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ON CONDUCTOR OVERRINGS OF A VALUATION DOMAIN

By

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Introduction. It is well known that every overring of a valuation domain V is of the form V_P for some prime ideal P of V. Hence, if I is an ideal of a valuation domain V with quotient field K, then the conductor overring $I:_K I$ is of the form V_P for some prime ideal P of V. In case $I:_K I=V_P$, is there any relation between I and P? The main purpose of this paper is to investigate this relation. In order to give a complete answer to the question stated above, we introduce the notion of "*recurrent closure*": If I is an ideal of an integral domain R with quotient field K, then the ideal $R:_R(I:_K I)$ of R is called the "*recurrent closure*" of I and is denoted by I_r. We prove, in Theorem 13, that if I is an ideal of a valuation domain V with quotient field K such that $I:_K I\neq V$, then I_r is always a prime ideal of V and if we set $I:_K I=V_P$ for some prime ideal P of V, then P is equal to the recurrent closure I_r.

In general, our terminology and notation will be the same as [3] and [6]. Throughout the paper, V denotes a valuation domain, with quotient field K.

THEOREM 1. *If P is a proper prime ideal of V, then $P:_K P=V_P$. In particular, if M is the unique maximal ideal of V, then $M:_K M=V$.*

PROOF. If $P=(0)$, then $(0):_K(0)=K=V_{(0)}$(cf. [9, Remark 1.2]) and hence our assertion is trivial. Thus we may assume that $P\neq(0)$. Then, by [3, Theorem 17.3], $P(x)=P$ for any $x\in V\setminus P$ and accordingly $1/xP\subseteq P$. Thus $1/x\in P:_K P$ for any $x\in V\setminus P$. From this fact it follows that $V_P\subseteq P:_K P$. Hence, if we put $P:_K P=V_Q$ for some prime ideal Q of V, then we have $V_P\subseteq P:_K P=V_Q$ and so $Q\subseteq P$. Assume now that $Q\neq P$. Then $Q:_K P$ is a nonmaximal prime ideal of $P:_K P$ by [9, Corollary 2.4]. On the other hand, $Q=QV_Q$ is a maximal ideal of V_Q by [3, Theorem 17.6]. Since $Q\subseteq Q:_K P$, we have $Q=Q:_K P$ and therefore $Q:_K P$ is a maximal ideal of $P:_K P$, a contradiction. Hence we must have $Q=P$, and accordingly $P:_K P=V_P$ as desired. Thus our first assertion is proved. The second assertion follows immediately from the first one.

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Before proving the next theorem, we first establish the following lemma.

LEMMA 2. *Let R be an integral domain with quotient field K and let I be a proper ideal of R. If $I:_K I=R_P$ for some prime ideal P of R, then we have $I\subseteq P$.*

PROOF. Assume the contrary. Then we can choose an element $a\in I\setminus P$. Then, by hypothesis, $1/a\in R_P=I:_K I$ since $a\notin P$. Therefore we have $1=a\cdot 1/a\in I(I:_K I)\subseteq I$, which implies that $I=R$. This clearly contradicts our assumption.

THEOREM 3. *If Q is a primary ideal of V, then $Q:_K Q=V_{\sqrt{Q}}$.*

PROOF. If $Q=(0)$, then $(0):_K(0)=K=V_{\sqrt{(0)}}$ and hence our assertion is evident. Therefore we may assume that $Q\neq(0)$. If we set $Q:_K Q=V_P$ with some prime ideal P of V, then $Q\subseteq P$ and hence $\sqrt{Q}\subseteq P$. We shall next show that $P\subseteq\sqrt{Q}$. By [3, Theorem 17.3], $Q(x)=Q$ for any element $x\in V\setminus\sqrt{Q}$, and accordingly $1/x\in Q:_K Q$ for any $x\in V\setminus\sqrt{Q}$. Thus we have $V_{\sqrt{Q}}\subseteq Q:_K Q=V_P$ and hence $P\subseteq\sqrt{Q}$, as required. This completes the proof.

COROLLARY 4. *If Q is a primary ideal of V, then $Q:_K Q=\sqrt{Q}:_K\sqrt{Q}$.*

PROOF. This follows immediately from Theorem 1 and Theorem 3.

DEFINITION 5. Let R be an integral domain with quotient field K and let I be a proper ideal of R. Then the ideal $R:_R(I:_K I)$ of R is called the "*recurrent closure*" of I and is denoted by I_r. An ideal I of R is said to be "*recurrent*" in case $I=I_r$.

