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GEOMETRY AND SYMMETRY ON SASAKIAN MANIFOLDS

By

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1. Introduction.

In [26] the second author has given a brief survey about the local generalization to arbitrary Riemannian manifolds of the notion of a reflection with respect to a point, a line or a linear subspace in a Euclidean space. Local symmetries with respect to a point (local geodesic symmetries) are well-known and these local diffeomorphisms are already used at many occasions to study and even to classify some particular classes of Riemannian manifolds (see for example [25]). Local symmetries with respect to a curve, in particular with respect to a geodesic, led to less well-known characterizations of locally symmetric manifolds and spaces of constant curvature [27]. Finally, local symmetries with respect to submanifolds are studied in [21] and they give some nice geometrical results in the theory of embedded minimal and totally geodesic submanifolds.

Local symmetries with respect to a curve may also be used to study contact geometry, in particular on Sasakian manifolds. On these manifolds, the integral curves of the characteristic vector field are geodesics and the study of the local symmetries with respect to these curves led in a natural way to a very geometrical treatment of the so-called φ-symmetric spaces introduced in [18]. These spaces are natural analogues of locally symmetric spaces. (See [4], [5], [6], [18].)

In this paper we continue our study of Sasakian geometry but now we focus on local symmetries with respect to geodesics which cut the integral curves of the characteristic vector field orthogonally. Such geodesics are usually called φ-geodesics. The main purpose is to use these local symmetries to give new characterizations of φ-symmetric spaces, Sasakian space forms and locally symmetric Sasakian manifolds. In sections 2, 3 and 4 we treat some general preliminaries about Sasakian manifolds and symmetries with respect to a curve thereby focussing on the central role of normal coordinates, Fermi coordinates and Jacobi vector fields. (For more details about contact geometry we refer to [1], [29].) In sections 5, 6 and 7 we prove our main results about symmetry.

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2. Contact Geometry and Sasakian Manifolds.

A smooth manifold M^{2n+1} is said to be an *almost contact manifold* if the structural group of its tangent bundle is reducible to $\mathcal{U}(n)\times 1$. It is well-known that such a manifold admits a tensor field φ of type (1, 1), a vector field ξ and a 1-form η satisfying

$$\eta(\xi)=1, \qquad \varphi^2=-I+\eta\otimes\xi.$$

These conditions imply that $\varphi\xi=0$ and $\eta\circ\varphi=0$. Moreover, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y)=g(X, Y)-\eta(X)\eta(Y)$$

for any tangent vector fields X and Y. Note that this implies that

$$\eta(X)=g(X, \xi).$$

M together with these structure tensors (φ, ξ, η, g) is said to be an *almost contact metric manifold*. If now these structure tensors satisfy

$$(\nabla_X\varphi)Y=g(X, Y)\xi-\eta(Y)X, \tag{1}$$

where ∇ denotes the Riemannian connection of g, M is said to be a *Sasakian manifold*. It is easy to see from (1) that

$$\nabla_X\xi=-\varphi X, \tag{2}$$

from which it follows that ξ is a Killing vector field. We refer to [1] for a lot of examples.

The curvature tensor

$$R_{XY}Z=\nabla_{[X,Y]}Z-[\nabla_X, \nabla_Y]Z$$

on a Sasakian manifold satisfies

$$R_{XY}\xi=\eta(X)Y-\eta(Y)X. \tag{3}$$

As a notational matter we write R_{XYZW} for $g(R_{XY}Z, W)$ and $(\nabla_U R)_{XYZW}$ for $g((\nabla_U R)_{XY}Z, W)$. Then we have

$$\begin{aligned} R_{XYZ\varphi W}+R_{XY\varphi ZW}=&\,d\eta(X, Z)g(Y, W)-d\eta(X, W)g(Y, Z)\\ &-d\eta(Y, Z)g(X, W)+d\eta(Y, W)g(X, Z), \end{aligned} \tag{4}$$

where

$$\Phi(X, Y)=(d\eta)(X, Y)=g(X, \varphi Y).$$

A plane section of the tangent space at a point of M is called a *φ-section* if it is spanned by vectors X and φX orthogonal to ξ. The sectional curvature of a φ-section is called a *φ-sectional curvature*. A Sasakian manifold of constant φ-sectional curvature c is called a Sasakian space form and its curvature tensor is given by

$$R_{XY}Z=\frac{c+3}{4}\{g(X, Z)Y-g(Y, Z)X\}+\frac{c-1}{4}\{\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y -g(X, Z)\eta(Y)\xi+g(Y, Z)\eta(X)\xi-g(Z, \varphi Y)\varphi X+g(Z, \varphi X)\varphi Y -2g(X, \varphi Y)\varphi Z\}. \tag{5}$$

Note that c is automatically globally constant for $\dim M\geqq 5$. (Here and in the rest of the paper we will suppose, if necessary, that M is a connected manifold.) Further, we have the following useful characterization.

THEOREM 1 [20]. *A connected Sasakian manifold M of dimension$\geqq 5$ is a Sasakian space form if and only if, for every vector field X orthogonal to ξ, we have*

$$R_{X\varphi XX}\sim\varphi X,$$

that is, $R_{X\varphi X}X$ is proportional to φX.

