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COVERINGS OVER d-GONAL CURVES

By

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§ 1. Introduction.

Let M be a compact Riemann surface and f be a meromorphic function on M. Let (f) be the principal divisor associated to f and $(f)_\infty$ be the polar divisor of f. We call f a meromorphic function of degree d if $d=\text{degree } (f)_\infty$. If d is the minimal integer in which a meromorphic function of degree d exists on M, then we call M a d-gonal curve.

Now we assume that M is d-gonal, and consider a covering map $\pi' : M' \to M$ that M' still remains d-gonal. The purpose of this paper is to show how such π' can be characterized.

The case that π' is a normal covering and $d=2$ (i.e., M is hyperelliptic) has been already studied ([2], [3], [4] and [7]). In this case the existence of the hyperelliptic involution v' on M' plays an important role. More precisely, as v' commutes with each element of the Galois group $G=Gal(M'/M)$, v' induces the hyperelliptic involution v on M and we can reduce π' to a normal covering $\pi : \boldsymbol{P}_1' \to \boldsymbol{P}_1$ with Galois group G, where $\boldsymbol{P}_1'$ and $\boldsymbol{P}_1$ are Riemann spheres isomorphic to quotient Riemann surfaces $M'/\langle v' \rangle$ and $M/\langle v \rangle$ respectively. On the other hand it is known that finite subgroups of the linear transformation group are cyclic, dihedral, tetrahedral, octahedral and icosahedral. Horiuchi [3] decided all the different normal coverings $\pi' : M' \to M$ over a hyperelliptic curve M that M' still remains a hyperelliptic curve by investigating each of above five types.

Let M be a d-gonal curve. In this paper we will show at first that a covering map $\pi' : M' \to M$ (not necessarily normal) with d-gonal M' canonically induces some covering map $\pi : \boldsymbol{P}_1' \to \boldsymbol{P}_1$ (Theorem 2.1 § 2). Moreover if both M and M' have unique linear system g_d^1 and π' is normal, then we can see that π is also normal (Cor. 2.3).

In § 3, § 4 and § 5 we assume that M is a cyclic p-gonal curve for a prime number p. We will determine all ramification types of normal coverings $\pi' : M' \to M$ with p-gonal M' by the same way as Horiuchi did in case $p=2$(§ 4),

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and we give some results about unramified coverings $\pi': M'\to M$, where π' is not necessarily normal (§ 5).

§ 2.

Let $\pi': M'\to M$ be a covering over an arbitrary compact Riemann surface M. Let $C(M)$ and $C(M')$ be the function fields of M and M' respectively and $Nm_{\pi'}=Nm: C(M')\to C(M)$ be the norm map. For a divisor $D=\sum_{i=1} n_iQ_i$ $(n_i\in Z)$ on M', we define a divisor $Nm_{\pi'}D=Nm\,D$ on M by

$$Nm_{\pi'}D=\sum n_i\pi'(Q_i).$$

Then the following equation of principal divisors holds ([1] Appendix B):

$$Nm_{\pi'}((f))=(Nm\,f).$$

If two divisors D' and E' are linearly equivalent, write $D'\sim E'$, the above equation means that $Nm\,D'\sim Nm\,E'$.

Let π'^*P denote a divisor on M' obtained by the inverse image of a point $P\in M$ with ramification points counted according to multiplicity. For a divisor $D=\sum n_iP_i$, $\pi'^*D:=\sum n_i\pi'^*P_i$. $|D|$ is the complete linear system of D and $\mathcal{L}(D)$ is the C-vector space consisting of 0 and meromorphic functions f satisfying $(f)+D>0$. $l(D)$ is the dimension of $\mathcal{L}(D)$ over C.

After this we assume that M is d-gonal. Then there exists a positive divisor D of degree d on M satisfying $l(D)\geqq 2$, and $l(E)=1$ for any positive divisor E of degree less than d. Actually on this D we can easily see that $l(D)=2$, and then the linear system $|D|$ defines a covering map of degree d;

$$\phi_{|D|}=\phi: M\longrightarrow P_1$$

where P_1 is a Riemann sphere. Explicitely $\phi(P)$ is defined by $\phi(P)=h(P)\in C\cup(\infty)$ for $P\in M$, where h is a non-trivial meromorphic function in $\mathcal{L}(D)$. ϕ is defined uniquely up to linear transformations of P_1. By the minimality of d, a divisor $\phi^*\phi(P)$ is uniquely determined not corresponding to the choice of h. For distinct points P and P' on M, $\phi^*\phi(P)$ and $\phi^*\phi(P')$ are linearly equivalent and having no common point in their supports.

Let $\pi': M'\to M$ be a covering of degree n over M that M' still remains d-gonal. Let D' be a positive divisor on M' of degree d satisfying $l(D')=2$. Then we have;

Theorem 2.1. *Put* $D=Nm_{\pi'}D'$. *Then*

i) *There exists a covering map* $\pi: \boldsymbol{P}_1' \to \boldsymbol{P}_1$ *satisfying the following diagram;*

$$\begin{array}{ccc} M' & \xrightarrow{\phi_{|D'|}=\phi'} & \boldsymbol{P}_1' \\ \pi' \downarrow & & \downarrow \pi \\ M & \xrightarrow[\phi_{|D|}=\phi]{} & \boldsymbol{P}_1 \end{array}$$

with $\deg \pi' = \deg \pi = n$ *and* $\deg \phi' = \deg \phi = d$.

ii) *Let* $\boldsymbol{C}(M')$, $\boldsymbol{C}(M)$, $\boldsymbol{C}(\boldsymbol{P}_1')$ *and* $\boldsymbol{C}(\boldsymbol{P}_1)$ *be the function fields. Then* $\boldsymbol{C}(M) \cap \boldsymbol{C}(\boldsymbol{P}_1') = \boldsymbol{C}(\boldsymbol{P}_1)$ *in* $\boldsymbol{C}(M')$ *and* $\boldsymbol{C}(M) \otimes_{\boldsymbol{C}(\boldsymbol{P}_1)} \boldsymbol{C}(\boldsymbol{P}_1') = \boldsymbol{C}(M')$.

To prove this Theorem we prepare some lemmas. Put $D' = \sum_{i=1}^{d} P_i$ (P_i are not necessarily distinct), and $\pi'^*\pi' P_i = \sum_{k=1}^{d} P_i^{(k)}$.

LEMMA 2.1.1. *For each* i,

$$Nm_{\pi'}\phi'^*\phi'(P_i^{(k)}) = Nm_{\pi'}\phi'^*\phi'(P_i) = Nm_{\pi'}D', \qquad k=1, 2, \cdots, n.$$

PROOF. $Nm_{\pi'}\phi'^*\phi'(P_i^{(k)})$ and $Nm_{\pi'}\phi'^*\phi'(P_i)$ are divisors of degree d on M, and they have a common point $\pi'(P_i^{(k)}) = \pi'(P_i)$. But they are linearly equivalent as $\phi'^*\phi'(P_i^{(k)}) \sim \phi'^*\phi'(P_i)$. Then we have $Nm_{\pi'}\phi'^*\phi'(P_i^{(k)}) = Nm_{\pi'}\phi'^*\phi'(P_i)$ by the minimality of d. □

As $l(D') > 1$, we may assume that $D' = \sum_{i=1}^{d} P_i$ $(= \phi'^*\phi'(P_1))$ satisfies the following conditions $*$);

$*$) $\quad$ P_i are distinct, π' is unramified over $\pi'(P_i)$, $1 \leqq i \leqq d$, and ϕ' is unramified over $\phi'(P_1^{(k)})$, $1 \leqq k \leqq n$.