REMARK 6. If I is a recurrent ideal of an integral domain R with quotient field K, then $I:_K I\neq R$. For, if $I:_K I=R$, then $I=I_r=R:_R(I:_K I)=R:_R R=R$, a contradiction. Moreover, if M is a maximal ideal of R, then the converse of the above statement also holds. In fact, if $M:_K M\neq R$, then $M\subseteq R:_R(M:_K M)\subsetneqq R$ and hence $M=R:_R(M:_K M)$, since M is a maximal ideal of R. Therefore M is a recurrent ideal of R as required.

REMARK 7. If M is the unique maximal ideal of V, then M is not recurrent. By Theorem 1, $M:_K M=V$ and therefore our assertion follows from Remark 6.

We first collect some facts about recurrent ideals that will be needed later.

LEMMA 8. *Let R be an integral domain with quotient field K. If I is an ideal of R such that $I:_K I\neq R$, then $I\subseteq I_r$ and I_r itself is recurrent.*

PROOF. By definition the containment $I \subseteq I_r$ is evident. Next, we shall establish the second assertion. First it should be noted that I_r is an ideal of $I:_K I$ (cf. [9, Lemma 1.1(2)]). It follows from this fact that if $x \in I:_K I$ and $a \in I_r$, then $xa \in I_r$. Thus we have $I:_K I \subseteq I_r:_K I_r$. Therefore $I_r = R:_R(I:_K I) \supseteq R:_R(I_r:_K I_r) \supseteq I_r$, whence $I_r = R:_R(I_r:_K I_r) = (I_r)_r$, completing the proof.

LEMMA 9. *Let R be an integral domain with quotient field K and let I be a proper ideal of R. Then*

(1) *If P is a prime ideal of R contained in I, then $I:_K I \subseteq P:_K P$.*

(2) *If I is a recurrent ideal of R, then, for any prime ideal P of R, $P \subseteq I$ if and only if $I:_K I \subseteq P:_K P$.*

PROOF. (1) Let $x \in I:_K I$ and $p \in P$. Since $x^2 \in I:_K I$ and $p \in I$, $x^2 p \in (I:_K I)I \subseteq I$, and accordingly $(xp)^2 = (x^2 p)p \in IP \subseteq P$, which implies that $xp \in P$ because $xp \in I \subset R$. Thus $(I:_K I)P \subseteq P$ and hence $I:_K I \subseteq P:_K P$ as required.

(2) The "only if" half is proved in (1). Conversely, assume that $I:_K I \subseteq P:_K P$. Then P is an ideal of $I:_K I$, since $P(I:_K I) \subseteq P(P:_K P) \subseteq P$. Hence, by [9, Lemma 1.1 (4)], $P \subseteq R:_R(I:_K I) = I_r$. Then we have $P \subseteq I_r = I$ because I is, by hypothesis, recurrent. This completes the proof.

REMARK 10. The part (1) of Lemma 9 is also found in [1, Lemma 2.2] or in [2, Lemma 3.7].

LEMMA 11. *Let R be an integral domain with quotient field K and let I be a proper ideal of R. If P is a recurrent prime ideal of R properly contained in I, then $I:_K I \subsetneqq P:_K P$.*

PROOF. By part (1) of Lemma 9, we have $I:_K I \subseteq P:_K P$. Hence, it suffices to show that $I:_K I \neq P:_K P$. Assume that $I:_K I = P:_K P$. Then I is an ideal of $P:_K P$ and therefore, by [9, Lemma 1.1 (4)], $I \subseteq P_r$. By hypothesis, $P_r = P$ and hence $I \subseteq P$, the desired contradiction. This completes the proof.

In the proof of Lemma 8, we showed that if I is an ideal of an integral domain R with quotient field K, then $I:_K I \subseteq I_r:_K I_r$. If P is a prime ideal of R, then it can be shown that $P:_K P = P_r:_K P_r$.

THEOREM 12. *Let R be an integral domain with quotient field K. If P is a prime ideal of R, then we have $P:_K P = P_r:_K P_r$.*

PROOF. We have already shown in Lemma 8 that $P:_K P \subseteq P_r:_K P_r$. Hence, we need only prove the reverse containment $P_r :_K P_r \subseteq P :_K P$. If $P=P_r$, then there is nothing to prove. Therefore we may assume that $P \neq P_r$. If we choose $t \in P_r \setminus P$, then, for any $x \in P_r :_K P_r$, we have $xt \in P_r \subset R$. Then we have $xtp \in P$ for any $p \in P$. But, since $xp \in (P_r :_K P_r)P \subseteq P_r \subset R$ and $t \in R \setminus P$, $(xp)t \in P$ implies that $xp \in P$. Thus $P_r :_K P_r \subseteq P :_K P$ as desired and our proof is complete.

