{24}(x_{44}-x_{22})+p_{24}x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Conversely, if the elements $a_{13}, a_{14}, a_{23}, a_{24}$; $p_{13}, p_{14}, p_{23}, p_{24}$, *belong to* $\boldsymbol{Z}(R)$, *then the mapping* d *defined by* $(*)$ *is an* R-*derivation of* $S_4(R)$.

The next proposition (as a direct consequence of Corollaries 4.8, 4.3) gives some properties of R-derivations of $S_4(R)$.

PROPOSITION 4.9. (1) *Let* d *be an* R-*derivation of* $S_4(R)$ *defined by* $(*)$, *then*

(a) d *is inner if and only if* $p_{13}-p_{23}=p_{14}-p_{24}$.

(b) $F(d)=0$, *where* $F(x)=x(x-p_{13})(x-p_{23})(x-p_{14})(x-p_{24})$.

(c) d *is nilpotent* (*i.e.* $d^s=0$, *for some natural* s) *if and only if the elements* $p_{13}, p_{14}, p_{23}, p_{24}$ *are nilpotents.*

(2) *If* d, d' *are* R-*derivations of* $S_4(R)$, *then* dd' *is an* R-*derivation too.*

5. Regular relations.

In this section, we give some sufficient and necessary conditions for a relation ρ (reflexive and transitive) to be regular over an abelian group G. We shall use homology and cohomology groups of some simplicial complexes to describe these conditions. Notations are standard (see [4]).

If ρ is a reflexive and transitive relation on I_n then a pair (I_n, ρ) will be denoted by Γ and called a graph. Elements of I_n will be called vertices of Γ and pairs (p, q), where $p\rho q$, $p\neq q$, will be called arrows. We say that a graph Γ is regular over a group G iff ρ is regular over G. Notice first, that we may reduce our consideration to the case where ρ is connected, because it is easy to show the following

LEMMA 5.1. *A graph* $\Gamma=(I_n, \rho)$ *is regular over* G *if and only if every connected component of* Γ *is a regular graph over* G.

Observe also, that it suffices to consider partially ordered graphs. Let $\sim$ be the equivalence relation on I_n defined by: $p\sim q$ iff $p\rho q$ and $q\rho p$. Denote by $[x]$ an equivalence class of $x\in I_n$ with respect to $\sim$, and let I_n^* be the set of all equivalence classes. We define the relation ρ^* on I_n^* as follows:

$$[x]\rho^*[y] \text{ iff } x\rho y.$$

ρ^* is obviously a partial order on I_n^* and ρ is connected if and only if ρ^* is connected. Moreover, we have

LEMMA 5.2. *A graph* (I_n, ρ) *is regular over* G *if and only if the graph* (I_n^*, ρ^*) *is regular over* G.

PROOF. Denote by $W=\{w_i\}$ some fixed set of representatives of the cosets in respect to $\sim$. Let ρ be regular over G and let $g:\rho^*\to G$ be a transitive mapping. Then the mapping $f:\rho\to G$, $f(x, y)=g([x], [y])$, is transitive, so (by assumption) $f(x, y)=\sigma(x)-\sigma(y)$, where $\sigma: I_n\to G$ is some function. Set $\sigma^*([w_i])=\sigma(w_i)$. Then $\sigma^*: I_n^*\to G$ and $g([x], [y])=\sigma^*([x])-\sigma^*([y])$.

Assume now, that Γ^* is regular over G. Let $f:\rho\to G$ be a transitive mapping. Then $g:\rho^*\to G$ defined by $g([w_i], [w_j])=f(w_i, w_j)$ is transitive, so (by assumption) $g([w_i], [w_j])=\tau([w_i])-\tau([w_j])$, where $\tau: I_n^*\to G$. If $p\in I_n$, then $p\in[w_i]$, for the unique $w_i\in W$. Therefore we put $\sigma(p)=f(p, w_i)+\tau([w_i])$. Then $f(p, q)=\sigma(p)-\sigma(q)$ i.e. f is trivial.

Further, to the end of this section, $\Gamma=(I_n, \rho)$ will be a connected and partially ordered graph.

Set vertices of the graph Γ in some Euclidean space E^s in such way that none four vertices don't belong to one affine plane. Therefore Γ is one-dimensional simplicial complex. Vertices of Γ will be denoted by (a), and arrows by $(a, b)(a\rho b, a\neq b)$. If $a\rho b$, $b\rho c$, then by (a, b, c) we denote the triangle (two-dimensional simplex) with vertices a, b, c.

