TSUKUBA J. MATH.
Vol. 18 Nol 1 (1994), 145–163

A REMARK ON THE SECOND HOMOTOPY GROUPS OF COMPACT RIEMANNIAN 3-SYMMETRIC SPACES

By

Takashi KODA

Abstract. In order to calculate the second Stiefel-Whitney class of a 1-connected compact Riemannian 3-symmetric space G/K by Borel-Hirzebruch's method, we have to know the second cohomology group $H^2(G/K, \boldsymbol{Z}_2)\cong Hom(\pi_2(G/K), \boldsymbol{Z}_2)$. In this paper, we shall describe precisely the connected Lie subgroup K and calculate explicitly the second homotopy group $\pi_2(G/K)$ in terms of the roots of G.

1. Introduction

A. Gray [3] introduced the notion of Riemannian 3-symmetric spaces which includes Hermitian symmetric spaces and he showed that every Riemannian 3-symmetric space is a homogeneous almost Hermitian manifold with the canonical almost complex structure associated to the Riemannian 3-symmetric structure. It is known that many compact Riemannian 3-symmetric spaces appear as the twistor spaces over even dimensional compact Riemannian symmetric spaces. So it is worth to study Riemannian 3-symmetric spaces.

An oriented Riemannian manifold (M, g) is a spin manifold if and only if the second Stiefel-Whitney class $w_2(M)$ of M vanishes. There are many compact Riemannian 3-symmetric spaces which are spin manifolds and also many ones which are not. Hence it seems interesting to determine compact Riemannian 3-symmetric spaces which are spin manifolds.

In order to calculate the second Stiefel-Whitney classes of a smooth manifold M, we have to know the second cohomology group $H^2(M, \boldsymbol{Z}_2)$. If M is 1-connected, $H^2(M, \boldsymbol{Z}_2)$ is isomorphic to the group $Hom(\pi_2(M), \boldsymbol{Z}_2)$. In this paper, we shall calculate the second homotopy groups $\pi_2(M)$ of all 1-conncted compact irreducible Riemannian 3-symmetric spaces $M=G/K$ in terms of the roots of G, and in the course of its calculation, we shall describe presicely the

This research was partially supported by the Grant-in-Aid for Scientific Research (No. 03740022), the Ministry of Education, Science and Culture.
Received August 10, 1992. Revised January 13, 1993.

connected Lie subgroup K by the elementary method. We shall show the following theorem.

THEOREM A. *Let $M=G/K$ be a connected simply connected irreducible compact Riemannian 3-symmetric space with a G-invariant Riemannian metric, where G is a compact connected centerless simple Lie group and K is the connected Lie subgroup of G with Lie algebra $\mathfrak{k}=\mathfrak{g}^{\theta}$ for some automorphism θ of $\mathfrak{g}$ of order 3. Then K, the second homotopy group $\pi_2(M)$ and the second cohomology group $H^2(M, \boldsymbol{Z}_2)$ are given by the following table.*

REMARK. We can see that a 6-dimensional connected, simply connected irreducible compact Riemannian 3-symmetric space M is not a spin manifold if and only if $M=SO(5)/\{SO(2)\times SO(3)\}$ or $M=Sp(2)/U(2)$. We are going to calculate $w_2(M)$ for all irreducible compact Riemannian 3-symmetric spaces in

Table 1

G	K	$\pi_2(G/K)$	$H^2(G/K, \boldsymbol{Z}_2)$
$SU(n)/\boldsymbol{Z}_n$ $(n\geqq 2)$	$S\{U(r_1)\times U(r_2)\times U(r_3)\}/\boldsymbol{Z}_n$ $0\leqq r_1\leqq r_2\leqq r_3$, $0<r_2$, $r_1+r_2+r_3=n$	$\boldsymbol{Z}\times\boldsymbol{Z}$ if $r_1=0$, $n=2$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
		$\boldsymbol{Z}$ if $r_1=0$, $n\geqq 3$	$\boldsymbol{Z}_2$
		$\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}$ if $r_1>0$, $n=3$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
		$\boldsymbol{Z}\times\boldsymbol{Z}$ if $r_1>0$, $n\geqq 4$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
$SO(2n+1)$ $(n\geqq 1)$	$U(r)\times SO(2n-2r+1)$ $(1\leqq r\leqq n)$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$
$Sp(n)/\boldsymbol{Z}_2$ $(n\geqq 1)$	$\{U(r)\times Sp(n-r)\}/\boldsymbol{Z}_2$ $(1\leqq r\leqq n)$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$
$SO(2n)/\boldsymbol{Z}_2$ $(n\geqq 3)$	$\{U(r)\times SO(2n-2r)\}/\boldsymbol{Z}_2$ $(1\leqq r\leqq n)$	$\boldsymbol{Z}\times\boldsymbol{Z}$ if $r=n-1$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
		$\boldsymbol{Z}$ if $1\leqq r<n-1$	$\boldsymbol{Z}_2$
		$\boldsymbol{Z}$ if $r=n$	$\boldsymbol{Z}_2$

G	K	$\pi_2(G/K)$	$H^2(G/K, \boldsymbol{Z}_2)$
G_2	$U(2)$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$
F_4	$\{Spin(7)\times T^1\}/\boldsymbol{Z}_2$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$
	$\{Sp(3)\times T^1\}/\boldsymbol{Z}_2$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$
$E_6/\boldsymbol{Z}_3$	$\{Spin(10)\times SO(2)\}/\boldsymbol{Z}_4$	$\boldsymbol{Z}_4\times\boldsymbol{Z}$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
	$\{[S(U(5)\times U(1))/\boldsymbol{Z}_3]\times SU(2)\}/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_5\times\boldsymbol{Z}$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
	$\{[SU(6)/\boldsymbol{Z}_3]\times T^1\}/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\times\boldsymbol{Z}$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
	$\{[Spin(8)\times SO(2)]/\boldsymbol{Z}_2\times SO(2)\}/\boldsymbol{Z}_2$	$\boldsymbol{Z}_3\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}\times\boldsymbol{Z}$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
$E_7/\boldsymbol{Z}_2$	$\{E_6\times T^1\}/\boldsymbol{Z}_3$	$\boldsymbol{Z}_3\times\boldsymbol{Z}$	$\boldsymbol{Z}_2$
	$\{[SU(2)\times(Spin(10)\times SO(2))/\boldsymbol{Z}_2]/\boldsymbol{Z}_2\}/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
	$\{[SO(2)\times Spin(12)]/\boldsymbol{Z}_2\}/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\times\boldsymbol{Z}$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
	$S\{U(7)\times U(1)\}/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\times\boldsymbol{Z}$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
E_8	$SO(14)\times SO(2)$	$\boldsymbol{Z}_2\times\boldsymbol{Z}$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
	$\{E_7\times T^1\}/\boldsymbol{Z}_2$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$
G_2	$SU(3)$	0	0
F_4	$\{SU(3)\times SU(3)\}/\boldsymbol{Z}_3$	$\boldsymbol{Z}_3$	0
$E_6/\boldsymbol{Z}_3$	$\{SU(3)\times SU(3)\times SU(3)\}/\{\boldsymbol{Z}_3\times\boldsymbol{Z}_3\}$	$\boldsymbol{Z}_3$	0
$E_7/\boldsymbol{Z}_2$	$\{SU(3)\times[SU(6)/\boldsymbol{Z}_2]\}/\boldsymbol{Z}_3$	$\boldsymbol{Z}_3$	0
E_8	$\{SU(3)\times E_6\}/\boldsymbol{Z}_3$	$\boldsymbol{Z}_3$	0
	$SU(9)/\boldsymbol{Z}_3$	$\boldsymbol{Z}_3$	0

G	K	$\pi_2(G/K)$	$H^2(G/K, \boldsymbol{Z}_2)$
$Spin(8)$	$SU(3)/\boldsymbol{Z}_3$	$\boldsymbol{Z}_3$	0
	G_2	0	0
$\{L\times L\times L\}/Z$ where L is compact simple and simply connected and Z is its center embedded diagonally.	L/Z where L is embedded diagonally in $L\times L\times L$ and Z is its center.	0	0

the forthcoming paper.