We are now in a position to prove the main theorem of this paper.

THEOREM 13. *Let V be a valuation domain with quotient field K. Then*

(1) *Every nonmaximal prime ideal P of V is recurrent.*

(2) *If I is an ideal of V such that $I :_K I \neq V$, then I_r is a prime ideal of V and we have $I :_K I = V_{I_r}$.*

(3) *If I is an ideal of V such that $I :_K I \neq V$, then $\sqrt{I} \subseteq I_r$.*

(4) *If Q is a primary ideal of V such that $\sqrt{Q}$ is not the unique maximal ideal M of V, then $\sqrt{Q} = Q_r$.*

PROOF. (1) First, by Theorem 1, $P :_K P = V_P \neq V$. Hence we get $P_r = V :_V (P :_K P) \neq V$. Indeed, if $P_r = V$ then $1 \in P_r$ and so $P :_K P \subseteq V$, a contradiction. Thus we get $P \subseteq P_r \neq V$. Next, by [9, Lemma 1.1 (2)], P_r is an ideal of $P :_K P = V_P$ and therefore $P_r \subseteq PV_P = P$. Accordingly, $P = P_r$, which implies that P is recurrent.

(2) By hypothesis, $I :_K I$ is a proper overring of V and so we can write $I :_K I = V_P$ with some nonmaximal prime ideal P of V. Since, by Theorem 1, $V_P = P :_K P$, it follows that $I :_K I = P :_K P$. Then we have $I_r = V :_V (I :_K I) = V :_V (P :_K P) = P$, since P is recurrent by (1). Thus, I_r is a prime ideal of V and moreover $I :_K I = V_{I_r}$ as required.

(3) Since $I \subseteq I_r$, we always have $\sqrt{I} \subseteq \sqrt{I_r}$. If $I :_K I \neq V$, then, by (2), I_r is prime and therefore $\sqrt{I} \subseteq \sqrt{I_r} = I_r$ as wanted.

(4) First, by Theorem 3, $Q :_K Q = V_{\sqrt{Q}}$. Moreover, $Q :_K Q \neq V$, since $\sqrt{Q}$ is not maximal. Hence, by (2), Q_r is prime and $Q :_K Q = V_{Q_r}$. Thus $V_{\sqrt{Q}} = V_{Q_r}$, and accordingly $\sqrt{Q} = Q_r$, completing the proof.

REMARK 14. Let R be an integral domain with quotient field K and let $P \subset I$ be ideals of R with P prime. Then we cannot in general expect that P is also prime in $I :_K I$. To show this, we shall give the following example.

EXAMPLE 15. Let $R = \boldsymbol{Z}[2X, X^2, X^3]$ be the subdomain of $T = \boldsymbol{Z}[X]$, where X is an indeterminate over $\boldsymbol{Z}$. Then $K = \boldsymbol{Q}(X)$ is the quotient field of R. If we set $M = 2\boldsymbol{Z}R + 2XR + X^2R + X^3R$, then $R/M = \boldsymbol{Z}/2\boldsymbol{Z}$ is a field and so M is a maximal ideal of R. Moreover, it is easy to see that $M :_K M = \boldsymbol{Z}[X]$. If we put $P = 2XR$

$+X^2R+X^3R$, then, since $R/P=\boldsymbol{Z}$, P is a prime ideal of R properly contained in M. But P is not a prime ideal of $M:_K M$, because $3X\in \boldsymbol{Z}[X]\setminus P$, but $(3X)^2\in P$.

COROLLARY 16. *If $P\subset I$ are ideals of V with P prime, then P is also prime in $I:_K I$ and $P=P:_K I$.*

PROOF. If $I:_K I=V$, then there is nothing to prove. Hence we may assume that $I:_K I\neq V$. Then, by Theorem 13 (2), $I:_K I=V_{I_r}$ and I_r is a prime ideal of V. Hence, by [3, Theorem 17.6 (b)], $P=PV_{I_r}$ is a prime ideal of V_{I_r}, since $P\subset I\subseteq I_r$. Thus, P is a prime ideal of $I:_K I$. Our second assertion follows then from [9, Corollary 1.5].

We close this paper with a characterization of primary ideals Q of V such that $Q:_K Q\neq V$.

We first prepare the following two lemmas.