We also note that the Sasakian space forms are classified completely and locally there are three classes according to $c+3>0$, $c+3=0$ or $c+3<0$ [19] (see also [1], [29]).

It is well-known that for a Sasakian manifold the condition of *local symmetry*, i.e. $\nabla R=0$, is a very strong condition. Indeed, we have

THEOREM 2 [17]. *A connected locally symmetric Sasakian manifold has constant sectional curvature $c=1$.*

For that reason T. Takahashi [18] introduced the notion of a *(locally) φ-symmetric Sasakian manifold.*

In the rest of this section we consider these spaces in more detail and state some important facts about them. Some will be needed later on and others are stated for reasons of completeness.

A geodesic γ on a Sasakian manifold is said to be a *φ-geodesic* if $\eta(\gamma')=0$. From (2) it is easy to see that a geodesic which is initially orthogonal to ξ remains orthogonal to ξ. A local diffeomorphism s_m of M, $m\in M$, is said to be a *φ-geodesic symmetry* if its domain U is such that, for every φ-geodesic $\gamma(s)$ such that $\gamma(0)$ lies in the intersection of U with the integral curve of ξ through m,

$$(s_m\circ\gamma)(s)=\gamma(-s)$$

for all s with $\gamma(\pm s)\in U$, s being the arc length [18]. Note that at the point m the differential s_{m*} of s_m is given by

$$s_{m*}(m)=-I+2\eta\otimes\xi. \tag{6}$$

In [18] Takahashi introduced the notion of a *locally φ-symmetric space* by requiring that

$$\varphi^2(\nabla_V R)_{XY}Z=0$$

for all vector fields V, X, Y, Z orthogonal to ξ. Moreover, he defined a *globally φ-symmetric space* by requiring that any φ-geodesic symmetry of the locally φ-symmetric space be extendable to a global automorphism of M and that the Killing vector field ξ generates a global one-parameter subgroup of isometries.

Finally, let $\tilde{\mathcal{U}}$ be a neighborhood on M on which ξ is *regular* (see [1]). Then, as is well-known, the fibration $\tilde{\mathcal{U}}\to\mathcal{U}=\tilde{\mathcal{U}}/\xi$ gives a Kähler structure (J, G) on the base manifold $\mathcal{U}$.

First we collect four results from [18].

THEOREM 3. *A Sasakian manifold is a locally φ-symmetric space if and only if each Kähler manifold, which is the base space of a local fibering, is a Hermitian locally symmetric space.*

THEOREM 4. *A complete, connected, simply connected Sasakian locally φ-symmetric space is a globally φ-symmetric space. It is a principal G^1-bundle over a Hermitian globally symmetric space.*

THEOREM 5. *A necessary and sufficient condition for a Sasakian manifold to be locally φ-symmetric is that it admits a φ-geodesic symmetry at every point, which is a local automorphism of the structure (φ, ξ, η, g).*

THEOREM 6. *A Sasakian manifold M is locally φ-symmetric if and only if, for all vector fields X, Y, Z, V, W,*

$$\begin{aligned}(\nabla_V R)_{XYZW}=&\eta(W)\{g(X, V)g(\varphi Y, Z)-g(Y, V)g(\varphi X, Z)+R_{XYV\varphi Z}\}\\&+\eta(X)\{g(Y, Z)g(\varphi V, W)-g(Y, W)g(\varphi V, Z)-R_{Y\varphi VZW}\}\\&+\eta(Y)\{g(X, W)g(\varphi V, Z)-g(X, Z)g(\varphi V, W)+R_{X\varphi VZW}\}\\&+\eta(Z)\{g(Y, V)g(\varphi X, W)-g(X, V)g(\varphi Y, W)-R_{XYV\varphi W}\}.\end{aligned} \tag{7}$$

Note that it is easily seen that a Sasakian space form is always locally φ-symmetric. Other examples may be constructed by using Theorem 4. See also [18].

In [4] D. E. Blair and the second author proved another characterization theorem.

THEOREM 7. *Let M be a Sasakian manifold with structure tensors (φ, ξ, η, g). Then M is a locally φ-symmetric space if and only if all the local φ-geodesic symmetries are*

a) *ϕ-preserving,*

or b) *φ-preserving,*

or c) *η-preserving,*

or d) *g-preserving.*

d) and a) correspond to the analogue characterizations of locally symmetric and Hermitian locally symmetric spaces (see [25]).

For dimension $M=3$ and 5 we also have

THEOREM 8 [4], [6]. *Let M be a connected Sasakian manifold of dimension* 3 *or* 5. *Then M is locally φ-symmetric if and only if all local φ-geodesic symmetries are volume-preserving.*

From this a complete classification of complete, simply connected three-dimensional locally φ-symmetric spaces is determined in [4].

For our purposes we shall need two other characterizations, proved in [5].

THEOREM 9. *A Sasakian manifold M is locally φ-symmetric if and only if*

$$\nabla_X R_{X\varphi XX\varphi X}=0$$

for all vector fields X orthogonal to ξ.