Let $Nm_{\pi'}D' = d_1R_1 + d_2R_2 + \cdots + d_tR_t$, $d_1 + d_2 + \cdots + d_t = d$, where R_i are distinct points in M and $\pi'(P_1) = R_1$. Changing the indeces of P_i, we may assume that

$$\pi'(P_1) = \cdots = \pi'(P_{d_1}) = R_1, \qquad \pi'(P_{d_1+1}) = \cdots = \pi'(P_{d_1+d_2}) = R_2, \cdots,$$

$$\pi'(P_{d_1+d_2+\cdots+d_{t-1}+1}) = \cdots = \pi'(P_{d_1+\cdots+d_t}) = R_t.$$

LEMMA 2.1.2. $d_1 | n$, $d_1 | d$ *and* $d_1 = d_2 = \cdots = d_t$.

PROOF. Put $\pi'^*R_i = \pi'^*\pi'(P_{d_1+\cdots+d_{i-1}+s_i}) = A_i^{(1)} + \cdots + A_i^{(n)}$, $s_i = 1, \cdots, d_i$, $i = 1, \cdots, t$. Then $A_i^{(k)}$ $(k=1, \cdots, n)$ are distinct by $*$). By Lemma 2.1.1 $Nm_{\pi'}\phi'^*\phi'(A_i^{(k)}) = d_1R_1 + \cdots + d_tR_t$. For ϕ' is unramified over $\phi'(A_1^{(k)})$, $\phi'^*\phi(A_1^{(1)})$ also consists of distinct d points. Changing the induces k of $A_j^{(k)}$ for each j, we may write;

$$\psi'^*\psi'(A_1^{(1)})=(A_1^{(1)}+\cdots+A_1^{(d_1)})+(A_2^{(1)}+\cdots+A_2^{(d_2)})+\cdots+(A_t^{(1)}+\cdots+A_t^{(d_t)}).$$

Especially $d_1\leqq n$. By the minimality of d, $\psi'^*\psi'(A_1^{(1)})=\cdots=\psi'^*\psi'(A_1^{(d_1)})$. If $d_1<n$, take a point over R_1, namely $A_1^{(d_1+1)}$, not equal to $A_1^{(k)}$, $1\leqq k\leqq d_1$. Then we may write;

$$\psi'^*\psi'(A_1^{(d_1+1)})=(A_1^{(d_1+1)}+\cdots+A_1^{(2d_1)})+\cdots+(A_t^{(d_t+1)}+\cdots+A_t^{(2d_t)})$$

and $\psi'^*\psi'(A_1^{(d_1+1)})=\cdots=\psi'^*\psi'(A_1^{(2d_1)})$. If still $2d_1<n$, repeat the same manner as above and finally we have the following sd_1+1 equations of divisors;

$$\begin{cases} \psi'^*\psi'(A_1^{(1)}) =(A_1^{(1)}+\cdots+A_1^{(d_1)})+\cdots+(A_t^{(1)}+\cdots+A_t^{(d_t)}) & (1.1)\\ \qquad\cdots\cdots\cdots\cdots & \cdots\\ \psi'^*\psi'(A_1^{(d_1)}) =(A_1^{(1)}+\cdots+A_1^{(d_1)})+\cdots+(A_t^{(1)}+\cdots+A_t^{(d_t)}) & (1.d_1)\end{cases}$$

$$\begin{cases} \psi'^*\psi'(A_1^{(d_1+1)})=(A_1^{(d_1+1)}+\cdots+A_1^{(2d_1)})+\cdots+(A_t^{(d_t+1)}+\cdots+A_t^{(2d_t)}) & (2.1)\\ \qquad\cdots\cdots\cdots\cdots & \cdots\\ \psi'^*\psi'(A_1^{(2d_1)}) =(A_1^{(d_1+1)}+\cdots+A_1^{(2d_1)})+\cdots+(A_t^{(d_t+1)})+\cdots A_t^{(2d_t)}) & (2.d_1)\end{cases}$$

$$\cdots\cdots\cdots\cdots$$

$$\begin{cases} \psi'^*\psi'(A_1^{((s-1)d_1+1)})=(A_1^{((s-1)d_1+1)}+\cdots+A_1^{(sd_1)})+\cdots+(A_t^{((s-1)d_t+1)}+\cdots+A_t^{(sd_t)}) & (s.1)\\ \qquad\cdots\cdots\cdots\cdots & \cdots\\ \psi'^*\psi'(A_1^{(sd_1)}) =(A_1^{((s-1)d_1+1)}+\cdots+A_1^{(sd_1)})+\cdots+(A_t^{((s-1)d_t+1)}+\cdots+A_t^{(sd_t)}) & (s.d_1)\end{cases}$$

and

$$\pi'^*R_1=(A_1^{(1)}+\cdots+A_1^{(d_1)})+\cdots+(A_1^{((s-1)d_1+1)}+\cdots+A_1^{(sd_1)})\,. \qquad (**)$$

Then $n=d_1\cdot s$. If $d_1>d_t$, then $n=d_1\cdot s>d_t\cdot s$. There exists a point over R_t, namely $A_t^{(n)}$, never appears in the right hand sides of the above equations $(1.1)\sim(s.d_1)$. On the other hand $\psi'^*\psi'(A_t^{(n)})$ has $A_1^{(k)}$ for some k in its support by Lemma 2.1.1. For the minimality of d, $\psi'^*\psi'(A_t^{(n)})=\psi'^*\psi'(A_t^{(k)})$. This is a contradiction. If $d_1<d_t$, then $n=d_1\cdot s<d_ts$. This also can not be happened. □

By Lemma 2.1.2, and the above equations $(1.1)\sim(s.d_1)$, $**$), we have;

Lemma 2.1.3.

$$\sum_{k=1}^{n}\psi'^*\psi'(P_1^{(k)})=\sum_{i=1}^{d}\pi'^*\pi'(P_i)=\pi'^*Nm(D')\,.$$

Proof of Theorem 2.1.

Let $E'=\sum Q_i$ and $E''=\sum S_i$ be in $|D'|$ satisfying the conditions $*$). Let h' be a non-constant function in $\mathcal{L}(D')$ and $h=Nm\,h'$.

$$
\begin{aligned}
div\,(h\circ\pi') &= \pi'^*Nm\,E' - \pi'^*Nm\,E'' \\
&= \sum_{k=1}^{n} \phi'^*\phi'(Q_1^{(k)}) - \sum_{k=1}^{n} \phi'^*\phi'(S_1^{(k)}) \qquad \text{by Lemma 2.1.3} \\
&= \sum_{k=1}^{n} [\{\phi'^*\phi'(Q_1^{(k)}) - \phi'^*\phi'(P_1)\} - \{\phi'^*\phi'(S_1) - \phi'^*\phi'(P_1)\}] \\
&= \sum_{k=1}^{n} [\{\phi'^*\phi'(Q_1) - D'\} - \{\phi'^*\phi'(S_1) - D'\}] \\
&= \sum_{k=1}^{n} \{(a_k h' + b_k) - (c_k h' + d_k)\} \\
&\left(\prod_{k=1}^{n} \frac{a_k h' + b_k}{c_k h' + d_k}\right).
\end{aligned}
$$

Then $h\circ\pi'$ is in $C(h')=C(P_1')$ and we have

$$
\begin{array}{ccc}
C(M') & \supset & C(P_1') \\
\cup & & \cup \\
C(M) & \supset & C(P_1),
\end{array}
\quad \text{with } [C(M'):C(M)]=[C(P_1'):C(P_1)]=n \text{ and}
$$

$$[C(M):C(P_1)]=[C(M'):C(P_1')]=d.$$

As $[C(P_1')\underset{C(P_1)}{\otimes}C(M):C(P_1')]=[C(M'):C(P_1')]$, we have ii). □

Conversely we have;

REMARK 2.2. Let $\phi: M\to P_1$ be a d-gonal curve with a d-th coverinng ϕ over a Riemann sphere P_1. Let $\pi': P_1'\to P_1$ be an arbitrary covering. Then function fields $C(M)$ and $C(P_1')$ are linearly disjoint over $C(P_1)$, and the Riemann surface M' obtained from the function field $C(M)\underset{C(P_1)}{\otimes}C(P_1')=C(M)\cdot C(P_1)$ is d-gonal.