LEMMA 17. *Let Q be a primary ideal of V. Then $Q:_K Q\neq V$ if and only if $\sqrt{Q}$ is not the unique maximal ideal of V.*

PROOF. Let M be the unique maximal ideal of V. First, suppose that $\sqrt{Q}=M$. Then, by Theorem 3, $Q:_K Q=V_{\sqrt{Q}}=V_M=V$. Thus, the "only if" half is proved. Conversely, suppose that $Q:_K Q=V$. Then, also by Theorem 3, $V=Q:_K Q=V_{\sqrt{Q}}$, and so $\sqrt{Q}=M$. Hence, the "if" half is also proved.

LEMMA 18. *Let I be a nonzero ideal of an integral domain R with quotient field K. Then, for any $x\in I:_K I$, x is a unit of $I:_K I$ if and only if $xI=I$.*

PROOF. First, assume that x is a unit of $I:_K I$. Then there is an element $y\in I:_K I$ such that $xy=1$. Then, $I=(xy)I=x(yI)\subseteq xI\subseteq I$, and so $I=xI$, as we required. Conversely, suppose that $I=xI$. Since $I\neq(0)$, x is a nonzero element of K, and so $x^{-1}\in K$. Hence, by hypothesis, $x^{-1}I=x^{-1}(xI)=(x^{-1}x)I=I$, and so $x^{-1}\in I:_K I$, which implies that x is a unit of $I:_K I$. This completes the proof.

THEOREM 19. *Let I be an ideal of V such that $I:_K I\neq V$. Then I is a primary ideal of V if and only if $\sqrt{I}=I_r$.*

PROOF. The "only if" half is proved in part (4) of Theorem 13. To prove the "if" half, suppose that I is not a primary ideal of V. By part (2) of Theorem 13, $I:_K I=V_{I_r}$, and therefore, to prove that $\sqrt{I}\neq I_r$, it suffices to show that $I:_K I\neq V_{\sqrt{I}}$. Now, since I is not primary, there exist $a, b\in V$ such that $a\notin I$, $b\notin\sqrt{I}$, but $ab \in I$. Then $b \notin \sqrt{I}$ implies that $I \subset (b)$, since V is a valuation domain. Then, since (b) is invertible, there exists an ideal J of V such that $I=J(b)$. Therefore, by hypothesis, $ab \in I=J(b)$, and so $a \in J$. Since $a \in J \setminus I$, $I=J(b) \subset J$ and therefore $bI=(b)I=J(b^2) \subset J(b)=I$. Thus, $bI \subset I$ and therefore it follows from Lemma 18 that b is not a unit of $I:_K I$. On the other hand, b is a unit of $V_{\sqrt{I}}$, since $b \notin \sqrt{I}$. Therefore $I:_K I \neq V_{\sqrt{I}}$, as we wanted and hence our proof is complete.

REMARK 20. If I is an ideal of V such that $I:_K I \neq V$, then $\sqrt{I}$ is not maximal in V. For, if $\sqrt{I}$ is maximal, then, by part (3) of Theorem 13, I_r is also maximal in V and therefore, by part (2) of Theorem 13, $I:_K I=V_{I_r}=V$, a contradiction.

COROLLARY 21. *Let I be an ideal of V such that $I:_K I \neq V$. Then I is recurrent if and only if I is prime.*

PROOF. First, assume that I is prime in V. Then it follows from Theorem 1 that I is not maximal in V, since $I:_K I \neq V$. Therefore the "if" half follows from part (1) of Theorem 13. Furthermore, the "only if" half follows immediately from part (2) of Theorem 13.

References

[1] Anderson, D. F., Comparability of ideals and valuation overrings, Houston J. Math., **5** (1979), 451-463.
[2] Fossum, R. M., The Divisor Class Group of a Krull domain, Springer-Verlag New York, 1973.
[3] Gilmer, R., Multiplicative Ideal Theory, Marcel Dekker, Inc., New York, 1972.
[4] ———, Multiplicative Ideal Theory, Queen's Papers in Pure and Applied Mathematics, No. 12, Queen's Univ. Press, Kingston, Ontario, 1968.
[5] Huckaba, J. A. and Papick, I. J., When the dual of an ideal is a ring, Manuscripta Math., **37** (1982), 67-85.
[6] Kaplansky, I., Commutative Rings, Allyn. and Bacon, Inc., Boston, 1970.
[7] Larsen, M. D. and McCarthy, P. J., Multiplicative thory of ideals, Academic Press, New York and London, 1971.
[8] Nagata, M., A treatise on the 14-th problem of Hilbert, Mem. Coll. Sci. Univ. Kyoto Ser. A Math., **30** (1956-57), 57-70.
[9] Okabe, A., On conductor overrings of an integral domain, preprint.