THEOREM 10. *A Sasakian manifold M is locally φ-symmetric if and only if for every point $m\in M$ and every φ-geodesic γ through m we have the following property: for every $p\in\gamma$ such that p and $s_m(p)$ lie in a normal neighborhood of m, the shape operators $T_p(m)$ and $T_{s_m(p)}(m)$ at m of the geodesic spheres of radius $d(m, p)$ centered at p and $s_m(p)$ commute with $s_{m*}(m)$, i.e.*

$$s_{m*}(m)\circ T_p(m)=T_{s_m(p)}(m)\circ s_{m*}(m). \tag{8}$$

To finish this section we note that *the local geodesic symmetries* of a locally φ-symmetric space have the remarkable property that they *are all volume-preserving* [28]. This is related to the fact that this volume-preserving property for the local geodesic symmetries holds for any naturally reductive homogeneous space [8], [14], [22]. We have

THEOREM 11 [5]. *Let M be a complete, connected, simply connected Sasakian* *manifold. Then M is a globally φ-symmetric space if and only if M is a naturally reductive homogeneous space with invariant Sasakian structure.*

The local version of this theorem may be stated and proved by using the theory of homogeneous structures [22], [23]. Finally we note that the volume-preserving property implies the following useful identity:

$$\nabla_X \rho_{XX}=0 \tag{9}$$

for all tangent vectors X. Here ρ denotes the Ricci tensor of (M, g). (See for example [25].)

3. Jacobi Vector fields on Sasakian Space Forms.

Let m be a point on a Sasakian manifold and let γ be a geodesic, parametrized by arc length s, through $m=\gamma(0)$ and with $\gamma'(0)=u$, $\|u\|=1$. In what follows we shall also write $\gamma'=u$ at any point of γ.

A Jacobi vector field X along γ is a vector field that satisfies the equation

$$\nabla_u \nabla_u X+R_{uX}u=0. \tag{10}$$

The Jacobi vector fields may be defined, as is well-known, by variations of geodesics, and hence it is clear that they play a very important role in the study of the geometry of a small neighborhood of a point or of a tubular neighborhood of a curve. So they are crucial in the theory of geodesic spheres and tubes and hence in the treatment of local geodesic symmetries and local symmetries with respect to a curve. (See for example [7], [11], [12], [24], [27].)

For our purposes we shall need the solutions of the Jacobi differential equation (10) when M is a Sasakian space form, i.e. when R is given by (5), c being a constant. We follow the method developed in [2], [3] but now we need the explicit solutions.

To solve (10) we take an orthonormal basis $\{e_i, i=1, \cdots, 2n+1\}$ at m such that $\gamma'(0)=u=e_1$ and we denote by $\{E_i, i=1, \cdots, 2n+1\}$ the orthonormal frame along γ obtained by parallel translation of the e_i along γ.

We start with the easy case $u=\xi$. Then by (3) and (10) we have

$$\nabla_\xi \nabla_\xi X+X=0$$

for X orthogonal to ξ and hence we obtain at once

THEOREM 12. *Let γ be an integral curve of the characteristic vector field ξ on a Sasakian manifold and let X be a Jacobi vector field orthogonal to ξ along γ. With respect to the parallel basis $\{u, E_a, a=2, \cdots, 2n+1\}$ we have*

$$X(s)=\sum_{a=2}^{2n+1}(A_a \sin s+B_a \cos s)E_a(s),$$

where A_a and B_a are constants.

Next, let γ be a geodesic through $m\in M$, tangent to $u\neq\xi$. This case is more complicated. Let $\alpha=\eta(u)$ and note that α is constant along γ. Then (5) implies

$$R_{uX}u=\frac{c+3}{4}\{X-g(u, X)u\}+\frac{c-1}{4}\{\alpha\eta(X)u-\alpha^2 X +\alpha g(u, X)\xi-\eta(X)\xi-3g(u, \varphi X)\varphi u\}. \tag{11}$$

Further, let $\{e_1, \cdots, e_{2n+1}\}$ be an orthonormal basis at m with $u=e_1$ and

$$e_{2n}=(1-\alpha^2)^{-1/2}(\xi-\alpha u), \qquad e_{2n+1}=(1-\alpha^2)^{-1/2}\varphi u.$$

Again, denote by $\{E_i, i=1, \cdots, 2n+1\}$ the basis obtained by parallel translation of the vectors e_i along γ. Then we obtain easily along γ:

$$\begin{cases} E_{2n}=(1-\alpha^2)^{-1/2}\{(\xi-\alpha u)\cos s+\varphi u \sin s\}, \\ E_{2n+1}=(1-\alpha^2)^{-1/2}\{-(\xi-\alpha u)\sin s+\varphi u \cos s\}, \end{cases} \tag{12}$$

where s denotes the arc length from m along γ. Note that span$\{\varphi u, \xi-\alpha u\}$ is parallel along γ. Further, from (11) and (12) we get

$$\begin{cases} R_{uE_a}u=\frac{1}{4}lE_a, \qquad a=2, \cdots, 2n-1, \\ R_{uE_{2n}}u=\{\cos^2 s+(p+1)\sin^2 s\}E_{2n}+\frac{1}{2}p\sin 2sE_{2n+1}, \\ R_{uE_{2n+1}}u=\frac{1}{2}p\sin 2sE_{2n}+\{\sin^2 s+(p+1)\cos^2 s\}E_{2n+1}, \end{cases} \tag{13}$$

where

$$l=c+3-\alpha^2(c-1), \qquad p=(c-1)(1-\alpha^2). \tag{14}$$

Now, let X be a Jacobi vector field orthogonal to the geodesic γ and put

$$X=\sum_{a=2}^{2n-1}f_a E_a+f_{2n}E_{2n}+f_{2n+1}E_{2n+1}. \tag{15}$$

Then, using (13), (14) and (15), (10) is equivalent to the following system of differential equations

$$f_a''+\frac{1}{4}lf_a=0, \qquad a=2, \cdots, 2n-1; \tag{16}$$

$$\begin{cases} f_{2n}''+f_{2n}+p\sin s(f_{2n}\sin s+f_{2n+1}\cos s)=0, \\ f_{2n+1}''+f_{2n+1}+p\cos s(f_{2n}\sin s+f_{2n+1}\cos s)=0. \end{cases} \tag{17}$$