2. Preliminaries

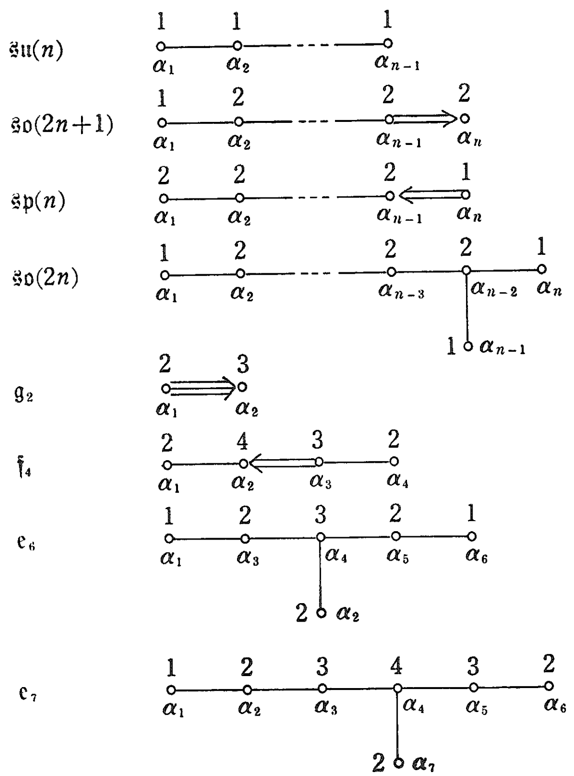
Let G be a compact connected centerless simple Lie group and T be a maximal torus of G. We denote by $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras of G and T respectively. Let $\Psi=\{\alpha_1, \cdots, \alpha_l\}$ be a simple root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Let σ be an automorphism of order 3 on G and put

$$K=G^{\sigma}=\{g\in G \mid \sigma(g)=g\}.$$

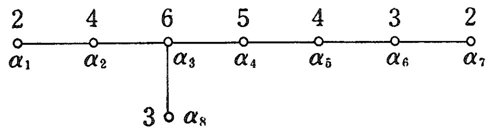
We denote by $\mu=\sum_{j=1}^{l} m_j\alpha_j$ the maximal root. Let $v_0, v_1, \cdots, v_l$ be the vectors in $\mathfrak{t}$ defined by

$$v_0=0, \qquad \alpha_i(v_j)=\frac{1}{m_i}\delta_{ij}.$$

In this paper, the simple roots of simple Lie algebras are numbered as follows:



$\mathfrak{e}_8$



J. A. Wolf and A. Gray [10] has given the complete classification of $(\mathfrak{g}, d\sigma, \mathfrak{k})$.

THEOREM 2.1 [10]. *Let φ be an inner automorphism of order 3 on a compact or complex simple Lie algebra $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{t}$ and let $\Psi=\{\alpha_1, \cdots, \alpha_l\}$ be a simple root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Then φ is conjugate (up to inner automorphism of $\mathfrak{g}$) to some $\theta=Ad(\exp 2\pi\sqrt{-1}x)$ where $x=(1/3)m_iv_i$ with $1\leqq m_i\leqq 3$ or $x=(1/3)(v_i+v_j)$ with $m_i=m_j=1$. A complete list of the possibilities for x is listed in the table below.*

THEOREM 2.2 [10]. *Let θ be an outer automorphism of order 3 on a compact or complex simple Lie algebra $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{k})$ is one of Table 3.*

Table 2

$\mathfrak{g}$	x	Ψ_x	$\mathfrak{g}^\theta$
$\mathfrak{su}(2)$	$\frac{1}{3}v_1$	empty	$\mathfrak{t}^1$
$\mathfrak{su}(n)$ $n\geqq 3$	$\frac{1}{3}v_i$	$\{\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_{n-1}\}$	$\mathfrak{su}(i)\oplus\mathfrak{su}(n-i)\oplus\mathfrak{t}^1$
	$\frac{1}{3}(v_i+v_j)$ $i<j$	$\{\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_{n-1}\}$	$\mathfrak{su}(i)\oplus\mathfrak{su}(j-i)\oplus\mathfrak{su}(n-j)\oplus\mathfrak{t}^2$
$\mathfrak{so}(2n+1)$ $n\geqq 2$	$\frac{1}{3}v_1$	$\{\alpha_2, \cdots, \alpha_n\}$	$\mathfrak{so}(2n-1)\oplus\mathfrak{t}^1$
	$\frac{2}{3}v_i$ $2\leqq i\leqq n$	$\{\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n\}$	$\mathfrak{su}(i)\oplus\mathfrak{so}(2(n-i)+1)\oplus\mathfrak{t}^1$
$\mathfrak{sp}(n)$ $n\geqq 2$	$\frac{2}{3}v_i$ $1\leqq i\leqq n-1$	$\{\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n\}$	$\mathfrak{su}(i)\oplus\mathfrak{sp}(n-i)\oplus\mathfrak{t}^1$
	$\frac{1}{3}v_n$	$\{\alpha_1, \cdots, \alpha_{n-1}\}$	$\mathfrak{su}(n)\oplus\mathfrak{t}^1$