PROOF. Consider the canonical surjective map $C(M)\underset{C(P_1)}{\otimes}C(P_1')\to C(M)\cdot C(P_1')$. Put $d'=[C(M)\cdot C(P_1'):C(P_1')]$. If $d'<d$, then M should be d''-gonal for some $d''\leqq d'$. This is a contradiction. □

Concerning about the digram in Theorem 2.1, π is not necessarily normal even if π' is normal. But we have;

COROLLARY 2.3. *If M' has unique linear system g_d^1 and π' is normal, then π is normal and $Gal(M'/M)\cong Gal(P_1'/P_1)$.*

PROOF. Let σ be an automorphism on M'. For the uniquness of g_d^1 there is an automorphism $\tilde{\sigma}$ on $\boldsymbol{P}_1'$ satisfying the following diagram:

$$\begin{array}{ccc} M' & \longrightarrow & \boldsymbol{P}_1' \\ \sigma \big| & & \big| \tilde{\sigma} \\ M' & \longrightarrow & \boldsymbol{P}_1' \end{array}$$

As $\boldsymbol{C}(M) \cap \boldsymbol{C}(\boldsymbol{P}_1') = \boldsymbol{C}(\boldsymbol{P}_1)$, $Gal(M'/M) \cong Gal(\boldsymbol{P}_1'/\boldsymbol{P}_1)$. □

REMARK 2.4. Under the two assumptions of corollary 2.3, (i.e., π' is normal and the uniqueness of g_d^1, we can prove Theorem 2.1. i) easier. In fact $Gal(M'/M)$ acts on $\boldsymbol{P}_1'$ as the proof of the corollary 2.3, and the fixed subfield of $\boldsymbol{C}(\boldsymbol{P}_1')$ by the action of $Gal(M'/M)$ is $\boldsymbol{C}(M) \cap \boldsymbol{C}(\boldsymbol{P}_1')$. This field is a function field of genus 0, and $[\boldsymbol{C}(M): \boldsymbol{C}(M) \cap \boldsymbol{C}(\boldsymbol{P}_1')] = d$ for the minimality of d.

REMARK 2.5. The condition that M' has unique g_d^1 is satisfied in the following case:

> M' is p-gonal of genus$\geqq (p-1)^2+1$ for a prime number p ([9], Cor. 2.4.5), especially M' is defined by the equation $D(u, y)=0$ (§ 3(1)) with $m \geqq 2p+1$ ([9], [8], [5]).

REMARK 2.6. Let p be a prime number. We assume that M has a p-th covering over $\boldsymbol{P}_1$. Then the condition that M is p-gonal is satisfied when genus of $M > (p-1)(p-2)$ ([9], Cor. 2.4.5).

§ 3.

Let p be a prime number and M be a Riemann surface defined by the equation

$$D(u, y) := y^p - (u-\alpha_1)^{k_1} \cdots (u-\alpha_m)^{k_m} = 0 \tag{1}$$

where α_i $(1 \leqq i \leqq m)$ are distinct and k_i are integers satisfying $1 \leqq k_i \leqq p-1$ and $\sum k_i \equiv 0 \ mod\ p$. Let $\phi: M \to \boldsymbol{P}_1$ be the cyclic normal covering of degree p over $\boldsymbol{P}_1$ defined by $(u, y) \mapsto u$. The branch points of ϕ are $\alpha_i \in \boldsymbol{P}_1$, and ϕ is completely ramified over α_i. Put $S = \{\alpha_i \mid 1 \leqq i \leqq m\}$. The genus of M is $\dfrac{(p-1)(m-2)}{2}$. Sometimes we use another equation $D'(u, y)$ for M

$$D'(u, y) := y^p - (u-\beta_1)^{k_1} \cdots (u-\beta_{m-1})^{k_{m-1}} = 0 \tag{2}$$

with $1\leqq k_i\leqq p-1$ and $\sum k_i\not\equiv 0 \ mod\ p$. ψ is defined as above and the set S of the branch points of $\psi=\{\beta_1, \cdots, \beta_{m-1}, \infty\}$. In this case let $k_m>0$ denote a minimal integer satisfying $k_m\equiv-\sum_{i=1}^{m-1} k_i \ mod\ p$, then we can get an equation of type (1) birational equivalent to (2).

We call M a cyclic p-gonal if M is p-gonal and defined by (1) or (2). Hereafter we assume that M is cyclic p-gonal and having unique g_p^1. If $m\geqq 2p-1$, M is p-gonal by Remark 2.6. If $m\geqq 2p+1$, M has unique g_p^1 by Remark 2.5. $\pi': M'\to M$ always means a covering map with p-gonal M'. Then a covering $\pi': M'\to M$ corresponds to a covering $\pi': \boldsymbol{P}_1'\to\boldsymbol{P}_1$ by Theorem 2.1.

In this section we show the method how to get the equation of M' and π' explicitely from the equation of M and π. Put $\boldsymbol{P}_1=Proj\ \boldsymbol{C}[z_0, z_1]$, $\boldsymbol{P}_1'=Proj\ \boldsymbol{C}[u_0, u_1]$, $z=z_1/z_0$ and $u=u_1/u_0$. Assume that π' is defined by $(z_0, z_1)\mapsto(F_0(z_0; z_1): F_1(z_0; z_1))$, where F_i $(i=1, 2)$ are relatively prime homogeneous polynomials of same degree n. $V=Spec\ \boldsymbol{C}[z]_{F_0(1;z)}$ and $U=Spec\ \boldsymbol{C}[u]$ are affine open subsets of $\boldsymbol{P}_1'$ and $\boldsymbol{P}_1$ respectively. Then $\pi': V\to U$ is represented by $z\mapsto u=\dfrac{F_1(1:z)}{F_0(1:z)}\overset{\text{put}}{=}f$. Assume M is defined by the equation $D(u, y)=0$ with $\alpha_i\in U=\boldsymbol{C}$ for all i. Put $A=\boldsymbol{C}[u, y]/(D(u, y))$. By Theorem 2.1,

$$\begin{aligned}\boldsymbol{C}(M')&=\boldsymbol{C}(M)\underset{\boldsymbol{C}(\boldsymbol{P}_1)}{\otimes}\boldsymbol{C}(\boldsymbol{P}_1')\\ &=A\underset{\boldsymbol{C}[u]}{\otimes}\boldsymbol{C}(\boldsymbol{P}_1')\\ &\supset A\underset{\boldsymbol{C}[u]}{\otimes}\boldsymbol{C}[z]_{(F_0(1:z))}\\ &=\boldsymbol{C}[z]_{(F_0(1:z))}[y]\Big/\Big(D\Big(\frac{F_1(1:z)}{F_0(1:z)}, y\Big)\Big)\\ &\overset{\text{put}}{=}B.\end{aligned}$$