Finally, put

$$\text{(18)}\qquad \begin{cases} z_{2n}=f_{2n}\sin s+f_{2n+1}\cos s\,, \\ z_{2n+1}=f_{2n}\cos s-f_{2n+1}\sin s\,. \end{cases}$$

Then (17) becomes

$$\text{(19)}\qquad \begin{cases} z''_{2n+1}+2z'_{2n}=0\,, \\ z''_{2n}-2z'_{2n+1}+pz_{2n}=0\,. \end{cases}$$

Now put

$$w=z'_{2n}\,.$$

Differentiation of the second equation in (19) gives

$$\text{(20)}\qquad \begin{cases} w''+lw=0\,, \\ z''_{2n+1}=-2w\,. \end{cases}$$

In this way we arrive at a system of differential equations which is easy to solve. We write down the solutions in the three cases we will need.

A. CASE $l=0$

$$f_a=A_a s+B_a\,,\qquad a=2,\cdots,2n-1\,,$$
$$f_{2n}=\beta\sin s+\gamma\cos s\,,$$
$$f_{2n+1}=\beta\cos s-\gamma\sin s\,,$$

where

$$\beta=\frac{1}{2}As^2+Bs+C\,,\qquad \gamma=-\frac{1}{3}As^3-Bs^2+\frac{1}{2}(A+pC)s+D\,;$$

B. CASE $l>0$

Put $k=\sqrt{l}$. Then we have

$$f_a=A_a\cos\frac{k}{2}s+B_a\sin\frac{k}{2}s\,,\qquad a=2,\cdots,2n-1\,,$$
$$f_{2n}=\bar\beta\sin s+\bar\gamma\cos s\,,$$
$$f_{2n+1}=\bar\beta\cos s-\bar\gamma\sin s\,,$$

where

$$\bar\beta=\frac{1}{k}(A\sin ks-B\cos ks)+C\,,$$

$$\bar\gamma=\frac{2}{k^2}(A\cos ks+B\sin ks)+\frac{1}{2}pCs+D\,;$$

C. CASE $l<0$

Put $k=\sqrt{-l}$. Then we have

$$f_a=A_a\cosh\frac{k}{2}s+B_a\sinh\frac{k}{2}s,\qquad a=2,\cdots 2n-1,$$

$$f_{2n}=\bar{\bar{\beta}}\sin s+\bar{\bar{\gamma}}\cos s,$$

$$f_{2n+1}=\bar{\bar{\beta}}\cos s-\bar{\bar{\gamma}}\sin s,$$

where

$$\bar{\bar{\beta}}=\frac{1}{k}A\sinh ks+B\cosh ks)+C,$$

$$\bar{\bar{\gamma}}=-\frac{2}{k^2}(A\cosh ks+B\sinh ks)+\frac{1}{2}pCs+D.$$

To prove our results in the following sections we shall need two particular systems of Jacobi vector fields along a geodesic γ.

I. The first system.

Let X_a be the Jacobi vector fields along γ with initial conditions

$$X_a(0)=0,\quad X_a'(0)=e_a,\quad a=2,\cdots,2n+1.$$

From the previous formulas we get

i) *γ is an integral curve of the field ξ.*

We have

$$X_a(s)=\sin sE_a(s),\qquad a=2,\cdots,2n+1.$$

ii) *γ is a geodesic not tangent to ξ.*

Here we have

	$l>0$	$l=0$	$l<0$
$X_a(s)$	$\frac{2}{k}\sin\frac{k}{2}sE_a(s)$	$sE_a(s)$	$\frac{2}{k}\sinh\frac{k}{2}sE_a(s)$

for $a=2,\cdots,2n-1$, and

$$X_{2n}(s)=(\mu\sin s+\nu\cos s)E_{2n}(s)+(\mu\cos s-\nu\sin s)E_{2n+1}(s),$$

$$X_{2n+1}(s)=(\lambda\sin s-\mu\cos s)E_{2n}(s)+(\lambda\cos s+\mu\sin s)E_{2n+1}(s),$$

where

	$l>0$	$l=0$	$l<0$
μ	$-\frac{2}{k^2}(\cos ks-1)$	s^2	$\frac{2}{k^2}(\cosh ks-1)$
ν	$\frac{4}{k^3}\sin ks+\frac{1}{k^2}ps$	$-\frac{2}{3}s^3+s$	$-\frac{4}{k^3}\sinh ks-\frac{1}{k^2}ps$
λ	$\frac{1}{k}\sin ks$	s	$\frac{1}{k}\sinh ks$

