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REPRESENTATIONS OF REDUCTIVE GROUP SCHEMES

By

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Introduction.

Let S be a reduced, irreducible scheme and G a reductive group scheme over S. A representation of G is, by definition, a pair (ρ, V) of a vector bundle V over S and a homomorphism $\rho: G\rightarrow GL(V)$. If $\bar{\eta}$ is the generic geometric point of S, we call (G, ρ, V) *an S-form* of $(G_{\bar{\eta}}, \rho_{\bar{\eta}}, V_{\bar{\eta}})$. The purpose of this paper is to describe the S-forms of an irreducible representation of $G_{\bar{\eta}}$, assuming that S is normal and locally noetherian.

As is well known, if S is the prime spectrum of a field, the S-forms of a given representation can be obtained by twisting *the* split S-form using the Galois cohomology. In the general case, the S-forms of a given representation can be also obtained by twisting the split ones using a non-abelian étale conomology, which is a natural generalization of the usual Galois cohomology. In contrast with the case where S is the prime spectrum of a field, there are possibly more than one split S-forms.

The results of this paper will be applied to a study of prehomogeneous vector spaces.

Conventions. Since we refer [5] very often, we shall write [Exp. X, Y.Z. …] for [5; Exp. X, Y.Z. …]. If we are considering an algebraic variety V over an algebraically close field K, we often identify V with the set of rational points $V(K)$. If a scheme X is considered as a scheme over another scheme S, we add suffix S and write X_S. If $S=\operatorname{Spec} A$, we write X_A for $X_{\operatorname{Spec} A}$.

1. Representations of Chevalley-Demazure group schemes.

The purpose of this section is to describe the irreducible representations of a Chevalley-Demazure group scheme. The main result of this section is (1.19).

1.1. Let K be an algebraically closed field, G_K a (connected) reductive algebraic group over K, T_K a maximal torus of G_K, B_K a Borel subgroup of

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G_K which contains T_K, R (resp. $R^\vee$) the root system (resp. the coroot system) of G_K with respect to T_K, $M=\mathrm{Hom}(T_K, K^\times)$ and $M^\vee=\mathrm{Hom}(K^\times, T_K)$. Let $\mathfrak{G}_K$ be the Lie algebra of G_K, $\mathfrak{G}_K^r=\mathfrak{G}_K(r)$ the root subspace of $\mathfrak{G}_K$ associated with a root $r(\in R)$, $R^+=\{r\in R \mid \mathfrak{G}_K(r)\subset \mathrm{Lie}(B_K)\}$ and R_0 the basis of R which is contained in R^+. Let $(X_r)_{r\in R}$ be a Chevalley system of G_K [Exp. 23, 6.1]. Define a homomorphism $p_r=p_{r,K}: K\to G_K$ by $p_r(x)=\exp(xX_r)$ and let $P_r=P_{r,K}$ $p_r(K)$. Let

$$w_r=p_r(1)p_{-r}(-1)p_r(1)$$
$$=p_{-r}(-1)p_r(1)p_{-r}(-1).$$

Let $w_0=w_{r_1}w_{r_2}\cdots w_{r_N}$ be "the longest element", i.e., the following conditions are satisfied:

$$R^+=\{r_N,\ w_{r_N}r_{N-1},\ w_{r_N}w_{r_{N-1}}r_{N-2},\ \cdots,\ w_{r_N}w_{r_{N-1}}\cdots w_{r_2}r_1\}$$

and

$$|R^+|=N.$$

Let $\mathfrak{U}_K$ be the enveloping algebra of $\mathfrak{G}_K$.

1.2. Let $\rho_K: G_K\to GL(V_K)$ be a representation of G_K on a finite dimensional vector space V_K, $V_K^\vee$ the dual vector space and $\rho_K^\vee$ the composition of

$$G_K \xrightarrow{\rho} GL(V_K) \xrightarrow{\sim} GL(V_K^\vee),$$

which is called the contragradient representation of ρ_K. Let $\langle\ \rangle: V_K^\vee\times V_K\to K$ be the natural pairing of $V_K^\vee$ and V_K, and $V_{\mu,K}=V_K(\mu)$ (resp. $V_{\mu,K}^\vee=V_K^\vee(\mu)$) the weight space of V_K (resp. $V_K^\vee$) which belongs to μ $(\in M)$.

If there is no fear of confusion, we refer to (ρ_K, V_K) or ρ_K as a representation. We refer to V_K as a G_K-module.

1.3. Let S be a reduced, irreducible scheme whose residue field at the generic point η is contained in K. Let $\bar\eta$ be the generic geometric point

$$\bar\eta: \mathrm{Spec}\, K\longrightarrow \eta\longrightarrow S.$$

Let $\mathcal{O}_S$ be the structure sheaf of S.