Then $Spec\ B=V\underset{U}{\times}Spec\ A$. If we have factorizations;

$$F_1(1:z)-F_0(1:z)\alpha_i=c_i\prod_{t=1}^{l(i)}(z-a_t^{(i)})^{e_t^{(i)}}$$

with some constants c_i, $a_t^{(i)}\in\boldsymbol{C}$ and $e_t^{(i)}\in\boldsymbol{N}$ satisfying $\sum_{t=1}^{l(i)} e_t^{(i)}\leqq n$, then

$Spec\ A\underset{\boldsymbol{P}_1}{\times}Spec\ \boldsymbol{C}[z]$ is defined by the equation

$$F_0(1:z)^{\sum_{i=1}^{m}k_i}y^p-\prod_{i=1}^{m}\Big(c_i\prod_{t=1}^{l(i)}(z-a_t^{(i)})^{e_t^{(i)}}\Big)^{k_i}=0\,,$$

Put

$$G(z)=F_0(1:z)^{(\sum_{i=1}^{m}k_i)/p}\cdot\prod_{i=1}^{m}\cdot\prod_{t=1}^{l(i)}(z-a_t^{(i)})^{-[e_t^{(i)}k_i/p]}(\prod_{i=1}^{m}c_i)^{(-\sum_{i=1}^{m}k_i)/p},$$

where $[a/b]$ is Gauss symbol. Changing $G(z)y$ by y we have an equation of type (1) for M'

$$y^p-\prod_{i=1}^{m}\prod_{t=1}^{l(i)}(z-a_t^{(i)})^{f_t^{(i)}}=0 \tag{3}$$

where $f_t^{(i)}$ are positive integers satisfying $0\leqq f_t^{(i)}<p$ and $f_t^{(i)}\equiv e_t^{(i)}k_i \ mod\ p$. π' is defined by

$$(y, z)\longmapsto(G(z)^{-1}y,\ F_1(1, z)/F_0(1, z)).$$

Let f_∞ be the integer satisfying $\sum_{i,t}f_t^{(i)}+f_\infty\equiv 0\ mod\ p$ and $0\leqq f_\infty<p$. The set S' of branch points of ϕ' consists of $a_t^{(i)}$ with $f_t^{(i)}\neq 0$ and ∞ if $f_\infty\neq 0$.

Next assume that M is defined by the equation $D'(u, y)=0$ in (2) and we have factorizations;

$$F_1(1:z)-\beta_iF_0(1:z)=c_i\prod_{t=1}^{l(i)}(z-b_t^{(i)})^{e_t^{(i)}} \qquad (1\leqq i\leqq m-1)$$

and

$$F_0(1:z)=c_m(z-\gamma_1)^{r_1}\cdot\cdots\cdot(z-\gamma_s)^{r_s}, \qquad r_1+\cdots+r_s\leqq n.$$

Let $f_t^{(i)}$ $(1\leqq i\leqq m-1)$ be numbers satisfying $e_t^{(i)}\cdot k_i\equiv f_t^{(i)}\ mod\ p$ and $0\leqq f_t^{(i)}<p$. Let g_j $(1\leqq j\leqq s)$ be numbers satisfying $r_j\cdot k_m\equiv g_j\ mod\ p$ and $0\leqq g_j<p$, where k_m is defined as before. By the same way as above we have an equation of M';

$$y^p-\Big(\prod_{i=1}^{m-1}\prod_{t=1}^{l(i)}(z-b_t^{(i)})^{f_t^{(i)}}\Big)(z-\gamma_1)^{g_1}\cdots(z-\gamma_s)^{g_s}=0 \tag{4}$$

π is defined by

$$(y, z)\longmapsto(G'(z)^{-1}y,\ F_1(1:z)/F_0(1:z))$$

where

$$G'(z)=F_0(1:z)^{(\sum_{i=1}^{m}k_i)/p}(\prod_{i=1}^{m}c_i)^{-\sum_{i=1}^{m}k_i/p}\cdot\prod_{i=1}^{m-1}\prod_{t=1}^{l(i)}(z-b_t^{(i)})^{-[e_t^{(i)}\cdot k_i/p]}$$

$$\times\prod_{j=1}^{s}(z-\gamma_j)^{-[r_j\cdot k_m/p]}$$

Let f_∞ be the integer satisfying $\sum_{i,t}f_t^{(i)}+\sum_j g_j+f_\infty\equiv 0\ (mod\ p)$ and $0\leqq f_\infty<p$. The set S' of branch points of ϕ' consists of $b_t^{(i)}$ $(f_t^{(i)}\neq 0)$, γ_j $(g_j\neq 0)$ and ∞ if $f_\infty\neq 0$.

LEMMA 3.1. *For a point $P\in M'$, put $\psi'(P)=a$ and $\pi\circ\psi'(P)=\alpha$. e (resp. e') denotes the ramification index of π (resp. π') at a (resp. P).*

(1) *Assume α is a branch point of ψ. Then $e'=e/p$ if $p\mid e$, and $e=e'$ if $p\nmid e'$.*

(2) *Assume α is not a branch point of ψ. Then $e=e'$.*

PROOF. We may assume that M is defined by the equation (1). If α is a branch point of ψ, then $\alpha=\alpha_i$, $a=a_t^{(i)}$ and $e=e_t^{(i)}$ for some i and t. If $p\mid e$, then $f_t^{(i)}=0$ and a is not a branch point of ψ' by (3). On the other hand the ramification index of ψ over α_i is p. As $\psi\circ\pi'=\pi\circ\psi'$, $p\cdot e'=e$. If $p\nmid e$, then $f_t^{(i)}\neq 0$ and $a=a_t^{(i)}$ is a branch point of ψ. $\therefore e=e'$. □

§4.

Let $\pi' : M'\to M$ be as in §3. Moreover we assume that π' is normal with Galois group G. By Corollary 2.3 in §2 π induced by π' is also normal with Galois group G. Then we use the following lemma to determine π';

LEMMA 4.1. ([6], [3]) *By choosing suitable coordinates z and u for $\boldsymbol{P}_1'$ and $\boldsymbol{P}_1$ respectively, any normal coverings $\pi' : \boldsymbol{P}_1'\to\boldsymbol{P}_1$ ($z\mapsto u=f(z)$) are one of the following five types;*

group		$\#G$	$u=f(x)$	$\begin{Bmatrix}\text{ramification indeces}\\ \text{branch points}\end{Bmatrix}$
I cyclic	C_n	n	$u=z^n$	$\begin{Bmatrix} n & n \\ 0 & \infty \end{Bmatrix}$
II dihedral	D_ν	2ν	$u=\dfrac{(z^\nu+1)^2}{4z^\nu}$	$\begin{Bmatrix} 2 & 2 & \nu \\ 0 & 1 & \infty \end{Bmatrix}$
III tetrahedral	A_4	12	$u=\dfrac{(z^4-2\sqrt{3}iz^2+1)^3}{-12\sqrt{3}iz^2(z^4-1)^2}$	$\begin{Bmatrix} 3 & 3 & 2 \\ 0 & 1 & \infty \end{Bmatrix}$
IV octahedral	S_4	24	$u=\dfrac{(z^8+14z^4+1)^3}{108z^4(z^4-1)^2}$	$\begin{Bmatrix} 3 & 2 & 4 \\ 0 & 1 & \infty \end{Bmatrix}$
V icosahedral	A_5	60	$u=\dfrac{(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3}{1728z^5(z^{10}+11z^5-1)^5}$	$\begin{Bmatrix} 3 & 2 & 5 \\ 0 & 1 & \infty \end{Bmatrix}$

where the symbol $\begin{Bmatrix} n_1 & n_2 & \cdots \\ \alpha_1 & \alpha_2 & \cdots \end{Bmatrix}$ means that π' is ramified over α_i with ramification index n_i.