Denote by $\tilde{\Gamma}$ the simplicial complex which we get as the union of Γ and all triangles of Γ. Let $C_0(\Gamma)$, $C_1(\Gamma)$ be free abelian groups with free generators which are vertices and arrows of Γ, respectively; and let $C_0(\tilde{\Gamma})$, $C_1(\tilde{\Gamma})$, $C_2(\tilde{\Gamma})$ be free abelian groups with free generators which are vertices, arrows and triangles of $\tilde{\Gamma}$. Clearly, $C_0(\Gamma)=C_0(\tilde{\Gamma})$, $C_1(\Gamma)=C_1(\tilde{\Gamma})$.

Moreover we use the following notations. If $a\bar{\rho}b$, then

$$\overline{(a, b)}=\begin{cases} (a, b), & \text{if } a\rho b \\ -(a, b), & \text{if } b\rho a \end{cases}$$

and if $a_1\bar{\rho}a_2$, $a_2\bar{\rho}a_3, \cdots, a_{s-1}\bar{\rho}a_s$, then

$$\langle a_1, \cdots, a_s\rangle=\overline{(a_1, a_2)}+\cdots+\overline{(a_{s-1}, a_s)}.$$

Elements of the form $\langle a_1, \cdots, a_s, a_1\rangle$ we call cycles of the length s. In particular cycles of the length 3 we call triangle cycles.

Let $A(\Gamma)$, $B(\Gamma)$ be the subgroups of $C_1(\Gamma)$ generated by all cycles and all triangle cycles of Γ, respectively.

Observe that every mapping $f:\rho\to G$ we can identify with a group homomorphism from $C_1(\Gamma)$ to G. Therefore $f: C_1(\Gamma)\to G$ is a transitive mapping iff $f(B(\Gamma))=0$. Moreover we have the following technical lemmas

LEMMA 5.3. *$f: C_1(\Gamma)\to G$ is a trivial transitive mapping iff $f(A(\Gamma))=0$.*

PROOF. ($\Rightarrow$) Let $f(a, b)=\sigma(a)-\sigma(b)$, where $\sigma: I_n \to G$ is a mapping. Then $f((\overline{a, b}))=\sigma(a)-\sigma(b)$, for $a\bar{\rho}b$, and we have

$$f(\langle a_1, \cdots, a_s, a_1\rangle=\sigma(a_1)-\sigma(a_2)+\sigma(a_2)- \cdots -\sigma(a_1)=0 .$$

($\Leftarrow$) Let p be a fixed vertex of Γ, and $x\in I_n$. Since Γ is connected then there exist $x_1, \cdots, x_s \in I_n$ such that $x\bar{\rho}x_1$, $x_1\bar{\rho}x_2, \cdots, x_s\bar{\rho}p$. We define $\sigma(x)=f(\langle x, x_1, \cdots, x_s, p\rangle)$. Since $f(A(\Gamma))=0$, then σ is well defined. Now we have $f(x, y)=\sigma(x)-\sigma(y)$, i.e. f is trivial.

LEMMA 5.4. *$f: C_1(\Gamma)\to R$ is a quasi-trivial transitive mapping iff $f(A(\Gamma))\subseteqq \boldsymbol{Z}(R)$.*

PROOF. If f is quasi-trivial, then $[f, -]$ is trivial, so (by Lemma 5.3) $[f, -](A(\Gamma))=0$. Therefore $f(x)r=rf(x)$, for any $r\in R$, i.e. $f(A(\Gamma))\subseteqq \boldsymbol{Z}(R)$. Conversely, if $f(A(\Gamma))\subseteqq \boldsymbol{Z}(R)$, then $[f, -](A(\Gamma))=0$, so (by Lemma 5.3) $[f, -]$ is trivial.

For Γ, $\tilde{\Gamma}$ we have two standard complexes of groups

$$\Gamma:\ 0 \longrightarrow C_1(\Gamma) \xrightarrow{d_1} C_0(\Gamma) \longrightarrow 0$$

$$\tilde{\Gamma}:\ 0 \longrightarrow C_2(\tilde{\Gamma}) \xrightarrow{d_2} C_1(\tilde{\Gamma}) \xrightarrow{d_1} C_0(\tilde{\Gamma}) \longrightarrow 0,$$

where

$$d_1(a, b)=(b)-(a)$$

$$d_2(a, b, c)=(a, b)+(b, c)-(a, c) .$$

Therefore

$$H_0(\Gamma)=H_0(\tilde{\Gamma})=Z$$

$$H_1(\Gamma)=\mathrm{Ker}\ d_1=A(\Gamma)$$

$$H_1(\tilde{\Gamma})=\mathrm{Ker}\ d_1/\mathrm{Im}\ d_2=A(\Gamma)/B(\Gamma) .$$

Moreover, by the Künneth formulas, we have

$$H^1(\tilde{\Gamma}, G)=\mathrm{Hom}(A(\Gamma)/B(\Gamma), G) .$$

We proceed to establish the lemma which be useful in the sequel

LEMMA 5.5. *For any homomorphism $g: A(\Gamma)\to G$ there exists a homomorphism $\bar{g}: C_1(\Gamma)\to G$ such that $\bar{g}|_{A(\Gamma)}=g$.*