1.4. An ($\mathcal{O}_S$-)lattice of V_K is, by definition, a pair $(V(\mathcal{O}_S), i)$ of a locally free $\mathcal{O}_S$-module $V(\mathcal{O}_S)$ of finite rank and an isomorphism $i: V(\mathcal{O}_S)_{\bar\eta}\xrightarrow{\sim}V_K$. Two lattices $(V(\mathcal{O}_S), i)$ and $(V'(\mathcal{O}_S), i')$ of V_K are isomorphic if there is an isomorphism $f: V(\mathcal{O}_S)\to V'(\mathcal{O}_S)$ and a (non-zero) homothety $c: V_K\to V_K$ such that the diagram

$$\begin{array}{ccc} V(\mathcal{O}_S)_{\bar\eta} & \xrightarrow{i} & V_K \\ f_{\bar\eta}\downarrow & & \downarrow c \\ V'(\mathcal{O}_S)_{\bar\eta} & \xrightarrow{i'} & V_K \end{array}$$

is commutative. Let $V^\vee(\mathcal{O}_S)$ be the dual $\mathcal{O}_S$-module of $V(\mathcal{O}_S)$, i.e., $V^\vee(\mathcal{O}_S)=\mathrm{Hom}_{\mathcal{O}_S}(V(\mathcal{O}_S), \mathcal{O}_S)$. The natural isomorphism $i^\vee: V^\vee(\mathcal{O}_S)_{\bar\eta}\to V_K^\vee$ which is induced by i, gives a lattice $(V^\vee(\mathcal{O}_S), i^\vee)$, which is called the dual lattice of $(V(\mathcal{O}_S), i)$. Let $S(V(\mathcal{O}_S))$ and $S(V^\vee(\mathcal{O}_S))$ be the symmetric algebras over $V(\mathcal{O}_S)$ and $V^\vee(\mathcal{O}_S)$, respectively. Then $V_S=\mathrm{Spec}\, S(V^\vee(\mathcal{O}_S))$ and $V_S^\vee=\mathrm{Spec}\, S(V(\mathcal{O}_S))$ are vector bundles over S, and their generic geometric fibres $V_{S,\bar\eta}$ and $V_{S,\bar\eta}^\vee$ are isomorphic to V_K and $V_K^\vee$, respectively. If there is no fear of confusion, we refer to $V(\mathcal{O}_S)$ as a lattice of V_K.

1.5. Let G_S be the Chevalley-Demazure group scheme over S such that $G_{S,\bar\eta}\cong G_K$. (In other words, $G_S=G_{\boldsymbol{Z}}\times S$, where $G_{\boldsymbol{Z}}$ is the $\boldsymbol{Z}$-group scheme which is constructed in [Exp. 25].) We can define an S-analogue of each object (or notion) which appears in (1.1) and (1.2). Especially we can define T_S, B_S etc., which are objects over S, corresponding to T_K, B_K etc. We may assume that $T_S\otimes K=T_K$, $B_S\otimes K=B_K$ etc. We assume that G_S is equipped with an épinglage [Exp. 23, 1.1].

REMARK. One of the advantages of the construction of [Exp. 25] is that we can treat representations which are not faithful.

1.6. Let $\rho_S: G_S\to GL(V_S)$ be a homomorphism such that $\rho_S\otimes K=\rho_K$, and $\rho_S^\vee$ the composition

$$G_S \xrightarrow{\rho_S} GL(V_S) \xrightarrow{\sim} GL(V_S^\vee)$$

We shall call ρ_S a representation of G_S and $\rho_S^\vee$ the contragradient representation of ρ_S. We say that ρ_S is (absolutely) irreducible if $\rho_S\otimes K$ is irreducible.

1.7. Let $\mathfrak{U}_{\boldsymbol{Z}}$ be the $\boldsymbol{Z}$-subalgebra of $\mathfrak{U}_C$, which is generated by the elements

$$X_r^m/m! \qquad (r\in R,\ m=0, 1, 2, \cdots).$$

Let $\mathfrak{U}_S$ be the $\mathcal{O}_S$-algebra defined by

$$U \longmapsto \Gamma(U, \mathcal{O}_S)\otimes\mathfrak{U}_{\boldsymbol{Z}},$$

where U is any open set of S. The $\mathcal{O}_S$-algebra $\mathfrak{U}_S$ has a graded $\mathcal{O}_S$-algebra structure of type M:

$$\deg(X_r^m/m!)=mr \qquad (\in M).$$

1.8. Let $V(\mathcal{O}_S)$ be a locally free $\mathcal{O}_S$-module of finite rank which has a (left) $\mathfrak{U}_S$-module structure. We say that $V(\mathcal{O}_S)$ is irreducible if $V(\mathcal{O}_S)\otimes K$ is an irreducible $\mathfrak{U}_K$-module. A graded $\mathfrak{U}_S$-module is, by definition, a (left) $\mathfrak{U}_S$-module equipped with a graded $\mathcal{O}_S$-module structure of type M which is compatible with the graded $\mathcal{O}_S$-algebra structure of $\mathfrak{U}_S$. A graded $\mathfrak{U}_S$-module is said to be irreducible, if it is irreducible as a $\mathfrak{U}_S$-module.