$\mathfrak{g}$	x	Ψ_x	$\mathfrak{g}^\theta$
$\mathfrak{so}(8)$	$\frac{1}{3}v_1$	$\{\alpha_2, \alpha_3, \alpha_4\}$	$\mathfrak{su}(4)\oplus\mathfrak{t}^1$
	$\frac{2}{3}v_2$	$\{\alpha_1, \alpha_3, \alpha_4\}$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ $\oplus\mathfrak{su}(2)\oplus\mathfrak{t}^1$
	$\frac{1}{3}(v_1+v_3)$	$\{\alpha_2, \alpha_4\}$	$\mathfrak{su}(3)\oplus\mathfrak{t}^2$
$\mathfrak{so}(2n)$ $n\geqq 5$	$\frac{1}{3}v_1$	$\{\alpha_2, \alpha_3, \cdots, \alpha_n\}$	$\mathfrak{so}(2n-2)\oplus\mathfrak{t}^1$
	$\frac{1}{3}v_n$	$\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$	$\mathfrak{su}(n)\oplus\mathfrak{t}^1$
	$\frac{2}{3}v_i$ $2\leqq i\leqq n-3$	$\{\alpha_1, \cdots, \alpha_{i-1},$ $\alpha_{i+1}, \cdots, \alpha_n\}$	$\mathfrak{su}(i)\oplus\mathfrak{so}(2n-2i)$ $\oplus\mathfrak{t}^1$
	$\frac{1}{3}(v_{n-1}+v_n)$	$\{\alpha_1, \alpha_2, \cdots, \alpha_{n-2}\}$	$\mathfrak{su}(n-1)\oplus\mathfrak{t}^2$
$\mathfrak{g}_2$	v_1	$\{\alpha_2, -\mu\}$	$\mathfrak{su}(3)$
	$\frac{2}{3}v_2$	$\{\alpha_1\}$	$\mathfrak{su}(2)\oplus\mathfrak{t}^1$
$\mathfrak{f}_4$	$\frac{2}{3}v_1$	$\{\alpha_2, \alpha_3, \alpha_4\}$	$\mathfrak{so}(7)\oplus\mathfrak{t}^1$
	v_3	$\{\alpha_1, \alpha_2, \alpha_4, -\mu\}$	$\mathfrak{su}(3)\oplus\mathfrak{su}(3)$
	$\frac{2}{3}v_4$	$\{\alpha_1, \alpha_2, \alpha_3\}$	$\mathfrak{sp}(3)\oplus\mathfrak{t}^1$
$\mathfrak{e}_6$	$\frac{1}{3}v_1$	$\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$	$\mathfrak{so}(10)\oplus\mathfrak{t}^1$
	$\frac{2}{3}v_3$	$\{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$	$\mathfrak{su}(2)\oplus\mathfrak{su}(5)\oplus\mathfrak{t}^1$
	$\frac{2}{3}v_2$	$\{\alpha_1, \alpha_3, \cdots, \alpha_6\}$	$\mathfrak{su}(6)\oplus\mathfrak{t}^1$
	v_4	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, -\mu\}$	$\mathfrak{su}(3)\oplus\mathfrak{su}(3)\oplus\mathfrak{su}(3)$
	$\frac{1}{3}(v_1+v_6)$	$\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$	$\mathfrak{so}(8)\oplus\mathfrak{t}^2$

$\mathfrak{g}$	x	Ψ_x	$\mathfrak{g}^\theta$
$\mathfrak{e}_7$	$\frac{1}{3}v_1$	$\{\alpha_2, \cdots, \alpha_7\}$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
	$\frac{2}{3}v_2$	$\{\alpha_1, \alpha_3, \cdots, \alpha_7\}$	$\mathfrak{su}(2)\oplus\mathfrak{so}(10)\oplus\mathfrak{t}^1$
	$\frac{2}{3}v_6$	$\{\alpha_1, \cdots, \alpha_5, \alpha_7\}$	$\mathfrak{so}(12)\oplus\mathfrak{t}^1$
	$\frac{2}{3}v_7$	$\{\alpha_1, \cdots, \alpha_6\}$	$\mathfrak{su}(7)\oplus\mathfrak{t}^1$
	v_3	$\{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, -\mu\}$	$\mathfrak{su}(3)\oplus\mathfrak{su}(6)$
$\mathfrak{e}_8$	$\frac{2}{3}v_1$	$\{\alpha_2, \cdots, \alpha_8\}$	$\mathfrak{so}(14)\oplus\mathfrak{t}^1$
	$\frac{2}{3}v_7$	$\{\alpha_1, \cdots, \alpha_6, \alpha_8\}$	$\mathfrak{e}_7\oplus\mathfrak{t}^1$
	v_6	$\{\alpha_7, -\mu, \alpha_1, \cdots, \alpha_5, \alpha_8\}$	$\mathfrak{su}(3)\oplus\mathfrak{e}_6$
	v_8	$\{\alpha_1, \cdots, \alpha_7, -\mu\}$	$\mathfrak{su}(9)$

Table 3

$\mathfrak{g}$	$\mathfrak{k}=\mathfrak{g}^\theta$
$\mathfrak{so}(8)$	$\mathfrak{g}_2$
	$\mathfrak{su}(3)$

3. Proof of the Main Theorem

By the universal coefficient theorem, we have an exact sequence

$$0 \longrightarrow Ext(H_1(M, Z), Z_2) \longrightarrow H^2(M, Z_2) \longrightarrow Hom(H_2(M, Z), Z_2) \longrightarrow 0.$$

Since M is simply connected, we have $H_1(M, Z)=0$. Hence we have

$$H^2(M, Z_2) \cong Hom(H_2(M, Z), Z_2).$$

Since M is 1-connected, by Hurewicz Theorem (cf. Whitehead [9], p. 169), we have

$$H_2(M, Z) \cong \pi_2(M).$$

So, in order to prove our Main Theorem, we have only to calculate the second homotopy group $\pi_2(M)$.

The homotopy exact sequence of the principal K-bundle $(G, K, M=G/K)$ is as follows:

$$\text{(3-1)} \qquad \pi_2(G) \longrightarrow \pi_2(G/K) \xrightarrow{f} \pi_1(K) \xrightarrow{h} \pi_1(G) \longrightarrow \pi_1(G/K) \longrightarrow \pi_0(K).$$

Let $\tilde{G}$ and $Z(\tilde{G})$ be the universal covering group of G and the center of G, respectively. Then G is isomorphic to the quotient group $\tilde{G}/Z(\tilde{G})$. Since the second homotopy group of a simply connected compact simple Lie group $\tilde{G}$ is trivial and $\pi_2(G)\cong\pi_2(\tilde{G})$, the homomorphism f is injective and $\pi_2(G/K)\cong\operatorname{Im} f=\ker h$. So we shall calculate the kernel of the homomorphism h.

Now we shall express $\pi_1(G)\cong Z(\tilde{G})$ in terms of the roots of $\tilde{G}$. Let T and $\mathfrak{t}$ be a maximal torus of $\tilde{G}$ and the Lie algebra of T, respectively. We denote by $\Psi=\{\alpha_1, \cdots, \alpha_l\}$ the simple root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and by $\exp: \mathfrak{g}\to\tilde{G}$ the exponential map. The central lattice Λ_1 and the unit lattice $\Lambda(\tilde{G})$ of $\tilde{G}$ are defined by

$$\Lambda_1(\tilde{G})=\exp^{-1}(Z(\tilde{G})),$$

$$\Lambda(\tilde{G})=\exp^{-1}(e),$$

respectively, where e denotes the identity element of $\tilde{G}$. We choose an $\mathrm{Ad}(\tilde{G})$-invariant inner product $(\ ,\ )$ on $\mathfrak{g}$. For each linear form $a\in\mathfrak{t}^*$, the element $\vec{a}\in\mathfrak{t}$ is defined by

$$(\vec{a}, v)=a(v) \qquad \text{for any} \quad v\in\mathfrak{t},$$

and for each root α, we define $\alpha^*\in\mathfrak{t}$ by

$$\alpha^*=\frac{2\vec{\alpha}}{(\alpha, \alpha)},$$

where the inner product (a, b) of two linear forms a and b is defined by $(a, b)=(\vec{a}, \vec{b})$. Then we have the following proposition (cf. [4] p. 479).

PROPOSITION 3.1. *Let $\tilde{G}$ be a compact semisimple Lie group and $\Psi=\{\alpha_1, \cdots, \alpha_l\}$ the simple root system of $\tilde{G}$ with respect to a maximal torus T of $\tilde{G}$. Then*

(1) $Z(\tilde{G})\cong\Lambda_1(\tilde{G})/\Lambda(\tilde{G})$.