II. The second system.

Let Y_a be the Jacobi vector fields with initial conditions

$$Y_a(0)=e_a, \quad Y'_a(0)=0, \quad a=2, \cdots, 2n+1.$$

As before, we obtain:

i) *γ is an integral curve of the field ξ.*

We have

$$Y_a(s)=\cos s E_a(s), \qquad a=2, \cdots, 2n+1.$$

ii) *γ is a geodesic not tangent to ξ.*

Here we get

	$l>0$	$l=0$	$l<0$
$Y_a(s)$	$\cos \frac{k}{2} s E_a(s)$	$E_a(s)$	$\cosh \frac{k}{2} s E_a(s)$

for $a=2, \cdots, 2n-1$, and

$$Y_{2n}(s)=(\mu \sin s+\lambda \cos s)E_{2n}(s)+(\mu \cos s-\lambda \sin s)E_{2n+1}(s),$$

$$Y_{2n+1}(s)=(\nu \sin s+\rho \cos s)E_{2n}(s)+(\nu \cos s-\rho \sin s)E_{2n+1}(s),$$

where

	$l>0$	$l=0$	$l<0$
μ	$\frac{1}{k}\sin ks$	s	$\frac{1}{k}\sinh ks$
λ	$\frac{1}{k^2}\{2\cos ks+k^2-2\}$	$1-s^2$	$\frac{1}{k^2}\{k^2+2-2\cosh ks\}$
ν	$\frac{1}{k^2}\{(k^2-2)\cos ks+2\}$	$1+s^2$	$\frac{1}{k^2}\{(k^2+2)\cosh ks-2\}$
ρ	$-\frac{2}{k^3}(k^2-2)\sin ks+\frac{1}{k^2}ps$	$-\frac{2}{3}s^3-s$	$-\frac{2}{k^3}(k^2+2)\sinh ks-\frac{1}{k^2}ps.$

4. Symmetries with Respect to a Curve.

Let $\sigma:[a, b]\to M$ be a smooth embedded curve in a Riemannian manifold and let $m=\sigma(t)$, $t\in[a, b]$. Further, let $\{F_i, i=1, \cdots, n=\dim M\}$ be an orthonormal frame along σ such that $\dot{\sigma}=F_1$ and $F_2, \cdots, F_n$ are parallel with respect to the normal connection along σ. We denote by $(x^1, \cdots, x^n)$ the system of Fermi coordinates with respect to m and the frame $(F_1, \cdots, F_n)$. Note that $F_1=\left.\frac{\partial}{\partial x^1}\right|_\sigma$. We refer to [12], [13], [15], [27] for more details.

Next, we consider the geodesic γ, parametrized by arc length s, such that $\gamma(0)=m$ and $\gamma'(0)=v\in T_mM$, $\|v\|=1$ and $g(v, \dot{\sigma})=0$. There always exists a sufficiently small tubular neighborhood $\mathcal{U}$ of σ such that if $p\in\gamma$ and $p\in\mathcal{U}$, γ is the unique geodesic through p cutting σ orthogonally at m and such that

$$\varphi_\sigma : p=\exp_m(sv) \longmapsto \exp_m(-sv)$$

is a local diffeomorphism. Then φ_σ is called the *local symmetry with respect to the curve* σ.

Now let ω be the volume form with respect to a chosen orientation of the tubular neighborhood $\mathcal{U}$ and put

$$\theta_\sigma=\omega\Big(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\Big).$$

This θ_σ is called the *volume density function* (see [27]). We have

THEOREM 13. *The local symmetry with respect to σ is volume-preserving (up to sign) if and only if*

$$\theta_\sigma(\exp_m(sv))=\theta_\sigma(\exp_m(-sv)) \tag{21}$$

for all $v\in T_m^\perp\sigma$, $m=\sigma(t)$ and $t\in[a, b]$.

In [27] T. J. Willmore and the second author derived a very useful formula for θ_σ using Jacobi vector fields and the Wronskian of two solutions of the Jacobi differential equations in matrix form. See also [13], [15]. We have

THEOREM 14. *Let (M, g) be a Riemannian manifold and σ a curve in M. Then we have, for $p=\exp_m(sv)$,*

$$\theta_\sigma(\exp_m(sv))=-\{g(S_p\dot{\sigma}(t), \dot{\sigma}(t))+\kappa_v(t)\}s\theta_m(p), \tag{22}$$

where S_p denotes the shape operator of the geodesic sphere with center p and radius $s=d(m, p)$. θ_m is the volume density function of the exponential map at m and

$$\kappa_v=g(\ddot{\sigma}(t), F_n(t))$$

where $v=F_n(t)$.

Using power series expansions with respect to the arc length s we get for (22) (see [7], [11], [27])

$$\theta_\sigma(p)=1-\kappa_v(t)s+0(s^2) \tag{23}$$

and hence, from (23), we obtain

THEOREM 15. *Let σ be a curve in (M, g) and suppose that the local symmetry with respect to σ is volume-preserving. Then σ is a geodesic.*

For this reason we will, in what follows, always restrict to local symmetries with respect to *geodesics* σ.