1.9. Let us show that, from a given representation ρ_S of G_S, we can canonically construct a graded $\mathfrak{U}_S$-module structure on $V(\mathcal{O}_S)$.

Let $\mathcal{O}_S M$ be the group algebra of the additive group M. The épinglage of G_S gives an identification of the character lattice of T_S with M. Hence we get a canonical identification of T_S with $\operatorname{Spec}\mathcal{O}_S M$. Thus the composition of morphisms

$$T_S \longrightarrow G_S \longrightarrow \mathrm{GL}(V_S) \longrightarrow \mathrm{End}(V_S)$$

induces an algebra homomorphism $V(\mathcal{O}_S)\otimes V^\vee(\mathcal{O}_S)\to\mathcal{O}_S M$, which induces an $\mathcal{O}_S$-linear mapping $q: V(\mathcal{O}_S)\to V(\mathcal{O}_S)\otimes\mathcal{O}_S M$. Let us define $\mathcal{O}_S$-linear mappings $q_\mu: V(\mathcal{O}_S)\to V(\mathcal{O}_S)$ $(\varphi\in M)$, by $q(v)=\sum_{\mu\in M}q_\mu(v)\otimes\mu$, where v is a local section of $V(\mathcal{O}_S)$. It is easy to see that q_μ's are mutually orthogonal projections onto submodules of $V(\mathcal{O}_S)$ and $\sum q_\mu=id$. (See [1; II, §2, 2.5].) Let $V_\mu(\mathcal{O}_S)$ be the image of q_μ. Then $V(\mathcal{O}_S)=\bigoplus_{\mu\in M}V_\mu(\mathcal{O}_S)$. Denote (symbolically) by $X_r^{[m]}|_{V_\mu(\mathcal{O}_S)}$ the composition of mappings

$$V_\mu(\mathcal{O}_S) \hookrightarrow V(\mathcal{O}_S) \xrightarrow{p_r(1)} V(\mathcal{O}_S) \xrightarrow{q_{\mu+mr}} V_{\mu+mr}(\mathcal{O}_S),$$

where $r\in R$ and $m=0, 1, 2, \cdots$. Let $X_r^{[m]}$ $(\in\mathrm{End}\,V(\mathcal{O}_S))$ be the direct sum of these mappings.

1.10. REMARK. Let us consider the case $S=\operatorname{Spec}K$. Let C be the coordinate ring of G_K. Then C has a K-coalgebra structure. See [2] for the definition of coalgebra and related notions. The left G_K-module structure on V_K induces a left C-module structure on $V_K^\vee$. The inclusion $T_K\to G_K$ induces a homomorphisms $KM\leftarrow C$ and

$$V_K^\vee \longrightarrow V_K^\vee\otimes C \longrightarrow V_K^\vee\otimes KM.$$

By the argument of (1.9), we can define a graded K-module structure on $V_K^\vee$ of type M. Also $\{p_r(1)\}\hookrightarrow G_K$ induces $K\leftarrow C$ and

$$V_K^\vee \longrightarrow V_K^\vee\otimes C \longrightarrow V_K^\vee.$$

Thus, to define the operator $X_r^{[m]}$ on $V_K^\vee$, we need only the C-module structure on $V_K^\vee$. In other words, we can define the operators $X_r^{[m]}$ for any C-module. Moreover, it is not difficult to generalize our argument to an arbitrary S other than Spec K.

1.11. LEMMA. *There is a (unique) $\mathfrak{U}_S$-module structure on $V(\mathcal{O}_S)$ such that $(X_r^m/m!)v=X_r^{[m]}v$ for every local section v of $V(\mathcal{O}_S)$.*

PROOF. Our task is to show that the operators $X_r^{[m]}$ ($r\in R$, $m=0, 1, \cdots$) satisfy the relations which they should satisfy. By an extension of scalars, we may assume that $S=\operatorname{Spec} K$ from the beginning. By (1.10), we can consider the same statement as above for any C-module W. To use the results of [2], we shall prove the statement in such a generalized form. By [2; 1.5a], we may assume that W is an injective C-module. By [2; 1.5h], we may assume further that W is indecomposable. By [2; 2.4c], we can reduce the proof to the case $S=\operatorname{Spec} L$, where L is an algebraic closure of the quotient field of the Witt ring of K. Since L is an algebraically closed field of characteristic zero, the above statement is clear from the construction of Chevalley groups.

1.12. Thus we get a $\mathfrak{U}_S$-module structure on $V(\mathcal{O}_S)$ and a graded $\mathcal{O}_S$-module structure $(V_\mu(\mathcal{O}_S))_{\mu\in M}$ of type M, which are clearly compatible.

Conversely, if we are given a graded $\mathfrak{U}_S$-module structure on $V(\mathcal{O}_S)$, we can define linear actions of $p_{r,S}$ ($r\in R$) and T_S on V_S. By a similar argument as above, we can show that these actions extend to a linear action of G_S on V_S. Thus we get the following lemma.