(2) $\Lambda_1(\tilde{G})=\{v\in\mathfrak{t}\,|\,\alpha_j(v)\in\boldsymbol{Z}$, for any $j=1, \cdots, l\}$.

(3) *Furthermore, if $\tilde{G}$ is simply connected, then $\Lambda(G)=\boldsymbol{Z}\alpha_1^*+\cdots+\boldsymbol{Z}\alpha_l^*$.*

By a straightforward calculation, we have

PROPOSITION 3.2. *The centers of* $SU(n)$, $Spin(n)$, $Sp(n)$, G_2, F_4, E_6, E_7 *and* E_8 *are given as follows;*

$$Z(SU(n))=\left\{\exp\left(\frac{j}{n}\sum_{i=1}^{n-1} i\alpha_i^*\right) \mid j=0,\ 1,\ \cdots,\ n-1\right\},$$

$$\begin{aligned} Z(Spin(2n+1)) &= Z(Spin(2n)) \\ &= \left\{\exp\left(\frac{j}{2}\sum_{i=1}^{n-2} i\alpha_i^* + \frac{j}{4}(n\alpha_{n-1}^* + (n-2)\alpha_n^*)\right.\right. \\ &\qquad \left.\left. + \frac{k(n-1)}{2}(\alpha_{n-1}^* + \alpha_n^*)\right) \mid j=0,\ 1,\ 2,\ 3,\ k=0,\ 1\right\}, \end{aligned}$$

$$Z(Sp(n))=\{e\},$$

$$Z(G_2)=\{e\},$$

$$Z(F_4)=\{e\},$$

$$Z(E_6)=\left\{\exp\left(\frac{j}{3}(\alpha_1^*+2\alpha_3^*+\alpha_5^*+2\alpha_6^*)\right) \mid j=0,\ 1,\ 2\right\},$$

$$Z(E_7)=\left\{\exp\left(\frac{j}{2}(\alpha_1^*+\alpha_3^*+\alpha_7^*)\right) \mid j=0,\ 1\right\},$$

$$Z(E_8)=\{e\}.$$

In the case where $\tilde{G}$ is a classical Lie group or $Z(\tilde{G})=1$, then we may calculate $\pi_2(G/K)$. So we shall deal with the case where $\tilde{G}=E_6$ or E_7.

First we shall show the following lemma.

LEMMA 3.3. *Let* $\mathfrak{k}$ *be the Lie algebra of a connected Lie group* $\tilde{K}$. *Suppose* $\mathfrak{k}$ *is a direct sum* $\mathfrak{k}_1 \oplus \mathfrak{k}_2$ *of two ideals* $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$. *We denote by* $\tilde{K}_i$ *the connected Lie subgroup of* $\tilde{K}$ *of Lie algebra* $\mathfrak{k}_i\,(i=1,\ 2)$. *Then* $\tilde{K}$ *is isomorphic to the quotient group* $\tilde{K}_1 \times \tilde{K}_2 / \tilde{K}_1 \cap \tilde{K}_2$.

PROOF. For any $X \in \mathfrak{k}_1$, $Y \in \mathfrak{k}_2$,

$$\begin{aligned} \exp Y \exp X (\exp Y)^{-1} &= \exp(Ad(\exp Y)X) \\ &= \exp(e^{ad(Y)}X) \\ &= \exp X. \end{aligned}$$

Hence we have $k_1k_2=k_2k_1$, for any $k_1 \in \tilde{K}_1$, $k_2 \in K_2$. We consider the homomorphism $\pi:\ \tilde{K}_1 \times \tilde{K}_2 \to \tilde{K}$ defined by $\pi(k_1,\ k_2)=k_1k_2$. Since

$$\begin{aligned} \ker\pi &= \{(k_1,\ k_2) \in \tilde{K}_1 \times \tilde{K}_2 \mid k_1k_2=e\} \\ &= \{(k,\ k^{-1}) \in \tilde{K}_1 \times \tilde{K}_2 \mid k \in \tilde{K}_1 \cap \tilde{K}_2\} \end{aligned}$$

$$\cong \tilde{K}_1 \cap \tilde{K}_2,$$

we obtain the lemma.

In the sequel, we shall adopt the following notation. Let $p: \tilde{G} \to G$ be the universal covering group of compact Lie group G, and $\tilde{K}$ (resp. K) the connected Lie subgroup of $\tilde{G}$ (resp. G) generated by the Lie subalgebra $\mathfrak{k}$. We denote by $\pi: \bar{K} \to \tilde{K}$ the universal covering group of $\tilde{K}$. Let $\bar{\gamma}: I \to \bar{K}$ be a path with $\bar{\gamma}(1) \in (p \circ \pi)^{-1}(e)$. We define a loop γ at e in K by $\gamma = p \circ \pi \circ \bar{\gamma}$. By the unique lifting property, the curve $\tilde{\gamma} := \pi \circ \bar{\gamma}$ is the lifting of γ starting at the identity of $\tilde{K}$.

Case (E6-1) $\mathfrak{g} = \mathfrak{e}_6$, $x = (1/3)v_1$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals;

$$\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{so}(10),$$

$$\mathfrak{k}_2 = \boldsymbol{R}(4\alpha_1^* + 3\alpha_2^* + 5\alpha_3^* + 6\alpha_4^* + 4\alpha_5^* + 2\alpha_6^*).$$

Put

$$v_1 = \frac{1}{2}(\alpha_2^* + \alpha_3^*),$$

$$w_1 = \frac{1}{4}(3\alpha_2^* + 5\alpha_3^* + 2\alpha_4^* + 2\alpha_6^*),$$

$$v_2 = 4\alpha_1^* + 3\alpha_2^* + 5\alpha_3^* + 6\alpha_4^* + 4\alpha_5^* + 2\alpha_6^*.$$

Then $\{w_1\}$ forms a basis of $\Lambda_1(\tilde{K}_1)$. We have

$$Z(\tilde{K}_1) = \{\exp(kw_1) \mid k = 0, 1, 2, 3\} \cong \boldsymbol{Z}_4,$$

$$\tilde{K}_1 = Spin(10).$$

Since the intersection $\tilde{K}_1 \cap \tilde{K}_2$ is equal to $\{\exp(k/4)v_2 \mid k = 0, 1, 2, 3\}$, we have

$$\tilde{K} = \{Spin(10) \times SO(2)\}/\boldsymbol{Z}_4.$$

If we put $\Gamma = Z(G) \cap \tilde{K}$, then K is isomorphic to $\tilde{K}/\Gamma$. In our case,

$$\begin{aligned} K &\cong \{[Spin(10) \times SO(2)]/\boldsymbol{Z}_4\}/\boldsymbol{Z}_3 \\ &= \{Spin(10) \times [SO(2)/\boldsymbol{Z}_3]\}/\boldsymbol{Z}_4 \\ &= \{Spin(10) \times SO(2)\}/\boldsymbol{Z}_4. \end{aligned}$$

Thus we have $\pi_1(K) = \boldsymbol{Z}_3 \times \boldsymbol{Z}_4 \times \boldsymbol{Z}$. We define paths $\bar{\gamma}_j (j=1, 2, 3)$ in $\bar{K} = Spin(10) \times \boldsymbol{R}$ by

$$\bar{\gamma}_1(t) = \left(e, \frac{t}{3} v_2\right),$$

$$\tilde{\gamma}_2(t)=(\exp(tw_1),\ 0),$$

$$\tilde{\gamma}_3(t)=(e,\ tv_2),$$

so that the corresponding paths $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ represent the generators (1, 0, 0), (0, 1, 0) and (0, 0, 1) of $\pi_1(\check{K})$ respectively. It is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K)\cong\ker h=\boldsymbol{Z}_4\times\boldsymbol{Z}$.