We shall use extensively expressions for S_p and $\theta_{\sigma(t)}$ and these may be determined by using the Jacobi vector fields X_i and Y_i along γ considered in section 3. First, put

$$X_i(s)=(AE_i)(s), \qquad i=1, \cdots, n-1,$$

where now E_n is tangent to γ. Then A satisfies

$$A''+R\circ A=0$$

with initial conditions $A(0)=0$, $A'(0)=I$. Further, put

$$Y_i(s)=(DE_i)(s), \qquad i=1, \cdots, n-1$$

D satisfies

$$D''+R\circ D=0$$

with initial conditions $D(0)=I$, $D'(0)=0$. In both cases, R is the endomorphism

$$R: x \longmapsto R_{\gamma' x}\gamma'$$

along γ. Finally, we note that

$$S_p(m)=(A^{-1}D)(s)$$

for $p=\exp_m(sv)$ [16] (See also [15], [27].) and

$$\theta_m(p)=s^{1-n}\det A.$$

Using all these relations we obtain

THEOREM 16 [27]. *Let σ be a geodesic in (M, g) and $p=\exp_m(sv)$. Then*

$$\theta_\sigma(p)=1-\frac{1}{6}s^2(\rho_{vv}+2R_{F_1uF_1u})(m)-\frac{1}{12}s^3(\nabla_v\rho_{vv}+\nabla_v R_{F_1vF_1v})(m)+0(s^4) \tag{24}$$

where ρ denotes the Ricci tensor of (M, g).

In order to be able to treat *isometric* local symmetries with respect to a geodesic σ, we need some expressions for the metric tensor g with respect to the Fermi coordinates we considered before. We follow the treatment of [9], [10], [12], [27]. Now we consider the Jacobi vector fields Z_i along γ with initial conditions

$$Z_1(0)=F_1(t), \quad Z_1'(0)=0,$$
$$Z_a(0)=0, \quad Z_a'(0)=F_a(t), \quad a=2, \cdots, n-1$$

where again $m=\sigma(t)$, $v=F_n$, $\dot{\sigma}(t)=F_1(t)$. Then we have

$$Z_1=\frac{\partial}{\partial x^1}, \quad Z_a=r\frac{\partial}{\partial x^a}, \quad a=2, \cdots, n-1$$

and from this we obtain

$$\begin{aligned} &g_{11}(p)=g(Z_1, Z_1), \\ &g_{1i}(p)=\frac{1}{s}g(Z_1, Z_i), \qquad i=2, \cdots, n, \\ &g_{ij}(p)=\frac{1}{s^2}g(Z_i, Z_j), \qquad i, j=2, \cdots n-1, \\ &g_{in}(p)=\delta_{in}, \qquad i=1, \cdots, n. \end{aligned}$$

Following again the method developed in [9], [27] we obtain

THEOREM 17. *Let σ be a geodesic in (M, g) and put $p=\exp_m(sv)$. Then, with respect to the Fermi coordinate system of above, we have*

$$(25) \quad \begin{cases} g_{11}(p)=1-s^2R_{F_1vF_1v}(m)-\frac{1}{3}s^3\nabla_vR_{F_1vF_1v}(m)+0(s^4), \\ g_{1i}(p)=-\frac{2}{3}s^2R_{F_1vF_iv}(m)-\frac{1}{4}s^3\nabla_vR_{F_1vF_iv}(m)+0(s^4), \\ g_{ij}(p)=\delta_{ij}-\frac{1}{3}s^2R_{F_ivF_jv}(m)-\frac{1}{6}s^3\nabla_vR_{F_ivF_jv}(m)+0(s^4), \end{cases}$$

where $i, j=2, \cdots, n-1$.

Based on Theorem 16 and Theorem 17, T. J. Willmore and the second author proved the following theorems [27] which we will need in the next sections.

THEOREM 18. *A Riemannian manifold (M, g) is locally symmetric if and only if the local symmetries with respect to all geodesics are volume-preserving.*

THEOREM 19. *A Riemannian manifold (M, g) is a space of constant curvature if and only if the local symmetries with respect to all geodesics are isometries.*

5. Symmetries with Respect to φ-Geodesics and Locally φ-Symmetric Spaces.

In this and in the next sections we will consider local symmetries with respect to φ-geodesics on a Sasakian manifold M^{2n+1} with structure tensors (φ, ξ, η, g).

We start with a new characterization of locally φ-symmetric spaces.

THEOREM 20. *A Sasakian manifold M is a locally φ-symmetric space if and only if the volume density function θ_σ has antipodal symmetry along φ-geodesics orthogonal to a φ-geodesic σ for any σ.*

PROOF. Let M be a locally φ-symmetric space and σ a φ-geodesic. From Theorem 13 it follows that we have to prove

$$\theta_\sigma(\exp_m(sv))=\theta_\sigma(\exp_m(-sv)) \tag{26}$$

for any unit vector v orthogonal to $u=\dot{\sigma}(t)$ and ξ. To do this we use (22) where $\kappa_v=0$.