1.13. LEMMA. *The functor*

$$V_S \longmapsto (V(\mathcal{O}_S), (V_\mu(\mathcal{O}_S))_{\mu\in M})$$

is an equivalence of the category of irreducible G_S-modules with that of irreducible, graded $\mathfrak{U}_S$-modules.

In the remainder of this section, we assume that ρ_K is an irreducible representation of G_K, $V(\mathcal{O}_S)$ is a lattice of V_K and $\rho_S\otimes K=\rho_K$.

1.14. As is well known, V_K has a highest weight μ_0 with respect to B_K, i.e., $B_K V_K(\mu_0)=V_K(\mu_0)$. It is also known that $\dim_K V_K(\mu_0)=1$. Let $\mathcal{V}_0=V_{\mu_0}(\mathcal{O}_S)$ be a subsheaf of $V_{\mu_0,K}$ which is locally free of rank 1. Here we are considering K as a constant sheaf on S. Since we can regard $\Gamma(U, \mathfrak{U}_S)$ as a subring of $\mathfrak{U}_K$ for any open set U of S, we can define a sheaf $V_{\min}(\mathcal{O}_S)$ by

$$U \longmapsto \Gamma(U, \mathfrak{U}_S)\cdot \mathcal{V}_0 \qquad (\subset V_K).$$

Let $i: V_{\min}(\mathcal{O}_S)_{\bar\eta} \to V_K$ be the natural morphism.

1.15. LEMMA. *The pair $(V_{\min}(\mathcal{O}_S), i)$ is an $\mathcal{O}_S$-lattice of V_K.*

PROOF. Follow the proof of Corollary 1 to Theorem 2 of [6; p. 17].

1.16. Let $\mu_0^\vee$ be the highest weight of the contragradient representation $\rho_K^\vee$ of ρ_K. We can prove that $\mu_0^\vee = -w_0\mu_0$. Hence $\langle\ \rangle$ defines a complete pairing between $w_0V_K(\mu_0)$ and $V_K^\vee(\mu_0^\vee)$. Let $\mathcal{V}_0^\vee$ be the dual $\mathcal{O}_S$-module of $w_0\mathcal{V}_0$. Then $\mathcal{V}_0^\vee$ is naturally a subsheaf of $V_K^\vee(\mu_0^\vee)$ and a locally free $\mathcal{O}_S$-module of rank 1. Thus we can define $(V_{\min}^\vee(\mathcal{O}_S), i^\vee)$ in the same way as above. Let $(V_{\max}(\mathcal{O}_S), i)$ be its dual lattice. Then $V_{\min}(\mathcal{O}_S)$ can be naturally considered as a submodule of $V_{\max}(\mathcal{O}_S)$ and $V_{\min,\mu_0}(\mathcal{O}_S) = V_{\max,\mu_0}(\mathcal{O}_S) = \mathcal{V}_0$. The graded $\mathcal{O}_S$-algebra structure of $\mathfrak{U}_S$ is inherited by graded $\mathfrak{U}_S$-module structures on $V_{\min}(\mathcal{O}_S)$ and $V_{\max}(\mathcal{O}_S)$. The inclusion $V_{\min}(\mathcal{O}_S) \subset V_{\max}(\mathcal{O}_S)$ is compatible with these structures.

If we need to state clearly the dependence on $\mathcal{V}_0$. we write $V_{\min}(\mathcal{O}_S;\mathcal{V}_0)$ (resp. $V_{\max}(\mathcal{O}_S;\mathcal{V}_0)$) for $V_{\min}(\mathcal{O}_S)$ (resp. $V_{\max}(\mathcal{O}_S)$). If $\mathcal{V}_0$ is a trivial $\mathcal{O}_S$-module and generated by a global section v_0, we write $V_{\min}(\mathcal{O}_S;v_0)$ (resp. $V_{\max}(\mathcal{O}_S;v_0)$) for $V_{\min}(\mathcal{O}_S;\mathcal{V}_0)$ (resp. $V_{\max}(\mathcal{O}_S;\mathcal{V}_0)$).

1.17. An S-form of V_K is, by definition, an $\mathcal{O}_S$-lattice $(V(\mathcal{O}_S), i)$ of V_K, equipped with a graded $\mathfrak{U}_S$-module structure which is compatible with the graded $\mathfrak{U}_K$-module structure of V_K. Let ξ be an element of $H^1(S, \mathcal{O}_S^\times)$. An S-form $(V(\mathcal{O}_S), i)$ is of type ξ, if $V_{\mu_0}(\mathcal{O}_S)$ is an invertible sheaf whose cohomology class is ξ.