PROOF. Since $C_1(\Gamma)/A(\Gamma)=C_1(\Gamma)/\mathrm{Ker}\ d_1=\mathrm{Im}\ d_1$, and because $\mathrm{Im}\ d_1$ is free subgroup of $C_0(\Gamma)$, then the following exact sequence

$$0 \longrightarrow A(\Gamma) \hookrightarrow C_1(\Gamma) \longrightarrow C_1(\Gamma)/A(\Gamma) \longrightarrow 0$$

splits.

REMARK 5.6. Now, we can give the second part of the proof of Theorem 3.10.

Let $f: C_1(\Gamma) \to R$ be a quasi-trivial transitive mapping and put $g = f|_{A(\Gamma)}$. Then, by Lemma 5.4, we have $f(A(\Gamma)) \subseteq Z(R)$ so g is a mapping from $A(\Gamma)$ to $Z(R)$. By Lemma 5.5, there exists $\bar{g}: C_1(\Gamma) \to Z(R)$ such that $\bar{g}|_{A(\Gamma)} = g = f|_{A(\Gamma)}$. Clearly, $\bar{g}$ is transitive so (by assumption) $\bar{g}$ is trivial. Therefore $\bar{g}(A(\Gamma)) = 0$ (by Lemma 5.3), so $f(A(\Gamma)) = 0$ and, by Lemma 5.3, f is trivial.

Now, if $d: P \to P$ is a derivation then the mapping f_d is trivial and, by Theorem 3.9, d is inner.

The next lemma is well-known (see [5]).

LEMMA 5.7. *Let A be a free abelian group of the rank s and let B be a nonzero subgroup of A. Then B is free of the rank $k \leqq s$ and there exist a basis $a_1, \cdots, a_s$ of A and a basis $b_1, \cdots, b_k$ of B such that $b_i = m_i a_i$, for $i = 1, \cdots, k$, and m_i divides m_{i+1} for $i = 1, \cdots, k-1$.*

Now we can prove the following theorems

THEOREM 5.8. *Let $\Gamma = (I_n, \rho)$ be a partially ordered connected graph. The following conditions are equivalent.*

(1) *Γ is regular over some nonzero group,*

(2) *Γ is regular over every torsion-free group,*

(3) *Γ is regular over some torsion-free group,*

(4) *Γ is regular over Z,*

(5) *The groups $A(\Gamma)$ and $B(\Gamma)$ are isomorphic,*

(6) *$H_1(\tilde{\Gamma})$ is finite,*

(7) *$H^1(\tilde{\Gamma}, G) = 0$, for any torsion-free group G.*

PROOP. The equivalences (5)$\Leftrightarrow$(6)$\Leftrightarrow$(7) are obvious. It suffices to prove (1)$\Rightarrow$(5) and (5)$\Rightarrow$(2).

Let $A = A(\Gamma)$, $B = B(\Gamma)$ and $a_1, \cdots, a_s$; $b_1, \cdots, b_k$ be basis as in Lemma 5.7.

(1)$\Rightarrow$(5). We show that $k = s$. Let $k < s$. Consider the mapping $f: A \to G$ defined by $f(a_{k+1}) = u$, where $0 \neq u \in G$, and $f(a_i) = 0$, for $i \neq k+1$. By Lemma 5.5, there exists a transitive mapping $\bar{f}: C_1(\Gamma) \to G$ such that $\bar{f}(A) \neq 0$, which contradicts with Lemma 5.3.

(5)$\Rightarrow$(2). If $A = B$, then, by Lemma 5.3, the proof is completed. Now let $B \subsetneqq A$. Then (Lemma 5.7) there exists $m \in N$ such that $mA \subseteq B$. Let G be a torsion-free group and $f: C_1(\Gamma)\to G$ be a transitive mapping. Then $f(B)=0$, so $f(A)=0$ (since $mf(A)=f(mA)\subseteq f(B)=0$), i.e. f is trivial (by Lemma 5.3).