Now we determine all ramification types of normal coverings $\pi': M' \to M$ for an arbitrary prime number p as Horiuchi did in case $p=2$.

As notations we use $P, P', P'', \cdots$ for ramification points of ϕ, and $Q_1, Q_2, \cdots, Q_p (Q'_1, \cdots, Q'_p; Q''_1, \cdots, Q''_p; \cdots)$ mean p distinct points with $\phi(Q_1)=\cdots=\phi(Q_p)$ $(\phi(Q'_1)=\cdots=\phi(Q'_p); \phi(Q''_1)=\cdots)$. The symbol $\begin{Bmatrix} m & \cdots \\ R & \cdots \end{Bmatrix}$ means that π' is ramified over R with ramification index m.

Proposition 4.2. *All the ramification types of normal coverings π' with Galois group $G \cong C_n$ are as follows;*

i) *If $p \nmid n$, then*

$$a) \begin{Bmatrix} n & n \\ P & P' \end{Bmatrix} \quad b) \begin{Bmatrix} n & n \cdots n \\ P & Q_1 \cdots Q_p \end{Bmatrix} \quad c) \begin{Bmatrix} n \cdots n & n \cdots n \\ Q_1 \cdots Q_p & Q'_1 \cdots Q'_p \end{Bmatrix}.$$

ii) *If $p \mid n$ and $p \neq n$, then*

$$a) \begin{Bmatrix} n/p & n/p \\ P & P' \end{Bmatrix} \quad b) \begin{Bmatrix} n/p & n \cdots n \\ P & Q_1 \cdots Q_p \end{Bmatrix} \quad c) \begin{Bmatrix} n \cdots n & n \cdots n \\ Q_1 \cdots Q_p & Q'_1 \cdots Q'_p \end{Bmatrix}.$$

iii) *If $p=n$, then*

$$a) \text{ unramified} \quad b) \begin{Bmatrix} n \cdots n \\ Q_1 \cdots Q_p \end{Bmatrix} \quad c) \begin{Bmatrix} n \cdots n & n \cdots n \\ Q_1 \cdots Q_p & Q'_1 \cdots Q'_p \end{Bmatrix}.$$

Proof. We may assume that the ramification type of π is $\begin{Bmatrix} n & n \\ 0 & \infty \end{Bmatrix}$. Let S be the set of branchpo ints of $\phi: M \to \boldsymbol{P}_1$. When $S \cap \{0, \infty\} = \{0, \infty\}$, we have i, ii, iii-*a*) by Lemma 3.1. When $S \cap \{0, \infty\} = \{0\}$ or $\{\infty\}$, we have i, ii, iii-*b*). When $S \cap \{0, \infty\} = \emptyset$, we have i, ii, iii-*c*). □

Proposition 4.3. *All the ramification types of normal coverings π' with Galois group $G \cong D_\nu$ are as follows;*

i) *If $p \nmid 2\nu$, then*

$$a) \begin{Bmatrix} 2 & 2 & \nu \\ P & P' & P'' \end{Bmatrix} \quad b) \begin{Bmatrix} 2 & 2 & \nu \cdots \nu \\ P & P' & Q_1 \cdots Q_p \end{Bmatrix} \quad c) \begin{Bmatrix} 2 \cdots 2 & 2 & \nu \\ Q_1 \cdots Q_p & P & P' \end{Bmatrix}$$

$$d) \begin{Bmatrix} 2 & 2 \cdots 2 & \nu \cdots \nu \\ P & Q_1 \cdots Q_p & Q'_1 \cdots Q'_p \end{Bmatrix} \quad e) \begin{Bmatrix} 2 \cdots 2 & 2 \cdots 2 & \nu \\ Q_1 \cdots Q_p & Q'_1 \cdots Q'_p & P \end{Bmatrix}$$

$$f) \begin{Bmatrix} 2 \cdots 2 & 2 \cdots 2 & \nu \cdots \nu \\ Q_1 \cdots Q_p & Q'_1 \cdots Q'_p & Q''_1 \cdots Q''_p \end{Bmatrix}$$

ii) *If $p|\nu$, $p\neq\nu$ and ν odd, then*

$$a)\ \begin{Bmatrix}2 & 2 & \nu/p\\ P & P' & P''\end{Bmatrix}\quad b)\ \begin{Bmatrix}2 & 2 & \nu & \cdots & \nu\\ P & P' & Q_1 & \cdots & Q_p\end{Bmatrix}\quad c)\ \begin{Bmatrix}2 & \cdots & 2 & 2 & \nu/p\\ Q_1 & \cdots & Q_p & P & P'\end{Bmatrix}$$

$$d)\ \begin{Bmatrix}2 & 2 & \cdots & 2 & \nu & \cdots & \nu\\ P & Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p\end{Bmatrix}\quad e)\ \begin{Bmatrix}2 & \cdots & 2 & 2 & \cdots & 2 & \nu/p\\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & P\end{Bmatrix}$$

$$f)\ \begin{Bmatrix}2 & \cdots & 2 & 2 & \cdots & 2 & \nu & \cdots & \nu\\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & Q''_1 & \cdots & Q''_p\end{Bmatrix}$$

iii) *If $p=\nu$ and ν is odd, then*

$$a)\ \begin{Bmatrix}2 & 2\\ P & P'\end{Bmatrix}\quad b)\ \begin{Bmatrix}2 & 2 & \nu & \cdots & \nu\\ P & P' & Q_1 & \cdots & Q_p\end{Bmatrix}\quad c)\ \begin{Bmatrix}2 & \cdots & 2 & 2\\ Q_1 & \cdots & Q_p & P\end{Bmatrix}$$

$$d)\ \begin{Bmatrix}2 & 2 & \cdots & 2 & \nu & \cdots & \nu\\ P & Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p\end{Bmatrix}\quad e)\ \begin{Bmatrix}2 & \cdots & 2 & 2 & \cdots & 2\\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p\end{Bmatrix}$$

$$f)\ \begin{Bmatrix}2 & \cdots & 2 & 2 & \cdots & 2 & \nu & \cdots & \nu\\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & Q''_1 & \cdots & Q''_p\end{Bmatrix}$$

iv) *If $p=2$ and ν is odd, then*

$$a)\ \begin{Bmatrix}\nu\\ P\end{Bmatrix}\quad b)\ \begin{Bmatrix}\nu & \nu\\ Q_1 & Q_2\end{Bmatrix}\quad c)\ \begin{Bmatrix}2 & 2 & \nu\\ Q_1 & Q_2 & P\end{Bmatrix}$$

$$d)\ \begin{Bmatrix}2 & 2 & \nu & \nu\\ Q_1 & Q_2 & Q'_1 & Q'_2\end{Bmatrix}\quad e)\ \begin{Bmatrix}2 & 2 & 2 & 2 & \nu\\ Q_1 & Q_2 & Q'_1 & Q'_2 & P\end{Bmatrix}$$

$$f)\ \begin{Bmatrix}2 & 2 & 2 & 2 & \nu & \nu\\ Q_1 & Q_2 & Q'_1 & Q'_2 & Q''_1 & Q''_2\end{Bmatrix}$$

v) *If $p=2$ and ν is even$\geqq 4$, then*

$$a)\ \begin{Bmatrix}\nu/2\\ P\end{Bmatrix}\quad b)\ \begin{Bmatrix}\nu & \nu\\ Q_1 & Q_2\end{Bmatrix}\quad c)\ \begin{Bmatrix}2 & 2 & \nu/2\\ Q_1 & Q_2 & P\end{Bmatrix}$$