Let us fix an invertible $\mathcal{O}_S$-submodule $\mathcal{V}_0$ of $V_K(\mu_0)$ whose cohomology class is ξ. We sometimes write $V_{\min}(\mathcal{O}_S;\xi)$ (resp. $V_{\max}(\mathcal{O}_S;\xi)$) for $V_{\min}(\mathcal{O}_S;\mathcal{V}_0)$ (resp. $V_{\max}(\mathcal{O}_S;\mathcal{V}_0)$).

1.18. LEMMA. *Let $V(\mathcal{O}_S)$ be a graded $\mathfrak{U}_S$-submodule of $V_{\max}(\mathcal{O}_S)$ which is locally free $\mathcal{O}_S$-module and contains $V_{\min}(\mathcal{O}_S)$. Denote by i the composition*

$$V(\mathcal{O}_S)_{\bar\eta} \longrightarrow V_{\max}(\mathcal{O}_S)_{\bar\eta} \xrightarrow{\ \sim\ } V_K.$$

Then $(V(\mathcal{O}_S), i)$ is an S-form of V_K and every S-form (up to isomorphims) of V_K of type ξ can be obtained uniquely in this way.

PROOF. If $(V(\mathcal{O}_S), i)$ is an S-form of type ξ, then, multiplying by a scalar if necessary, we may assume that $V_{\mu_0}(\mathcal{O}_S)=\mathcal{O}v_0$. Then $V(\mathcal{O}_S)\supset V_{\min}(\mathcal{O}_S)$ and $V^\vee(\mathcal{O}_S)\supset V^\vee_{\min}(\mathcal{O}_S)$, where $V^\vee(\mathcal{O}_S)$ is the dual $\mathcal{O}_S$-module of $V(\mathcal{O}_S)$. Hence $V_{\max}(\mathcal{O}_S)\supset V(\mathcal{O}_S)\supset V_{\min}(\mathcal{O}_S)$. The remaining is clear.

1.19. PROPOSITION. (1) *The correspondence*

$$\phi: V(\mathcal{O}_S)\longmapsto V_{\mu_0}(\mathcal{O}_S)$$

defines a surjective mapping of the set of isomorphims classes of S-forms of V_K *onto* $\mathrm{H}^1(S, \mathcal{O}_S^\times)$.

(2) *Let* $\xi\in\mathrm{H}^1(S, \mathcal{O}_S^\times)$. *There is a one-to-one correspondence between the set* $\phi^{-1}(\xi)$ *and the set of graded* $\mathfrak{U}_S$*-submodules of* $V_{\max}(\mathcal{O}_S;\xi)$ *which is a locally free* $\mathcal{O}_S$*-module and contains* $V_{\min}(\mathcal{O}_S;\xi)$.

(3) *For any two cohomology classes* $\xi_1, \xi_2\in\mathrm{H}^1(S, \mathcal{O}_S^\times)$, *there is a one-to-one correspondence* $\phi^{-1}(\xi_1)\simeq\phi^{-1}(\xi_2)$.

PROOF. The first and second parts are already proved. Let us prove the last part. Assume that $\xi\in\mathrm{H}^1(S, \mathcal{O}_S^\times)$ and

$$V_{\max}(\mathcal{O}_S;\xi)\supset V(\mathcal{O}_S)\supset V_{\min}(\mathcal{O}_S;\xi).$$

Then there is an open covering $\{U_\alpha\}$ of S such that $\xi|_{U_\alpha}$ is trivial for every α. Then $V_\mu(\mathcal{O}_S)$ has a section v_α on U_α which does not intersect the zero section. There is a unique graded $\mathfrak{U}_S$-automorphism $\phi_{\alpha\beta}$ of $V(\mathcal{O}_S)|_{U_\alpha\cap U_\beta}$ such that

$$\phi_{\alpha\beta}(v_\beta|_{U_\alpha\cap U_\beta})=v_\alpha|_{U_\alpha\cap U_\beta}.$$

If we patch the $\mathfrak{U}_S$-modules $\{V(\mathcal{O}_S)|_{U_\alpha}\}$ according to the patching data $\{\phi_{\alpha\beta}\}$, we get a graded $\mathfrak{U}_S$-module $V_0(\mathcal{O}_S)$ such that

$$V_{\max}(\mathcal{O}_S;\xi_0)\supset V_0(\mathcal{O}_S)\supset V_{\min}(\mathcal{O}_S;\xi_0)$$

and which is a locally free $\mathcal{O}_S$-module. Here ξ_0 is the trivial class of $\mathrm{H}^1(S, \mathcal{O}_S^\times)$. Thus we get a correspondence $\phi^{-1}(\xi)\to\phi^{-1}(\xi_0)$, which is clearly bijective.

2. Vector bundles.

To treat representations of general reductive groups, we need to show that a quasi-coherent $\mathcal{O}_S$-module of finite type is étale locally free if and only if it is Zariski locally free.