Case (E6-2) $\mathfrak{g}=\mathfrak{e}_6$, $x=(2/3)v_3$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals;

$$\mathfrak{k}_1=[\mathfrak{k},\ \mathfrak{k}]\cong\mathfrak{su}(2)\oplus\mathfrak{su}(5),$$

$$\mathfrak{k}_2=\boldsymbol{R}(5\alpha_1{}^*+6\alpha_2{}^*+10\alpha_3{}^*+12\alpha_4{}^*+8\alpha_5{}^*+4\alpha_6{}^*).$$

Put

$$v_1=\frac{1}{2}\alpha_1{}^*,$$

$$w_1=\frac{1}{5}(4\alpha_2{}^*+3\alpha_4{}^*+2\alpha_5{}^*+\alpha_6{}^*),$$

$$v_2=5\alpha_1{}^*+6\alpha_2{}^*+10\alpha_3{}^*+12\alpha_4{}^*+8\alpha_5{}^*+4\alpha_6{}^*.$$

Then $\{v_1,\ w_1\}$ forms a basis of $\Lambda_1(\tilde{K}_1)$. We have

$$\begin{aligned} Z(\tilde{K}_1)&=\{\exp(jv_1)\,|\,j=0,\ 1\}\times\{\exp(kw_1)\,|\,k=0,\ 1,\ 2,\ 3,\ 4\}\\ &\cong\boldsymbol{Z}_2\times\boldsymbol{Z}_5\\ &\cong Z(SU(2)\times SU(5)),\end{aligned}$$

$$\tilde{K}_1\cong SU(2)\times SU(5).$$

Since the intersection $\tilde{K}_1\cap\tilde{K}_2$ is equal to $\{\exp(k/10)v_2\,|\,k=0,\ 1,\ \cdots,\ 9\}=\{\exp(j/5)v_2\,|\,j=0,\ 1,\ 2,\ 3,\ 4\}\times\{\exp(k/2)v_2\,|\,k=0,\ 1\}$, we have

$$\begin{aligned}\tilde{K}&\cong\{SU(2)\times[SU(5)\times U(1)]/\boldsymbol{Z}_5\}/\boldsymbol{Z}_2\\ &\cong\{SU(2)\times S(U(5)\times U(1))\}/\boldsymbol{Z}_2.\end{aligned}$$

If we put $\Gamma=Z(\tilde{G})\cap\tilde{K}$, then K is isomorphic to $\tilde{K}/\Gamma$. In our case,

$$\begin{aligned}K&\cong\{[SU(2)\times S(U(5)\times U(1))]/\boldsymbol{Z}_2\}/\boldsymbol{Z}_3\\ &=\{SU(2)\times[S(U(5)\times U(1))/\boldsymbol{Z}_3]\}/\boldsymbol{Z}_2.\end{aligned}$$

Thus we have $\pi_1(K)=\boldsymbol{Z}_3\times\boldsymbol{Z}_2\times\boldsymbol{Z}_5\times\boldsymbol{Z}$. We define paths $\tilde{\gamma}_j(j=1,\ 2,\ 3,\ 4)$ in $\bar{K}=\{SU(2)\times SU(5)\}\times\boldsymbol{R}$ by

$$\tilde{\gamma}_1(t)=\left(e,\ \frac{2t}{3}v_2\right).$$

$$\tilde{\gamma}_2(t)=\left(\exp\frac{1}{2}v_2, \frac{-t}{2}v_2\right),$$

$$\tilde{\gamma}_3(t)=\left(\exp\frac{1}{5}v_2, \frac{-t}{5}v_2\right),$$

$$\tilde{\gamma}_4(t)=(e, tv_2),$$

so that the corresponding paths $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, $\tilde{\gamma}_3$ and $\tilde{\gamma}_4$ represent the generators (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1) of $\pi_1(\tilde{K})$ respectively. It is easily seen that γ_2, γ_3 and γ_4 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K)\cong\ker h=\boldsymbol{Z}_2\times\boldsymbol{Z}_5\times\boldsymbol{Z}$.

Case (E6-3) $\mathfrak{g}=\mathfrak{e}_6$, $x=(2/3)v_2$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\mathfrak{k}_1=[\mathfrak{k}, \mathfrak{k}]\cong\mathfrak{su}(6),$$

$$\mathfrak{k}_2=\boldsymbol{R}(\alpha_1{}^*+2\alpha_2{}^*+2\alpha_3{}^*+3\alpha_4{}^*+2\alpha_5{}^*+\alpha_6{}^*).$$

Put

$$v_1=\frac{1}{6}(5\alpha_1{}^*+4\alpha_3{}^*+3\alpha_4{}^*+2\alpha_5{}^*+\alpha_6{}^*)\in\mathfrak{k}_1,$$

$$v_2=\alpha_1{}^*+2\alpha_2{}^*+2\alpha_3{}^*+3\alpha_4{}^*+2\alpha_5{}^*+\alpha_6{}^*\in\mathfrak{k}_2.$$

Then $\{v_1\}$ forms a basis of $\Lambda_1(\tilde{K}_1)$. We have

$$\begin{aligned} Z(K_1)&=\exp\Lambda_1(\tilde{K}_1) \\ &=\{\exp(jv_1)\,|\,j=0, 1, \cdots, 5\} \\ &\cong\boldsymbol{Z}_6\cong Z(SU(6)), \end{aligned}$$

$$\tilde{K}_1\cong SU(6).$$

Since the intersection $\tilde{K}_1\cap\tilde{K}_2$ is equal to $\{\exp((j/2)v_2)\,|\,j=0, 1\}\cong\boldsymbol{Z}_2$, we have

$$\tilde{K}\cong\{SU(6)\times T^1\}/\boldsymbol{Z}_2.$$

If we put $\Gamma=Z(\tilde{G})\cap\tilde{K}$, then K is isomorphic to $\tilde{K}/\Gamma$. In our case,

$$K\cong\{[SU(6)/\boldsymbol{Z}_3]\times T^1\}/\boldsymbol{Z}_2.$$

Thus we have $\pi_1(K)=\boldsymbol{Z}\times\boldsymbol{Z}_3\times\boldsymbol{Z}_2$. We define paths $\tilde{\gamma}_j\,(j=1, 2, 3)$ in $\bar{K}=SU(6)\times\boldsymbol{R}$ by

$$\tilde{\gamma}_1(t)=(e, tv_2),$$

$$\tilde{\gamma}_2(t)=(\exp(2tv_1), 0),$$

$$\tilde{\gamma}_3(t)=\left(\exp\frac{1}{2}v_2, -\frac{t}{2}v_2\right),$$

so that the corresponding paths $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ represent the generators (1, 0, 0), (0, 1, 0) and (0, 0, 1) of $\pi_1(\tilde{K})$ respectively. It is easily seen that γ_1 and γ_3 are null-homotopic and γ_2 is not. Therefore we have $\pi_2(G/K)\cong\ker h=\boldsymbol{Z}\times\boldsymbol{Z}_2$.