$$d)\ \begin{Bmatrix}2 & 2 & \nu & \nu\\ Q_1 & Q_2 & Q'_1 & Q'_2\end{Bmatrix}\quad e)\ \begin{Bmatrix}2 & 2 & 2 & 2 & \nu/2\\ Q_1 & Q_2 & Q'_1 & Q'_2 & P\end{Bmatrix}$$

$$f)\ \begin{Bmatrix}2 & 2 & 2 & 2 & \nu & \nu\\ Q_1 & Q_2 & Q'_1 & Q'_2 & Q''_1 & Q''_2\end{Bmatrix}$$

vi) *If $p=v=2$ (Theorem 2′* [3], [4]),

$a)$ *unramified* $\quad b)$ $\begin{Bmatrix} 2 & 2 \\ Q_1 & Q_2 \end{Bmatrix}$ $\quad c)$ $\begin{Bmatrix} 2 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q'_1 & Q'_2 \end{Bmatrix}$

$d)$ $\begin{Bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q'_2 & Q'_2 & Q''_1 & Q''_2 \end{Bmatrix}$.

PROOF. The ramification type of π is $\begin{Bmatrix} 2 & 2 & \nu \\ 0 & 1 & \infty \end{Bmatrix}$. The cases i$\sim$v $a)$, i$\sim$v $b)$, i$\sim$v $c)$, i$\sim$v $d)$, i$\sim$v $e)$ and i$\sim$v $f)$ are corresponding to $S\cap\{0, 1, \infty\} =^{a)}\{0, 1, \infty\}$, $^{b)}\{0, 1\}$, $^{c)}\{0, \infty\}$ or $\{1, \infty\}$, $^{d)}\{0\}$ or $\{1\}$, $^{e)}\{\infty\}$ and $^{f)}\emptyset$ respectively. In case vi), $a)$, $b)$, $c)$ and $d)$ are corresponding to $S\cap\{0, 1, \infty\}=$ $^{a)}\{0, 1, \infty\}$, $^{b)}\{0, 1\}$ or $\{0, \infty\}$ or $\{1, \infty\}$, $^{c)}\{0\}$ or $\{1\}$ or $\{\infty\}$ and $^{d)}\emptyset$ respectively.

PROPOSIN 4.4. *All the ramification types of normal coverings π' with $G\cong A_A$ are as follows;*

i) *If $p\geqq 5$, then*

$a)$ $\begin{Bmatrix} 3 & 3 & 2 \\ P & P' & P'' \end{Bmatrix}$ $\quad b)$ $\begin{Bmatrix} 3 & 3 & 2 & \cdots & 2 \\ P & P' & Q_1 & \cdots & Q_p \end{Bmatrix}$ $\quad c)$ $\begin{Bmatrix} 3 & \cdots & 3 & 3 & 2 \\ Q_1 & \cdots & Q_p & P & P' \end{Bmatrix}$

$d)$ $\begin{Bmatrix} 3 & 3 & \cdots & 3 & 2 & \cdots & 2 \\ P & Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p \end{Bmatrix}$ $\quad e)$ $\begin{Bmatrix} 3 & \cdots & 3 & 3 & \cdots & 3 & 2 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & P \end{Bmatrix}$

$f)$ $\begin{Bmatrix} 3 & \cdots & 3 & 3 & \cdots & 3 & 2 & \cdots & 2 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & Q''_1 & \cdots & Q''_p \end{Bmatrix}$

ii) *If $p=3$, then*

$a)$ $\begin{Bmatrix} 2 \\ P \end{Bmatrix}$ $\quad b)$ $\begin{Bmatrix} 2 & 2 & 2 \\ Q_1 & Q_2 & Q_3 \end{Bmatrix}$ $\quad c)$ $\begin{Bmatrix} 3 & 3 & 3 & 2 \\ Q_1 & Q_2 & Q_3 & P \end{Bmatrix}$

$d)$ $\begin{Bmatrix} 3 & 3 & 3 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 \end{Bmatrix}$ $\quad e)$ $\begin{Bmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 2 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 & P \end{Bmatrix}$

$f)$ $\begin{Bmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 & Q''_1 & Q''_2 & Q''_p \end{Bmatrix}$

iii) *If $p=2$, then*

$a)$ $\begin{Bmatrix} 3 & 3 \\ P & P' \end{Bmatrix}$ $\quad b)$ $\begin{Bmatrix} 3 & 3 & 2 & 2 \\ P & P' & Q_1 & Q_p \end{Bmatrix}$ $\quad c)$ $\begin{Bmatrix} 3 & 3 & 3 \\ P & Q_1 & Q_2 \end{Bmatrix}$

$d)\ \begin{Bmatrix} 3 & 3 & 3 & 2 & 2 \\ P & Q_1 & Q_2 & Q_1' & Q_2' \end{Bmatrix}$ $e)\ \begin{Bmatrix} 3 & 3 & 3 & 3 \\ Q_1 & Q_2 & Q_1' & Q_2' \end{Bmatrix}$

$f)\ \begin{Bmatrix} 3 & 3 & 3 & 3 & 2 & 2 \\ Q_1 & Q_2 & Q_1' & Q_2' & Q_1'' & Q_2'' \end{Bmatrix}.$

PROOF. The ramification type of π is $\begin{Bmatrix} 3 & 3 & 2 \\ 0 & 1 & \infty \end{Bmatrix}$. The cases i~iii a), i~iii b), i~iii c), i~iii d), i~iii e) and i~iii f) are corresponding to $S\cap\{0, 1, \infty\} = {}^{a)}\{0, 1, \infty\}$, ${}^{b)}\{0, 1\}$, ${}^{c)}\{0, \infty\}$ or $\{1, \infty\}$, ${}^{d)}\{0\}$ or $\{1\}$, ${}^{e)}\{\infty\}$ and ${}^{f)}\emptyset$ respectively. □

PROPOSITION 4.5. *All the ramification types of normal coverings π' with $G\cong S_4$ are as follows;*

i) *If $p\geqq 5$, then*

$a)\ \begin{Bmatrix} 3 & 2 & 4 \\ P & P' & P'' \end{Bmatrix}$ $b)\ \begin{Bmatrix} 3 & 2 & 4 \cdots 4 \\ P & P' & Q_1 \cdots Q_p \end{Bmatrix}$ $c)\ \begin{Bmatrix} 3 & 2 \cdots 2 & 4 \\ P & Q_1 \cdots Q_p & P' \end{Bmatrix}$

$d)\ \begin{Bmatrix} 3 \cdots 3 & 2 & 4 \\ Q_1 \cdots Q_p & P & P' \end{Bmatrix}$ $e)\ \begin{Bmatrix} 3 & 2 \cdots 2 & 4 \cdots 4 \\ P & Q_1 \cdots Q_p & Q_1' \cdots Q_p' \end{Bmatrix}$

$f)\ \begin{Bmatrix} 3 \cdots 3 & 2 & 4 \cdots 4 \\ Q_1 \cdots Q_p & P & Q_1' \cdots Q_p' \end{Bmatrix}$ $g)\ \begin{Bmatrix} 3 \cdots 3 & 2 \cdots 2 & 4 \\ Q_1 \cdots Q_p & Q_1' \cdots Q_p' & P \end{Bmatrix}$