2.1. LEMMA. *Let A and B be local rings, $A\to B$ a local homomorphism, M an A-module and $N=M\otimes_A B$. If B is a finite étale over A and N is a free B-module of finite type, then M is a free A-module of finite type.*

PROOF. If is known that B is a free A-module of finite type [3; Chapter 4, 18.2.3]. Hence, if we ragard N as an A-module, it is free of finite type. If

$$B=Ax_1\oplus\cdots\oplus Ax_n,$$

then 1 ($\in B$) can be expressed as $1=a_1x_1+\cdots+a_nx_n$ ($a_i\in A$). Let $R(A)$ (resp. $R(B)$) be the radical of A (resp. B). Since $1\notin R(B)$, at least one of a_i is not contained in $R(A)$. If $a_1\notin R(A)$, $\{1, x_2, \cdots, x_n\}$ is a free A-basis of B. Hence

$$0\longrightarrow A\longrightarrow B\longrightarrow Ax_2\oplus\cdots\oplus Ax_n\longrightarrow 0$$

and

$$0=\mathrm{Tor}_1^A(M, Ax_2\oplus\cdots\oplus Ax_n)\longrightarrow M\longrightarrow N$$

are exact. Since N is a free A-module of finite type and A is a local ring, M is also a free A-module of finite type.

2.2. LEMMA. *Let $f: S'\to S$ be a surjective, étale morphism and $\mathcal{V}$ an $\mathcal{O}_S$-module such that $f^*\mathcal{V}$ is a free $\mathcal{O}_S$-module of finite type. Then $\mathcal{V}$ is a locally free $\mathcal{O}_S$-module of finite type.*

PROOF. Let $x\in S$, y be a point of S' sucn that $f(y)=x$, $A=\mathcal{O}_{S,x}$ and $B=\mathcal{O}_{S',y}$. We may assume that $S=\mathrm{Spec}\,A$ and $S'=\mathrm{Spec}\,B$. By a descent argument [4], we can show that $\mathcal{V}$ is quasi-coherent. Thus we have reduced the proof to (2.1).

3. Representations of reductive group schemes.

The purpose of this section is to describe the irreducible representations of reductive group schemes over normal, locally noetherian schemes. The main result of this section is (3.10).

3.1. Let S be a scheme.

DEFINITION. Let G be a group scheme over S. If G is affine and smooth over S, and its geometric fibres are all connected and reductive, then G is said to be reductive.

3.2. Let $\mathcal{R}=(M, M^\vee, R, R^\vee, R_0)$ be a reduced root datum with an épinglage (i.e., donnée radicielle reduite épinglée). In other words, M is a $\boldsymbol{Z}$-lattice, $M^\vee$ is the dual lattice of M, R is a (reduced) root system which is contained in M, $R^\vee$ is the dual root system of R and R_0 is a basis of R. Assume that a reduced root datum $\mathcal{R}$ is given.

3.3. DEFINITION ([Exp. 23, 1.1]). Let us consider a family $e=(i, (X_r)_{r\in R_0})$ of

(i) an isomorphism i of Spec $\mathcal{O}_S M$ onto a maximal torus T of G, where $\mathcal{O}_S M$ is the group ring of the additive group M, such that R (resp. $R^\vee$) is identified with the root system (resp. the coroot system) of G with respect to T via this isomorphism i,

and

(ii) a section $X_r \in \Gamma(S, \mathfrak{G}^r)^\times$ for each root subspace $\mathfrak{G}^r$ ($r\in R_0$) of $\mathfrak{G}=\mathrm{Lie}(G)$, where $\Gamma(S, \mathfrak{G}^r)^\times$ is the set of global sections which does not intersect the zero section.

Such a family e is called an épinglage of G of type $\mathcal{R}$.

3.4. The followings are known:

(1) If $(\mathcal{R}, e)$ and $(\mathcal{R}, e')$ are two épinglages of a group scheme G over S, then there exists a unique inner automorphism of G over S which transforms the former épinglage to the latter [Exp. 24, 1.5].

(2) A reductive group shceme G over S has an épinglage of type $\mathcal{R}$, if and only if it is split and of type $\mathcal{R}$ [Exp. 22, 2.7].

(3) Assume further that S is locally noetherian and normal. For any point s of S, there exists an open set U of S containing s and a surjective finite étale morphism $S'\to U$ such that $G_{S'}=G\times_S S'$ is split [Exp. 24, 4.1.6].

3.5. Let (G, ρ, V) be a triple of a reductive group scheme G over S, a vector bundle V over S and a homomorphism $\rho: G\to GL(V)$. We call such a triple a representation of G. If there is no fear of confusion, we refer to (ρ, V), ρ or V as a representation of G. We also refer to V as a G-module. If S is irreducible and $\bar{\eta}$ is a generic geometric point of S, then we say that (G, ρ, V) is an S-form of $(G_{\bar{\eta}}, \rho_{\bar{\eta}}, V_{\bar{\eta}})$. If $S=\mathrm{Spec}\,A$, we call it an A-form.

A representation (G, ρ, V) is said to be split if G is split and $V_{\mu_0}(\mathcal{O}_S)$ is isomorphic to $\mathcal{O}_S$ as an $\mathcal{O}_S$-module.