Case (E6-4) $\mathfrak{g}=\mathfrak{e}_6$, $x=v_4$.

The center of $\mathfrak{k}$ is 0, and $\mathfrak{k}$ is semisimple. We denote by $\alpha_0=-\mu$ the negative of the maximal root. Then we have

$$Z(\tilde{K})=\left\{\exp\frac{j}{3}(\alpha_1{}^*+2\alpha_3{}^*)\,|\,j=0,1,2\right\}\times\left\{\exp\frac{k}{3}(\alpha_5{}^*+2\alpha_6{}^*)\,|\,k=0,1,2\right\}$$
$$\cong\boldsymbol{Z}_3\times\boldsymbol{Z}_3,$$
$$\tilde{K}\cong\{SU(3)\times SU(3)\times SU(3)\}/\boldsymbol{Z}_3,$$

If we put $\Gamma=\boldsymbol{Z}(\tilde{G})\cap\tilde{K}$, then K is isomorphic to $\tilde{K}/\Gamma$. In our case

$$K\cong\{SU(3)\times SU(3)\times SU(3)\}/\{\boldsymbol{Z}_3\times\boldsymbol{Z}_3\}.$$

Thus we have $\pi_1(K)\cong\boldsymbol{Z}_3\times\boldsymbol{Z}_3$. We define paths $\bar{\gamma}_j\,(j=1,2)$ in $\bar{K}=SU(3)\times SU(3)\times SU(3)$ by

$$\bar{\gamma}_1(t)=\left(\exp\frac{t}{3}(\alpha_1{}^*+2\alpha_3{}^*),\ \exp\frac{t}{3}(\alpha_0{}^*+2\alpha_2{}^*),\ \exp\frac{2t}{3}(\alpha_5{}^*+2\alpha_6{}^*)\right),$$

$$\bar{\gamma}_2(t)=\left(\exp\frac{t}{3}(\alpha_1{}^*+2\alpha_3{}^*),\ e,\ \exp\frac{t}{3}(\alpha_5{}^*+2\alpha_6{}^*)\right),$$

so that the corresponding paths $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ represent the generators (1, 0) and (0, 1) of $\pi_1(\tilde{K})$ respectively. It is easily seen that γ_1 is null-homotopic and γ_2 is not. Therefore we have $\pi_2(G/K)\cong\ker h=\boldsymbol{Z}_3$.

Case (E6-5) $\mathfrak{g}=\mathfrak{e}_6$, $x=(1/3)(v_1+v_6)$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\mathfrak{k}_1=[\mathfrak{k},\mathfrak{k}]\cong\mathfrak{so}(8),$$
$$\mathfrak{k}_2=\boldsymbol{R}(4\alpha_1{}^*+\alpha_2{}^*+3\alpha_3{}^*+2\alpha_4{}^*-2\alpha_6{}^*)$$
$$\oplus\boldsymbol{R}(-2\alpha_1{}^*-\alpha_3{}^*+\alpha_5{}^*+2\alpha_6{}^*).$$

Put

$$v_1=\frac{1}{2}(\alpha_2{}^*+\alpha_3{}^*),$$

$$w_1=\frac{1}{2}(\alpha_2{}^*+\alpha_5{}^*),$$

$$v_2=4\alpha_1{}^*+\alpha_2{}^*+3\alpha_3{}^*+2\alpha_4{}^*-2\alpha_6{}^*,$$

$$w_2=-2\alpha_1{}^*-\alpha_3{}^*+\alpha_5{}^*+2\alpha_6{}^*.$$

Then $\{v_1, w_1\}$ forms a basis of $\Lambda_1(\tilde{K}_1)$. We have

$$\begin{aligned} Z(\tilde{K}_1) &= \{\exp(jv_1)\,|\,j=0, 1\} \times \{\exp(kw_1)\,|\,k=0, 1\} \\ &\cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \\ &\cong Z(Spin(8)), \\ \tilde{K}_1 &\cong Spin(8). \end{aligned}$$

Since the intersection $\tilde{K}_1 \cap \tilde{K}_2$ is equal to $\{\exp(j/2)v_2\,|\,j=0, 1\} \times \{\exp(k/2)(v_2+w_2)\,|\,k=0, 1\}$, we have

$$\tilde{K} \cong \{[Spin(8)\times SO(2)]/\boldsymbol{Z}_2 \times SO(2)\}/\boldsymbol{Z}_2.$$

If we put $\Gamma = Z(G) \cap \tilde{K}$, then K is isomorphic to $\tilde{K}/\Gamma$. In our case,

$$\begin{aligned} K &\cong \{\{[Spin(8)\times SO(2)]/\boldsymbol{Z}_2 \times SO(2)\}/\boldsymbol{Z}_2\}/\boldsymbol{Z}_3 \\ &= \{[Spin(8)\times SO(2)]/\boldsymbol{Z}_2 \times [SO(2)/\boldsymbol{Z}_3]\}/\boldsymbol{Z}_2 \\ &= \{[Spin(8)\times SO(2)]/\boldsymbol{Z}_2 \times SO(2)\}/\boldsymbol{Z}_2. \end{aligned}$$

Thus we have $\pi_1(K) \cong \boldsymbol{Z}_3 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \times \boldsymbol{Z} \times \boldsymbol{Z}$. We define paths $\bar{\gamma}_j (j=1, \cdots, 5)$ in $\bar{K} = Spin(8) \times \boldsymbol{R} \times \boldsymbol{R}$ by

$$\begin{aligned} \bar{\gamma}_1(t) &= \left(\exp(v_1+w_1),\ 0,\ -\frac{t}{6}w_2\right), \\ \bar{\gamma}_2(t) &= \left(\exp v_1,\ -\frac{t}{2}v_2,\ 0\right), \\ \bar{\gamma}_3(t) &= \left(\exp w_1,\ -\frac{t}{2}v_2,\ -\frac{t}{2}w_2\right), \\ \bar{\gamma}_4(t) &= (e,\ tv_2,\ 0), \\ \bar{\gamma}_5(t) &= (e,\ 0,\ tw_2), \end{aligned}$$

so that the corresponding paths $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4$ and $\tilde{\gamma}_5$ represent the generators (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0) and (0, 0, 0, 0, 1) of $\pi_1(\tilde{K})$ respectively. It is easily seen that $\gamma_2, \gamma_3, \gamma_4$ and γ_5 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \times \boldsymbol{Z} \times \boldsymbol{Z}$.