First we note that

$$\theta_m(\exp_m(sv))=\theta_m(\exp_m(-sv))$$

since all local geodesic symmetries are volume-preserving on M. Secondly, we apply Theorem 10 to get a property for the shape operator S_p in (22). From (8) we get

$$s_{m*}(m)\circ S_p(m)\dot{\sigma}(t)=S_{-p}(m)\circ s_{m*}(m)\dot{\sigma}(t) \tag{27}$$

where $p=\exp_m(sv)$ and $-p=\exp_m(-sv)$. Now using (6) and $\eta(\dot{\sigma}(t))=0$, we obtain from (27)

$$g(S_p(m)\dot{\sigma}(t),\ \dot{\sigma}(t))=g(S_{-p}(m)\dot{\sigma}(t),\ \dot{\sigma}(t))$$

which proves now immediately the required result.

Conversely, suppose (26) holds. Then, from this and (24) we have

$$\nabla_v\rho_{vv}+\nabla_v R_{vuvu}=0 \tag{28}$$

for all vectors v orthogonal to u and ξ and all u orthogonal to ξ. Next, let $\{e_i, \xi, i=1, \cdots, 2n\}$ be an orthonormal basis at $m\in M$ with $v=e_{2n}$. Put $u=e_i$, $i=1, \cdots, 2n-1$ in (28) and sum with respect to i. This gives

$$2n\nabla_v\rho_{vv}-\nabla_v R_{v\xi v\xi}=0.$$

But by (2) and (3) $\nabla_v R_{v\xi v\xi}=0$ and so we obtain

$$\nabla_v\rho_{vv}=0.$$

Now (28) becomes

$$\nabla_v R_{vuvu}=0 \tag{29}$$

for all v orthogonal to u and ξ, and ξ orthogonal to u. Finally, put $u=\varphi v$ in (29). Then we obtain

$$\nabla_v R_{v\varphi v v\varphi v}=0$$

for all v orthogonal to ξ. The result follows now from this and Theorem 9.

6. Symmetries with Respect to φ-Geodesics and Sasakian Space Forms.

In section 5 we considered local symmetries with respect to φ-geodesics such that the volume density function θ_σ was only partly preserved. Now we shall consider the case where this function is completely preserved by the local symmetries with respect to φ-geodesics.

THEOREM 21. *A connected Sasakian manifold is a Sasakian space form if and only if the local symmetries with respect to all φ-geodesics are volume-preserving.*

PROOF. First let M be a Sasakian space form. With the same notation as before we have

$$\theta_m(\exp_m(sv))=\theta_m(\exp_m(-sv))$$

for any v since M is locally φ-symmetric. Next we use

$$S_p(m)=(A^{-1}D)(s)$$

from section 4 and the formulas from section 3 to compute A and D. An easy calculation shows

$$g((A^{-1}D)(s)\dot{\sigma},\ \dot{\sigma})(t)=-g((A^{-1}D)(-s)\dot{\sigma},\ \dot{\sigma})(t)$$

and so the result follows from (22) with $\kappa_v=0$.

Conversely, suppose that the local symmetries with respect to any φ-geodesic are volume-preserving. First, Theorem 20 implies that M is locally φ-symmetric.

Let $\dim M=3$. Since the scalar curvature is constant (by a consequence of (9)) we get at once that M is a Sasakian space form.

Further, suppose $\dim M\geqq 5$. Then, since (9) holds, we obtain from (24)

$$\nabla_v R_{vuvu}=0$$

for all v orthogonal to u and all u orthogonal to ξ. Next, we use (4) and (7) to get

$$0=\nabla_v R_{vuvu}=-\eta(v)(R_{vuv\varphi u}+R_{vuu\varphi v})=2\eta(v)R_{vu\varphi vu}.$$

So

$$\eta(v)R_{vu\varphi vu}=0. \tag{30}$$

Put $v=\alpha X+\beta Y+\gamma Z$, with X, Y, Z orthogonal to u, in (30) and expand. Since the coefficient of $\alpha\beta\gamma$ must vanish, we get

$$\begin{aligned}\eta(X)(R_{Yu\varphi Zu}+R_{Zu\varphi Yu})+\eta(Y)(R_{Xu\varphi Zu}+R_{Zu\varphi Xu})\\+\eta(Z)(R_{Xu\varphi Yu}+R_{Yu\varphi Xu})=0.\end{aligned} \tag{31}$$

Take $X=\xi$, $Z=\varphi u$ and Y orthogonal to ξ. Then (31) gives

$$R_{u\varphi uu\varphi Y}=0$$

for all Y orthogonal to u and ξ. Using this and (3) we easily see that

$$R_{u\varphi u}u \sim \varphi u$$

and hence, Theorem 1 implies the result.

7. Isometric Local Symmetries with Respect to φ-Geodesics.

In this final section we derive two characterizations of connected *locally symmetric* Sasakian manifolds M when $\dim M\geqq 5$. Note that such spaces have constant curvature 1 (Theorem 2). Also we study isometric symmetries on three-dimensional Sasakian space forms.

First we start with a definition introduced in [12] and refer to [24] for more information about harmonic manifolds.

DEFINITION. A Riemannian manifold M is said to be *harmonic with respect to a geodesic* σ provided that the volume density function θ_σ depends only on the distance from the geodesic σ.

In [12] A. Gray and the second author proved

THEOREM 22. *A Riemannian manifold M has constant curvature if and only if M is harmonic with respect to all geodesics.*

Here we prove

THEOREM 23. *A Sasakian manifold M has constant curvature* 1 *if and only if M is harmonic with respect to each φ-geodesic.*

PROOF. First suppose that M has constant curvature $(=1)$. Then the result follows from Theorem 22.