Let (G, ρ, V) and (G', ρ', V') be two representations. A homomorphism of (G, ρ, V) to (G', ρ', V') is, by definition, a pair (ϕ, ψ) of a homomorphism $\phi: G\to G'$ and a morphism $\psi: V\to V'$ of vector bundles which are compatible.

Assume that S is an irreducible scheme. Let η be the generic point of S, K be an algebraically closed field which contains the residue field at η and $\bar{\eta}$ be the geometric point Spec $\bar{K}\to S$. We say that ρ is (absolutely) irreducible if $\rho_{\bar{\eta}}: G_{\bar{\eta}}\to GL(V_{\bar{\eta}})$ is irreducible. Hereafter, we assume that ρ is irreducible. We say that ρ is an S-form of $\rho_{\bar{\eta}}$. If the representation ρ is split, we say that ρ is a split S-form of $\rho_{\bar{\eta}}$.

3.6. Consider a triple $\varepsilon=(\mathcal{R}, \boldsymbol{e}, v_0)$ of a reduced root datum with an épinglage $\mathcal{R}$, an épinglage $\boldsymbol{e}$ of G of type $\mathcal{R}$ and a global section $v_0 \in \Gamma(S, V_{\mu_0}(\mathcal{O}_S))^{\times}$. Such a triple $\varepsilon=(\mathcal{R}, \boldsymbol{e}, v_0)$ is called an épinglage of the representation (G, ρ, V).

The following lemma can be obtained from (3.4).

3.7. LEMMA. (1) *A representation has an épinglage if and only if it is split.*

(2) *Given two épinglages of a representation, there exists a unique automorphism of the representation which transforms one épinglage to another épinglage and induces the identity mapping on M.*

3.8. Hereafter, we shall assume that S is irreducible, normal and locally noetherian. In that case, (G, ρ, V) has étale locally an épinglage. Let $\mathcal{S}$ be the totality of étale neighbourhoods S' such that $\rho \times_S S'$ is split. Fix an épinglage for each $\rho \times_S S'$ $(S' \in \mathcal{S})$. If S_α, $S_\beta \in \mathcal{S}$, there are two épinglages ε_α and ε_β of $\rho \times_S (S_\alpha \times_S S_\beta)$ which come from $\rho \times_S S_\alpha$ and $\rho \times_S S_\beta$ respectively. By (3.7), there is a unique automorphism $(\phi_{\alpha\beta}, \psi_{\alpha\beta})$ of $\rho \times_S (S_\alpha \times_S S_\beta)$ which transforms ε_β to ε_α. By a usual descent argument, we can show that $\rho \times_S S_\alpha$ $(S_\alpha \in \mathcal{S})$ can be patched together according to the patching data $\{(\phi_{\alpha\beta}, \psi_{\alpha\beta})\}$ and give a split representation (G_0, ρ_0, V_0). (See [4] for the descent.) Here we used the results of section 2.

If ρ has an épinglage, we can show that the patching data $\{(\phi_{\alpha\beta}, \psi_{\alpha\beta})\}$ is a coboundary. Hence ρ is isomorphic to ρ_0. Hence every irreducible representation can be uniquely obtained by twisting a split irreducible representation ρ_0 by using $\mathrm{H}^1(S, \mathcal{A}ut\, \rho_0)$. Here $\mathcal{A}ut\, \rho_0$ is the étale sheaf

$$S' \longmapsto \mathrm{Aut}(\rho_0 \times_S S'),$$

i.e., the étale sheaf represented by $\mathrm{Aut}\, \rho_0$. Let us restate our results.

3.9. Let S be an irreducible, normal, locally noetherian scheme and $\bar{\eta}: \mathrm{Spec}\, K \to S$ its generic geometric point. Let (G_K, ρ_K, V_K) be an irreducible representation of a reductive algebraic group with an épinglage $(\mathcal{R}, \boldsymbol{e}, v_0)$. Let

$\mathcal{F}$ = the set of isomorphism classes of S-forms of ρ_K (see (1.17) for an S-form)

$\mathcal{F}_0$ = the set of isomorphism classes of split S-forms of ρ_K

$\mathcal{F}_1$ = the set of graded $\mathfrak{U}_S$-modules $V(\mathcal{O}_S)$ which are locally free $\mathcal{O}_S$-modules and

$$V_{\min}(\mathcal{O}_S\,;\,v_0)\subset V(\mathcal{O}_S)\subset V_{\max}(\mathcal{O}_S\,;\,v_0)\,.$$

See (1.14)-(1.16), for $V_{\min}(\mathcal{O}_S\,;\,v_0)$ and $V_{\max}(\mathcal{O}_S\,;\,v_0)$. By (3.8), we can define a mapping

$$\Phi:\mathfrak{F}\longrightarrow\mathfrak{F}_0$$

such that $\Phi|_{\mathfrak{F}_0}$ is the identity mapping.