Case (E7-1) $\mathfrak{g} = \mathfrak{e}_7$, $x = (1/3)v_1$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\begin{aligned} \mathfrak{k}_1 &= [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{e}_6, \\ \mathfrak{k}_2 &= \boldsymbol{R}(3\alpha_1^* + 4\alpha_2^* + 5\alpha_3^* + 6\alpha_4^* + 4\alpha_5^* + 2\alpha_6^* + 3\alpha_7^*). \end{aligned}$$

Put

$$v_1=\frac{1}{3}(\alpha_2^*+2\alpha_3^*+\alpha_5^*+2\alpha_6^*),$$

$$v_2=(3\alpha_1^*+4\alpha_2^*+5\alpha_3^*+6\alpha_4^*+4\alpha_5^*+2\alpha_6^*+3\alpha_7^*).$$

Then $\{v_1\}$ forms a basis of $\Lambda_1(\tilde{K}_1)$. We have

$$Z(\tilde{K}_1)=\{\exp(jv_1)\mid j=0, 1, 2\}\cong \boldsymbol{Z}_3\cong Z(E_6),$$

$$\tilde{K}_1\cong E_6.$$

Since the intersection $\tilde{K}_1\cap K_2$ is equal to $\{\exp(k/3)v_2\mid k=0, 1, 2\}$, we have

$$\tilde{K}\cong\{E_6\times T^1\}/\boldsymbol{Z}_3.$$

If we put $\Gamma=Z(\tilde{G})\cap K$, then K is isomorphic to $\tilde{K}/\Gamma$. In our case,

$$\begin{aligned} K &\cong \{[E_6\times T^1]/\boldsymbol{Z}_3\}/\boldsymbol{Z}_2 \\ &=\{E_6\times[T^1/\boldsymbol{Z}_2]\}/\boldsymbol{Z}_3 \\ &\cong\{E_6\times T^1\}/\boldsymbol{Z}_3. \end{aligned}$$

Thus we have $\pi_1(K)\cong\boldsymbol{Z}_2\times\boldsymbol{Z}_3\times\boldsymbol{Z}$. We defined paths $\bar{\gamma}_j(j=1, 2, 3)$ in $\bar{K}=E_6\times \boldsymbol{R}$ by

$$\bar{\gamma}_1(t)=\left(\exp\frac{1}{3}(\alpha_2^*+2\alpha_3^*+\alpha_5^*+2\alpha_6^*),\ \frac{t}{6}v_2\right),$$

$$\bar{\gamma}_2(t)=\left(\exp\frac{1}{3}(\alpha_2^*+2\alpha_3^*+\alpha_5^*+2\alpha_6^*),\ -\frac{t}{3}v_2\right),$$

$$\bar{\gamma}_3(t)=(e,\ tv_2),$$

so that the corresponding paths $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ represent the generators (1, 0, 0), (0, 1, 0) and (0, 0, 1) of $\pi_1(\tilde{K})$ respectively. If is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K)\cong\ker h=\boldsymbol{Z}_3\times\boldsymbol{Z}$.

Case (E7-2) $\mathfrak{g}=\mathfrak{e}_7$, $x=(2/3)v_2$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\mathfrak{k}_1=[\mathfrak{k}, \mathfrak{k}]\cong\mathfrak{su}(2)\oplus\mathfrak{so}(10),$$

$$\mathfrak{k}_2=\boldsymbol{R}(2\alpha_1^*+4\alpha_2^*+5\alpha_3^*+6\alpha_4^*+4\alpha_5^*+2\alpha_6^*+3\alpha_7^*).$$

Put

$$v_1=\frac{1}{2}\alpha_1^*,$$

$$w_1=\frac{1}{4}(\alpha_3^*+2\alpha_4^*+2\alpha_6^*+3\alpha_7^*),$$

$$v_2=2\alpha_1^*+4\alpha_2^*+5\alpha_3^*+6\alpha_4^*+4\alpha_5^*+2\alpha_6^*+3\alpha_7^*.$$

Then $\{v_1, w_1\}$ forms a basis of $\Lambda_1(\tilde{K}_1)$. We have

$$\begin{aligned} \boldsymbol{Z}(\tilde{K}_1) &= \{\exp(jv_1) \mid j=0, 1\} \times \{\exp(kw_1) \mid k=0, 1, 2, 3\} \\ &\cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_4 \\ &\cong Z(SU(2) \times Spin(10)), \\ \tilde{K}_1 &\cong SU(2) \times Spin(10). \end{aligned}$$

Since the intersection $\tilde{K}_1 \cap \tilde{K}_2$ is equal to $\{\exp(k/4)v_2 \mid k=0, 1, 2, 3\}$, we have

$$\begin{aligned} K &\cong \{[SU(2) \times Spin(10)] \times T^1\} / \boldsymbol{Z}_4 \\ &\cong \{SU(2) \times [Spin(10) \times T^1] / \boldsymbol{Z}_2\} / \boldsymbol{Z}_2. \end{aligned}$$

If we put $\Gamma = Z(\tilde{G}) \cap \tilde{K}$, then K is isomorphic to $\tilde{K}/\Gamma$. In our case,

$$K \cong \{[SU(2) \times (Spin(10) \times SO(2))/\boldsymbol{Z}_2]/\boldsymbol{Z}_2\}/\boldsymbol{Z}_2.$$

Thus we have $\pi_1(K) \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_4 \times \boldsymbol{Z}$. We define paths $\gamma_j (j=1, 2, 3)$ in $\bar{K} = SU(2) \times Spin(10) \times \boldsymbol{R}$ by

$$\begin{aligned} \bar{\gamma}_1(t) &= \left(\exp(v_1), \frac{t}{2} v_2\right), \\ \bar{\gamma}_2(t) &= \left(\exp(v_1 + w_1), -\frac{t}{4} v_2\right), \\ \bar{\gamma}_3(t) &= (e, tv_2), \end{aligned}$$

so that the corresponding paths $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ represent the generators (1, 0, 0), (0, 1, 0) and (0, 0, 1) of $\pi_1(\tilde{K})$ respectively. It is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \boldsymbol{Z}_4 \times \boldsymbol{Z}$.

Case (E7-3) $\mathfrak{g} = \mathfrak{e}_7$, $x = (2/3)v_6$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\begin{aligned} \mathfrak{k}_1 &= [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{so}(12), \\ \mathfrak{k}_2 &= \boldsymbol{R}(\alpha_1^* + 2\alpha_2^* + 3\alpha_3^* + 4\alpha_4^* + 3\alpha_5^* + 2\alpha_6^* + 2\alpha_7^*). \end{aligned}$$

Put

$$\begin{aligned} v_1 &= \frac{1}{2}(\alpha_1^* + 3\alpha_3^* + 3\alpha_5^*), \\ w_1 &= \frac{1}{2}(\alpha_5^* + \alpha_7^*), \\ v_2 &= \alpha_1^* + 2\alpha_2^* + 3\alpha_3^* + 4\alpha_4^* + 3\alpha_5^* + 2\alpha_6^* + 2\alpha_7^*. \end{aligned}$$

Then $\{v_1, w_1\}$ forms a basis of $\Lambda_1(\tilde{K}_1)$. We have

$$Z(\tilde{K}_1) = \{\exp(jv_1) \mid j=0, 1\} \times \{\exp(kw_1) \mid k=0, 1\}$$

$$\cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$$

$$\cong Z(Spin(12)),$$

$$\tilde{K}_1 \cong Spin(12).$$

Since the intersection $\tilde{K}_1 \cap \tilde{K}_2$ is equal to $\{\exp(k/2)v_2 \mid k=0, 1\}$, we have

$$\tilde{K} \cong \{Spin(12) \times T^1\}/\boldsymbol{Z}_2.$$

If we put $\Gamma = Z(\tilde{G}) \cap \tilde{K}$, then K is isomorphic to $\tilde{K}/\Gamma$. In our case,

$$K \cong \{[Spin(12) \times SO(2)]/\boldsymbol{Z}_2\}/\boldsymbol{Z}_2.$$

Thus we have $\pi_1(K) \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}$. We define paths $\bar{\gamma}_j (j=1, 2, 3)$ in $\bar{K} = Spin(12) \times \boldsymbol{R}$ by

$$\bar{\gamma}_1(t) = \left(\exp\frac{t}{2}(\alpha_1^* + \alpha_3^* + \alpha_7^*),\ 0\right),$$

$$\bar{\gamma}_2(t) = \left(\exp\frac{1}{2}v_2,\ -\frac{t}{2}v_2\right),$$

$$\bar{\gamma}_3(t) = (e,\ tv_2),$$

so that the corresponding paths $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ represent the generators (1, 0, 0), (0, 1, 0) and (0, 0, 1) of $\pi_1(\tilde{K})$ respectively. It is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K) \cong \ker h = \boldsymbol{Z}_2 \times \boldsymbol{Z}$.