$h)\ \begin{Bmatrix} 3 \cdots 3 & 2 \cdots 2 & 4 \cdots 4 \\ Q_1 \cdots Q_p & Q_1' \cdots Q_p' & Q_1'' \cdots Q_p'' \end{Bmatrix}.$

ii) *If $p=3$, then*

$a)\ \begin{Bmatrix} 2 & 4 \\ P & P' \end{Bmatrix}$ $b)\ \begin{Bmatrix} 2 & 4 & 4 & 4 \\ P & Q_1 & Q_2 & Q_3 \end{Bmatrix}$ $c)\ \begin{Bmatrix} 2 & 2 & 2 & 4 \\ Q_1 & Q_2 & Q_3 & P \end{Bmatrix}$

$d)\ \begin{Bmatrix} 3 & 3 & 3 & 2 & 4 \\ Q_1 & Q_2 & Q_3 & P & P' \end{Bmatrix}$ $e)\ \begin{Bmatrix} 2 & 2 & 2 & 4 & 4 & 4 \\ Q_1 & Q_2 & Q_3 & Q_1' & Q_2' & Q_3' \end{Bmatrix}$

$f)\ \begin{Bmatrix} 3 & 3 & 3 & 2 & 4 & 4 & 4 \\ Q_1 & Q_2 & Q_3 & P & Q_1' & Q_2' & Q_3' \end{Bmatrix}$ $g)\ \begin{Bmatrix} 3 & 3 & 3 & 2 & 2 & 2 & 4 \\ Q_1 & Q_2 & Q_3 & Q_1' & Q_2' & Q_3' & P \end{Bmatrix}$

$h)\ \begin{Bmatrix} 3 & 3 & 3 & 2 & 2 & 2 & 4 & 4 & 4 \\ Q_1 & Q_2 & Q_3 & Q_1' & Q_2' & Q_3' & Q_1'' & Q_2'' & Q_3'' \end{Bmatrix}.$

iii) *If $p=2$, then*

$a)\ \begin{Bmatrix} 3 & 2 \\ P & P' \end{Bmatrix}$ $b)\ \begin{Bmatrix} 3 & 4 & 4 \\ P & Q_1 & Q_2 \end{Bmatrix}$ $c)\ \begin{Bmatrix} 3 & 2 & 2 & 2 \\ P & Q_1 & Q_2 & P' \end{Bmatrix}$

$d)\ \begin{Bmatrix} 3 & 3 & 2 \\ Q_1 & Q_2 & P \end{Bmatrix}$ $e)\ \begin{Bmatrix} 3 & 2 & 2 & 4 & 4 \\ P & Q_1 & Q_2 & Q_1' & Q_2' \end{Bmatrix}$

$f)\ \begin{Bmatrix} 3 & 3 & 4 & 4 \\ Q_1 & Q_2 & Q_1' & Q_2' \end{Bmatrix}$ $g)\ \begin{Bmatrix} 3 & 3 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q_1' & Q_2' & P \end{Bmatrix}$

$h)\ \begin{Bmatrix} 3 & 3 & 2 & 2 & 4 & 4 \\ Q_1 & Q_2 & Q_1' & Q_2' & Q_1'' & Q_2'' \end{Bmatrix}.$

PROOF. The ramification type of π is $\begin{Bmatrix} 3 & 2 & 4 \\ 0 & 1 & \infty \end{Bmatrix}$. The cases i~iii *a*), i~iii *b*), i~iii *c*), i~iii *d*), i~iii *e*), i~iii *f*), i~iii *g*) and i~iii *h*) are corresponding to $S \cap \{0, 1, \infty\} = {}^{a)}\{0, 1, \infty\}$, ${}^{b)}\{0, 1\}$, ${}^{c)}\{0, \infty\}$, ${}^{d)}\{1, \infty\}$, ${}^{e)}\{0\}$, ${}^{f)}\{1\}$, ${}^{g)}\{\infty\}$ and ${}^{h)}\emptyset$ respectively. □

PROPOSITION 4.6. *All the ramification types of normal coverings π' with $G \cong A_5$ are as follows,*

i) *If $p \geqq 7$, then*

$a)\ \begin{Bmatrix} 3 & 2 & 5 \\ P & P' & P'' \end{Bmatrix}$ $b)\ \begin{Bmatrix} 3 & 2 & 5 \cdots 5 \\ P & P' & Q_1 \cdots Q_p \end{Bmatrix}$ $c)\ \begin{Bmatrix} 3 & 2 \cdots 2 & 5 \\ P & Q_1 \cdots Q_p & P' \end{Bmatrix}$

$d)\ \begin{Bmatrix} 3 \cdots 3 & 2 & 5 \\ Q_1 \cdots Q_p & P & P' \end{Bmatrix}$ $e)\ \begin{Bmatrix} 3 & 2 \cdots 2 & 5 \cdots 5 \\ P & Q_1 \cdots Q_p & Q_1' \cdots Q_p' \end{Bmatrix}$

$f)\ \begin{Bmatrix} 3 \cdots 3 & 2 & 5 \cdots 5 \\ Q_1 \cdots Q_p & P & Q_1' \cdots Q_p' \end{Bmatrix}$ $g)\ \begin{Bmatrix} 3 \cdots 3 & 2 \cdots 2 & 5 \\ Q_1 \cdots Q_p & Q_1' \cdots Q_p' & P \end{Bmatrix}$

$h)\ \begin{Bmatrix} 3 \cdots 3 & 2 \cdots 2 & 5 \cdots 5 \\ Q_1 \cdots Q_p & Q_1' \cdots Q_p' & Q_1'' \cdots Q_p'' \end{Bmatrix}$

ii) *If $p=5$, then*

$a)\ \begin{Bmatrix} 3 & 2 \\ P & P' \end{Bmatrix}$ $b)\ \begin{Bmatrix} 3 & 2 & 5 \cdots 5 \\ P & P' & Q_1 \cdots Q_p \end{Bmatrix}$ $c)\ \begin{Bmatrix} 3 & 2 \cdots 2 \\ P & Q_1 \cdots Q_p \end{Bmatrix}$

$d)\ \begin{Bmatrix} 3 \cdots 3 & 2 \\ Q_1 \cdots Q_p & P \end{Bmatrix}$ $e)\ \begin{Bmatrix} 3 & 2 \cdots 2 & 5 \cdots 5 \\ P & Q_1 \cdots Q_p & Q_1' \cdots Q_p' \end{Bmatrix}$

$f)\ \begin{Bmatrix} 3 & \cdots & 3 & 2 & 5 & \cdots & 5 \\ Q_1 & \cdots & Q_p & P & Q'_1 & \cdots & Q'_p \end{Bmatrix}$ $\quad g)\ \begin{Bmatrix} 3 & \cdots & 3 & 2 & \cdots & 2 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p \end{Bmatrix}$

$h)\ \begin{Bmatrix} 3 & \cdots & 3 & 2 & \cdots & 2 & 5 & \cdots & 5 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & Q''_1 & \cdots & Q''_p \end{Bmatrix}.$

iii) *If $p=3$, then*

$a)\ \begin{Bmatrix} 2 & 5 \\ P & P' \end{Bmatrix}$ $\quad b)\ \begin{Bmatrix} 2 & 5 & 5 & 5 \\ P & Q_1 & Q_2 & Q_3 \end{Bmatrix}$ $\quad c)\ \begin{Bmatrix} 2 & 2 & 2 & 5 \\ Q_1 & Q_2 & Q_p & P \end{Bmatrix}$

$d)\ \begin{Bmatrix} 3 & 3 & 3 & 2 & 5 \\ Q_1 & Q_2 & Q_3 & P & P' \end{Bmatrix}$ $\quad e)\ \begin{Bmatrix} 2 & 2 & 2 & 5 & 5 & 5 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 \end{Bmatrix}$