3.10. Under the notations of (3.9), we have the following theorem.

MAIN THEOREM. (1) *There is a bijection* $\mathfrak{F}_1\xrightarrow{\sim}\mathfrak{F}_0$.

(2) *For each* $\rho\in\mathfrak{F}_0$, *there is a bijection* $\mathrm{H}^1(S,\,\mathcal{A}ut\,\rho)\xrightarrow{\sim}\Phi^{-1}(\rho)$.

(The first assertion is a restatement of the second part of (1.19) with $\xi=0$. The second assertion has been proved in (3.8).)

3.11. REMARK. If $S=\operatorname{Spec}A$, we define $V_{\min}(A)$ and $V_{\max}(A)$ to be the set of global sections of $V_{\min}(\mathcal{O}_S)$ and $V_{\max}(\mathcal{O}_S)$, respectively. It is known that the category of quasi-coherent $\mathcal{O}_S$-modules and that of A-modules are equivalent. Hence to give an element $V(\mathcal{O}_S)$ of $\mathfrak{F}_1$ is equivalent to give a graded $(\mathfrak{U}_Z\otimes A)$-module $V(A)$ which is a projective A-module and

$$V_{\min}(A)\subset V(A)\subset V_{\max}(A)\,.$$

3.12. REMARK. If $A=k$ is a field, then $V_{\min}(k)=V_{\max}(k)$. Hence the choice of $V(k)$ is unique. Hence there is a one-to-one correspondence

$$\mathfrak{F}\xrightarrow{\sim}\mathrm{H}^1(\mathrm{Gal}\,(k_{\mathrm{sep}}/k),\,\mathrm{Aut}\,(\rho\otimes k_{\mathrm{sep}}))\,,$$

where k_{sep} is a separable closure of k.

4. Automorphism group of a representation.

In the statement of the Main Theorem, we have met with the sheaf $\mathcal{A}ut\,\rho_0$. To determine this sheaf, it suffices to determine $\mathcal{A}ut\,\rho$ for every split irreducible representation ρ, which is our purpose of this section. Our result of this section is (4.3.1) and (4.3.2).

4.1. Let S be a reduced, irreducible scheme, G a split reductive group scheme over S and (ρ,V) a split irreducible representation of G. Let (ϕ,ψ) be an automorphism of the representation (G,ρ,V). Then $\rho\circ\phi$ is isomorphic to ρ. Let us fix an épinglage $(\mathfrak{R},e)$ of G. By [Exp. 24, 1.3], ϕ is uniquely expressed as $\phi_2\phi_1$, where ϕ_1 is an automorphism of $(G,\mathfrak{R},e)$ and ϕ_2 is an inner automorphism [Exp. 24, 1.1]. Then $\rho\circ\phi_1$ is locally isomorphic to $\rho\circ\phi$ for the *fpqc* topology. Hence ϕ_1 fixes the highest weight μ_0 of ρ.

Conversely, assume that an automorphism ϕ_1 of $(G, \mathfrak{R}, e)$ fixes μ_0. Since ϕ_1 preserves $\{X_r\}_{r\in R_0}$, ϕ_1 also preserves the $\mathcal{O}_S$-algebra of operators on $V(\mathcal{O}_S)$ generated by $X_r^{[m]}$ $(r\in R,\ m=0, 1, 2, \cdots)$. (See (1.9).) Hence $\rho\circ\phi_1$ corresponds to the same graded $\mathfrak{U}_S$-module as ρ. Hence $\rho\circ\phi_1$ is isomorphic to ρ. If $\phi=\phi_2\phi_1$ with an inner automorphism ϕ_2, $\rho\circ\phi$ is also locally isomorphic to ρ for the *fpqc*-topology. Hence for each point s of S, we can find an *fpqc*-neighbourhood S' of s and an automorphism ψ of $V\times_S S'$ such that $(\phi\times_S S', \psi)$ is an automorphism of $\rho\times_S S'$. This automorphism ψ is uniquely determined by ϕ up to homothety.

4.2. Let H be the sheaf theoretical image of $\mathrm{Aut}\,\rho\rightarrow\mathrm{Aut}\,G$. Here we identify a scheme on S with the *fpqc*-sheaf represented by it. Define a homomorphism $G_m\rightarrow\mathrm{Aut}\,\rho$ by

$$c\longmapsto(\text{identity, multiplication by } c).$$

The automorphism group of G can be expressed as a semi-direct product

$$\mathrm{Aut}\,G=\mathrm{ad}(G)\rtimes\mathrm{Aut}(G,\ \mathfrak{R},\ e).$$

4.3. Using the notations of (4.2), results of (4.1) can be stated as follows:

(4.3.1) $$1\longrightarrow G_m\longrightarrow\mathrm{Aut}\,\rho\longrightarrow H\longrightarrow 1$$

is exact.

(4.3.2) $$H=\mathrm{ad}(G)\rtimes\mathrm{Aut}(G,\ \mathfrak{R},\ e,\ \mu_0).$$

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