Case (E7-4) $\mathfrak{g} = \mathfrak{e}_7$, $x = (2/3)v_7$.

Take a direct sum decomposition of $\mathfrak{k}$ by the following two ideals:

$$\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(7),$$

$$\mathfrak{k}_2 = \boldsymbol{R}(3\alpha_1^* + 6\alpha_2^* + 9\alpha_3^* + 12\alpha_4^* + 8\alpha_5^* + 4\alpha_6^* + 7\alpha_7^*).$$

Put

$$v_1 = \frac{1}{7}(\alpha_1^* + 2\alpha_2^* + 3\alpha_3^* + 4\alpha_4^* + 5\alpha_5^* + 6\alpha_6^*),$$

$$v_2 = (3\alpha_1^* + 6\alpha_2^* + 9\alpha_3^* + 12\alpha_4^* + 8\alpha_5^* + 4\alpha_6^* + 7\alpha_7^*).$$

Then $\{v_1\}$ forms a basis of $\Lambda_1(\tilde{K}_1)$. We have

$$Z(\tilde{K}_1) = \{\exp(jv_1) \mid j=0, 1, \cdots, 6\} \cong \boldsymbol{Z}_7 \cong Z(SU(7)),$$

$$\tilde{K}_1 \cong SU(7).$$

Since the intersection $\tilde{K}_1 \cap \tilde{K}_2$ is equal to $\{\exp(k/7)v_2 \mid k=0, 1, \cdots, 6\}$, we have

$$\tilde{K} \cong \{SU(7) \times T^1\}/\boldsymbol{Z}_7 \cong S\{U(7) \times U(1)\}.$$

If we put $\Gamma = Z(\tilde{G}) \cap \tilde{K}$, then K is isomorphic to $\tilde{K}/\Gamma$. In our case,

$$K \cong S(U(7)\times U(1))/\boldsymbol{Z}_2.$$

Thus we have $\pi_1(K)\cong \boldsymbol{Z}_2\times\boldsymbol{Z}_7\times\boldsymbol{Z}$. We define paths $\bar{\gamma}_j(j=1, 2, 3)$ in $\bar{K}=SU(7)\times\boldsymbol{R}$ by

$$\bar{\gamma}_1(t)=\left(e, \frac{t}{2}v_2\right),$$

$$\bar{\gamma}_2(t)=\left(\exp(3v_1), -\frac{1}{7}v_2\right),$$

$$\bar{\gamma}_3(t)=(e, tv_2),$$

so that the corresponding paths $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ represent the generators (1, 0, 0), (0, 1, 0) and (0, 0, 1) of $\pi_1(\tilde{K})$ respectively. It is easily seen that γ_2 and γ_3 are null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K)\cong\ker h=\boldsymbol{Z}_2\times\boldsymbol{Z}$.

Case (E7-5) $\mathfrak{g}=\mathfrak{e}_7$, $x=v_3$.

The center of $\mathfrak{k}$ is 0, and $\mathfrak{k}$ is semisimple. We denote by $\mu=-\alpha_0$ the maximal root $\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7$ of $\mathfrak{g}$. Put

$$v_1=\frac{1}{3}(\alpha_1{}^*+2\alpha_2{}^*),$$

$$w_1=\frac{1}{6}(\alpha_0{}^*+2\alpha_6{}^*+3\alpha_5{}^*+4\alpha_4{}^*+5\alpha_7{}^*)$$

$$=\frac{1}{6}(-\alpha_1{}^*-2\alpha_2{}^*-3\alpha_3{}^*+3\alpha_7{}^*).$$

Then $\{w_1\}$ forms a basis of $\Lambda_1(\tilde{K})$. We have

$$Z(\tilde{K})=\{\exp(kw_1)\mid k=0, 1, \cdots, 5\}\cong\boldsymbol{Z}_6,$$

$$\tilde{K}\cong\{SU(3)\times SU(6)\}/\boldsymbol{Z}_3,$$

If we put $\Gamma=Z(G)\cap\tilde{K}$, then K is isomrphic to $\tilde{K}/\Gamma$. In our case,

$$K\cong\{[SU(3)\times SU(6)]/\boldsymbol{Z}_3\}/\boldsymbol{Z}_2$$

$$=\{SU(3)\times[SU(6)/\boldsymbol{Z}_2]\}/\boldsymbol{Z}_3.$$

Thus we have $\pi_1(K)\cong\boldsymbol{Z}_2\times\boldsymbol{Z}_3$. We define paths $\bar{\gamma}_j(j=1, 2)$ in $\bar{K}=SU(3)\times SU(6)$ by

$$\bar{\gamma}_1(t)=(e, \exp(3tw_1)),$$

$$\bar{\gamma}_2(t)=(\exp(tv_1), \exp(2tw_1)),$$

so that the corresponding paths $\tilde{\gamma}_1$ the $\tilde{\gamma}_2$ represent the generators (1, 0) and (0, 1) of $\pi_1(\tilde{K})$ respectively. It is easily seen that γ_2 is null-homotopic and γ_1 is not. Therefore we have $\pi_2(G/K)\cong\ker h=\boldsymbol{Z}_3$.

Acknowledgement The author wishes to express his sincere thanks to Professor K. Sekigawa for his hearty guidance and many valuable suggestions and to the referee for many valuable suggestions.

References

[1] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. of Math. **57** (1953), 115-207.

[2] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces I, Amer. J. Math. **30** (1958), 458-538.

[3] A. Gray. Riemannian manifolds with geodesic symmetries of order 3, J. Differential Geom. **7** (1972), 343-369.

[4] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.

[5] T. Koda, The first Chern classes of compact Riemannian 3-symmetric spaces, Math. Rep. Toyama Univ. **12** (1989), 113-138.

[6] T. Koda, The first Chern class of Riemmanian 3-symmetric spaces: the classical case, Note di Matematica **10** (1990), 141-156.

[7] N. Steenrod, The Topology of Fibre Bundles, Princeton Univ. Press, 1972.

[8] M. Takeuchi, On Pontrjagin classes of compact symmetric spaces, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **9** (1961-63), 313-328.

[9] G.W. Whitehead, Elements of Homotopy Theory, Springer-Verlag, New York, 1978.

[10] J.A. Wolf and A. Gray, Homogeneous spaces defined by Lie group automorphisms I, II, J. Differential Geom. **2** (1968), 77-114, 115-159.

Department of Mathematics,
Faculty of Science,
Toyama University,
Gofuku, Toyama 930, Japan