THEOREM 5.9. *Let $\Gamma=(I_n, \rho)$ be a partially ordered connected gragh. The following conditions are equivalent*

(1) *Γ is regular over any group,*
(2) *Γ is regular over Q/Z,*
(3) $A(\Gamma)=B(\Gamma)$,
(4) $H_1(\tilde{\Gamma})=0$,
(5) $H^1(\tilde{\Gamma}, G)=0$, *for any group G.*

PROOF. The implication (3)$\Rightarrow$(1) follows by Lemma 5.3.

(2)$\Rightarrow$(3). By Theorem 5.8, the groups $A(\Gamma)$, $B(\Gamma)$ have the same rank. Let $A=A(\Gamma)$, $B=B(\Gamma)$. Let $B\subsetneqq A$ and let $b_1, \cdots, b_s$; $a_1, \cdots, a_s$ be basis of groups B, A, respectively; as in Lemma 5.7. In this case, we have $m_s>1$. Consider the mapping $f: A\to Q/Z$, defined by $f(a_i)=0$, for $i=1, \cdots, s-1$, and $f(a_s)=$the coset of $1/m_s$ in Q/Z. Then, by Lemma 5.5, there exists the transitive mapping $\bar{f}: C_1(\Gamma)\to Q/Z$ such that $\bar{f}(A)\neq 0$, which contradicts with Lemma 5.3. The rest of the proof is obvious.

It remains to consider the case where $H_1(\tilde{\Gamma})$ is a nonzero finite group.

THEOREM 5.10. *Let $\Gamma=(I_n, \rho)$ be a partially ordered connected graph such that the order of $H_1(\tilde{\Gamma})$ is equal to $m>1$, and let G be an abelian group.*

The following conditions are equivalent

(1) *Γ is regular over G,*
(2) *G is an m-torsion-free group,*
(3) $H^1(\tilde{\Gamma}, G)=0$.

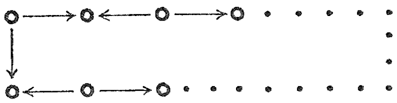
Proof is similar to the proofs of Theorems 5.8, 5.9.

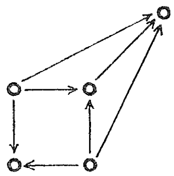
By the three above theorems immediately we have

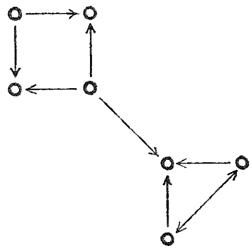
COROLLARY 5.11. *Let $\Gamma=(I_n, \rho)$ and let G be an abelian group. Then Γ is regular over G iff $H^1(\tilde{\Gamma}, G)=0$.*

We end this paper with some examples, which we easily get by the above theorems.

1. The following graphs

(a)  (i. e. the graph $\tilde{A}_{2n}$),

(b) 

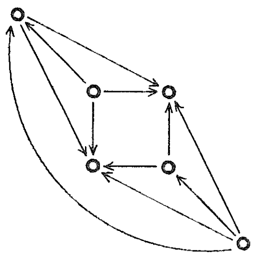
(c) 

are not regular over any group.

2. The graphs

(a) not having subgraphs of the type $\tilde{A}_{2n}$ (in particular trees),

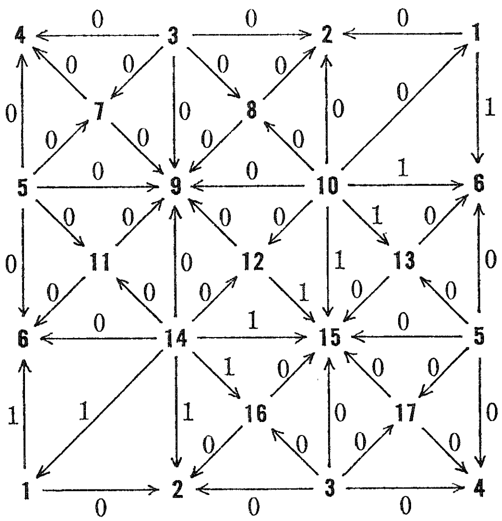
(b) cones (i.e. the graphs (I_n, ρ) for which there exists $b \in I_n$ such that $b\bar{\rho}a$, for any $a \in I_n$)

(c) 

(d) all graphs, for $n \leqq 3$,

are regular over any group.

3. By Theorem 5.10, it follows that there exist graphs which are regular over some groups and which are not regular over another groups. For example, the following graph (which is one-dimensional triangulation of the projective plane)



is regular over any 2-torsion-free group, and is not regular over remaining groups. The numbers at the arrows define the non-trivial transitive mapping to Z_2.

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