$f)\ \begin{Bmatrix} 3 & 3 & 3 & 2 & 5 & 5 & 5 \\ Q_1 & Q_2 & Q_3 & P & Q'_1 & Q'_2 & Q'_3 \end{Bmatrix}$ $\quad g)\ \begin{Bmatrix} 3 & 3 & 3 & 2 & 2 & 2 & 5 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 & P \end{Bmatrix}$

$h)\ \begin{Bmatrix} 3 & 3 & 3 & 2 & 2 & 2 & 5 & 5 & 5 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 & Q''_1 & Q''_2 & Q''_3 \end{Bmatrix}$

iv) *If $p=2$, then*

$a)\ \begin{Bmatrix} 3 & 5 \\ P & P' \end{Bmatrix}$ $\quad b)\ \begin{Bmatrix} 3 & 5 & 5 \\ P & Q_1 & Q_2 \end{Bmatrix}$ $\quad c)\ \begin{Bmatrix} 3 & 2 & 2 & 5 \\ P & Q_1 & Q_2 & P \end{Bmatrix}$

$d)\ \begin{Bmatrix} 3 & 3 & 5 \\ Q_1 & Q_2 & P \end{Bmatrix}$ $\quad e)\ \begin{Bmatrix} 2 & 3 & 3 & 5 & 5 \\ P & Q_1 & Q_2 & Q'_1 & Q'_2 \end{Bmatrix}$

$f)\ \begin{Bmatrix} 3 & 3 & 5 & 5 \\ Q_1 & Q_2 & Q'_1 & Q'_2 \end{Bmatrix}$ $\quad g)\ \begin{Bmatrix} 3 & 3 & 2 & 2 & 5 \\ Q_1 & Q_p & Q'_1 & Q'_p & P \end{Bmatrix}$

$h)\ \begin{Bmatrix} 3 & 3 & 2 & 2 & 5 & 5 \\ Q_1 & Q_2 & Q'_1 & Q'_2 & Q''_1 & Q''_2 \end{Bmatrix}.$

PROOF. The ramification type of π is $\begin{Bmatrix} 3 & 2 & 5 \\ 0 & 1 & \infty \end{Bmatrix}$. The cases i~iv a), i~iv b), i~iv c), i~iv d), i~iv e), i~iv f), i~iv g) and i~iv h) are corresponding to $S\cap\{0, 1, \infty\}={}^{a)}\{0, 1, \infty\}$, ${}^{b)}\{0, 1\}$, ${}^{c)}\{0, \infty\}$, ${}^{d)}\{1, \infty\}$, ${}^{e)}\{0\}$, ${}^{f)}\{1\}$, ${}^{g)}\{\infty\}$ and ${}^{h)}\ \emptyset$ respectively. □

REMARK 4.7. There exists unique covering π' that attains each type in proposition 4.2~4.6. If we appoints branch points $P, P', \cdots; Q_1, Q_2, \cdots; Q'_1, \cdots.$

By Lemma 4.1, § 3.(4) and Proposition 4.2～4.6 we have;

THEOREM 4.8. *Let M be a cyclic p-gonal curve. All the unramified normal coverings $\pi': M' \to M$ with a p-gonal curve M' are obtained by the following manners;*

i) *Let p be an arbitrary prime number. Take two ramification points P, P' of $\phi: M \to \boldsymbol{P}_1$. Let $\pi: \boldsymbol{P}_1' \to \boldsymbol{P}_1$ be a normal covering with Gaiois group C_p ramified over $\phi(P)$ and $\phi(P')$. Then π' as in Theorem 2.1 is unramified. Moreover if M and π are defined by $y^p - u^{m_1}(u-a_2)^{m_2} \cdots (u-a_{r-1})^{m_{r-1}} = 0$ ($a_i \in \boldsymbol{C} - \{0\}$, $\sum m_i \not\equiv 0 \bmod p$) and $\pi: z \mapsto z^p$, then M' and π' are defined by*

$$y^p - (z^p - a_2)^{m_2} \cdots (z^p - a_{r-1})^{m_{r-1}} = 0 \quad \text{and} \quad \pi': (z, y) \longmapsto (z^p, z^{-m_1} y).$$

ii) $p=2$. ([3], [4]) *Take three ramification points P, P', P'' of ϕ and a normal covering π of degree 4 with Galois group D_2 ramified over $\phi(P)$, $\phi(P')$, $\phi(P'')$. Then π' is unramified. Moreover if M and ϕ are defined by $y^2 - u(u-1)(u-a_3) \cdots (u-a_{r-1}) = 0$, $r-1 \not\equiv 0 \bmod 2$, $a_i \in \boldsymbol{C} - \{0\}$ and $\pi: z \mapsto u = (z^2+1)^2/4z^2$, then M' and ϕ' are defined by*

$$y^2 - \{(z^2+1)^2 - 4a_3 z^2\} \cdots \{(z^2+1)^2 - 4a_{r-1} z^2\} = 0 \qquad \text{and}$$

$$\pi': (z, y) \longmapsto \left(\frac{(z^2+1)^2}{4z^2}, \frac{(z^2+1)(z^2-1)}{(2z)^{r-1}} y \right).$$

§ 5.

Let M be a cyclic p-gonal curve with $m \geqq 2p+1$ and $\pi': M' \to M$ be as before, but we do not assume that π' is normal. We consider the condition that π' is unramified (if π' is normal, all umramified π' are obtained by Theorem 4.8). By Lemma 3.1 we have:

LEMMA 5.1. *Let $\pi: \boldsymbol{P}_1' \to \boldsymbol{P}_1$ and $\phi: M \to \boldsymbol{P}_1$ be as in Theorem 2.1. Then the followings are equivalent;*

i) *π' is unramified.*

ii) *Any branch points of π are also branch points of ϕ and any ramification indeces of π are equal to p.*

Finally we give an example of an unramified covering π' that is not normal.

EXAMPLE 5.2. Let $\pi: \boldsymbol{P}_1' \to \boldsymbol{P}_1$ be defined by

$$z \longmapsto \frac{(z-1)^2(z-k)^2}{z^2}, \qquad \text{where } k \neq 0, \pm 1$$

Then the ramification points ($\in P_1'$) of π are 1, k, 0, ∞ and $\pm\sqrt{k}$ with ramification index p. $\pi(1)=\pi(k)=0$, $\pi(0)=\pi(\infty)=\infty$, $\pi(\sqrt{k})=(1-\sqrt{k})^4$ and $\pi(-\sqrt{k})=(1+\sqrt{k})^4$. Thus π is not normal. Let M be a hyperelliptive curve defined by

$$y^2-u\{u-(1-\sqrt{k})^4\}\{u-(1+\sqrt{k})^4\}(u-a_5)\cdots(u-a_{2g+2})=0.$$

Then $\pi': M'\to M$ as in Theorem 2.1 is unramified. Explicitely M' and π' are represented by

$$y^2-\{z^2-(2-2\sqrt{k}+2k)z+k\}\{z^2-(2+2\sqrt{k}+2k)z+k\}$$
$$\cdot\{(z-1)^2(z-k)^2-a_5z^2\}\times\cdots\times\{(z-1)^2(z-k)^2-a_{2g+2}z^2\}=0$$

and

$$\pi': (z, y) \longmapsto \left(\frac{(z-1)^2(z-k)^2}{z^2}, z^{-(2g+2)}(z+\sqrt{k})(z-\sqrt{k})y\right).$$

References

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