Conversely, let σ be a φ-geodesic and suppose M is harmonic with respect to σ. From (24) we then get

$$\rho_{vv}+2R_{vuvu}=\alpha(u) \tag{32}$$

for any unit vector v orthogonal to u and since this must be true for all σ, (32) must also hold for all u orthogonal to ξ. Put $v=\xi$ in (32). Then we get

$$\alpha(u)=2n+2 \tag{33}$$

since $\rho_{\xi\xi}=2n$. Next, let $\{e_i, i=1, \cdots, 2n+1\}$ be an orthonormal basis such that

$v=e_1$, $\xi=e_{2n+1}$, and put $u=e_a$, $a=2, \cdots, 2n$ in (32). With (33) we obtain after summation with respect to a:

$$\rho_{vv}=2n. \tag{34}$$

Hence (32), (33) and (34) give

$$R_{vuvu}=1$$

for all u orthogonal to v and both unit vectors orthogonal to ξ. This gives at once the required result.

Next we have

THEOREM 24. *Let M^{2n+1} be a Sasakian manifold of dimension$\geqq 5$. Then M has constant curvature 1 if and only if the local symmetries with respect to all φ-geodesics are isometries.*

PROOF. First, let M be a space of constant curvature. Then the result follows from Theorem 19.

Conversely, suppose that all the local symmetries with respect to φ-geodesics are isometries. Theorem 21 implies that M is a Sasakian space form. Further, from (25) and the hypothesis we derive

$$R_{vuvX}=0 \tag{35}$$

for all v, X orthogonal to u and all u orthogonal to ξ. This, together with (5) implies

$$0=R_{vuvX}=-\frac{3}{4}(c-1)g(v, \varphi u)g(\varphi v, X). \tag{36}$$

Since $\dim M\geqq 5$ we can choose u orthogonal to ξ and X orthogonal to u, φu, ξ. Then put $v=\varphi u+\varphi X$ in (36). This gives at once $c=1$ and the result is proved.

For three-dimensional manifolds we have

THEOREM 25. *Let M be a three-dimensional Sasakian space from. Then any local symmetry with respect to any φ-geodesic is an isometry.*

PROOF. Again, let σ denote the φ-geodesic, $m=\sigma(t)$, $u=\dot\sigma(t)$, $p=\exp_m(sv)$, $\|v\|=1$ and $g(v, u)=0$.

We proceed as in section 3 and use the method described after Theorem 16 in section 4. Here also we consider two cases:

i) $v=\xi$

Here we get at once

$$Z_1(s)=\cos sE_1(s), \qquad Z_2(s)=\sin sE(s)$$

and hence

$$g_{11}(p)=\cos^2 s, \quad g_{12}(p)=0, \quad g_{22}(p)=\frac{1}{s^2}\sin^2 s$$

which proves the result.

ii) $v\neq\xi$

At m we choose the following basis:

$$f_1=(1-\alpha^2)^{-1/2}(\xi-\alpha v), \quad f_2=(1-\alpha^2)^{-1/2}\varphi v, \quad f_3=v$$

where $\alpha=\eta(v)$. Let $\{E_1, E_2, E_3\}$ be the parallel basis obtained by parallel translating the vectors f_i, $i=1, 2, 3$ along the geodesic γ. Note that

$$e_1=u=\mp f_2, \quad e_2=\pm f_1, \quad e_3=v=f_3$$

at m. So the initial conditions are

$$\begin{aligned} Z_1(0)&=\mp f_2, & Z_1'(0)&=0, \\ Z_2(0)&=0, & Z_2'(0)&=\pm f_1 \end{aligned}$$

and hence the solutions of the Jacobi equation are

$$Z_1(s)=\pm\{(\mu\sin s+\lambda\cos s)E_1(s)+(\mu\cos s-\lambda\sin s)E_2(s)\},$$

$$Z_2(s)=\pm\{(\nu\sin s+\rho\cos s)E_1(s)+(\nu\cos s-\rho\sin s)E_2(s)\}$$

where

	$l>0$	$l=0$	$l<0$
μ	$-\frac{1}{k^2}\{(k^2-2)\cos ks+2\}$	$-s^2-1$	$\frac{1}{k^2}\{-(k^2+2)\cosh ks+2\}$
λ	$\frac{2}{k^3}(k^2-2)\sin ks-\frac{1}{k^2}ps$	$\frac{2}{3}s^3+s$	$\frac{2}{k^3}(k^2+2)\sinh ks+\frac{1}{k^2}ps$
ν	$-\frac{2}{k^2}(\cos ks-1)$	s^2	$\frac{2}{k^2}(\cosh ks-1)$
ρ	$\frac{4}{k^3}\sin ks+\frac{1}{k^2}ps$	$-\frac{2}{3}s^3+s$	$-\frac{4}{k^3}\sinh ks-\frac{1}{k^2}ps$

From this we obtain the expressions

$$\begin{cases} g_{11}(p)=\mu^2+\lambda^2, \\ g_{12}(p)=\dfrac{1}{s}(\mu\nu+\lambda\rho), \\ g_{22}(p)=\dfrac{1}{s^2}(\nu^2+\rho^2), \end{cases}$$

which again give the result.

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