TSUKUBA J. MATH.
Vol. 18 No. 1 (1994), 217–221

A NOTE ON QUOTIENT SPACES OF SUPERCOMPACT SPACES

By

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Abstract A space is called supercompact if it has an open subbase such that every cover consisting of elements of the subbase has a subcover consisting of two elements. In this paper we prove that the quotient space of a supercompact space obtained by identifying a finite set or a closed G_δ-set to a point is also supercompact thus answering a question of M. G. Bell.

AMS Subj. Class. 54D30
Key words. supercompact, quotient.

1. Introduction.

All spaces in this paper are assumed to be Hausdorff. *Supercompact spaces,* introduced by de Groot [4], are spaces X which possess an open subbase $\mathcal{G}$ such that every cover of X consisting of members of $\mathcal{G}$ has a subcover of at most 2 members. For our purposes it is more elegant to work with closed subbase. A collection of sets $\mathcal{G}$ is *linked* if every 2 members of $\mathcal{G}$ has a non-empty intersection. A collection of sets $\mathcal{G}$ is *binary* if every linked subcollection of $\mathcal{G}$ has a non-empty intersection. So, X is supercompact if and only if it has a binary closed subbase.

Many compact spaces, but not all, are supercompact. For example, all compact metric spaces are supercompact [3, 6]; all continuous images of compact ordered spaces are supercompact [2]. On the other hand, the author recently proved that every cluster point of a countable subset of a supercompact space is the limit of a nontrivial sequence [7]; therefore there exist many non-supercompact compact spaces. In 1990, Bell [1] gave a negative answer for the question of whether all dyadic spaces ($=$continuous images of 2^κ) are supercompact. In fact, Bell proved that there exists a supercompact subset $A\subset 2^{\omega_3}$ such that the quotient space obtained by identifying A to a point is not

Received January 27, 1993.

supercompact. Thus the question of whether the quotient space of a supercompact space obtained by identifying a finite set to a point is supercompact was raised [1]. Bell himself proved that it is true if X is zero-dimensional. In the present paper we give a positive answer to this question. Moreover we show that it is also true if a finite set is replaced by a closed G_δ-set. We finally want to note that there exists a non-supercompact space which is an at most 2 to 1 irreducible image of a supercompact space [5].

2. Preliminaries.

It is trivial that the smallest collection which contains a binary collection and is closed with respect to arbitrary intersections is also binary. Thus we assume throughout this paper that all closed subbase are closed with respect to arbitrary intersections. For a collection $\mathcal{G}$ of sets let

$$\mathcal{G}^{<\omega}=\{\mathcal{A}\subset\mathcal{G} : \mathcal{A} \text{ is finite}\}.$$

The following Lemma 1 has a short proof which has been mentioned in many papers [e.g., 1, 3 and 7].

LEMMA 1. *Let X be a compact space and $\mathcal{G}$ a closed subbase for X. Then for every closed set F and every open set $U\supset F$ there exists $\mathcal{A}\in\mathcal{G}^{<\omega}$ such that $F\subset\bigcup\mathcal{A}\subset U$.*

LEMMA 2. *Under the assumptions of Lemma 1, there exists $\mathcal{A}\in\mathcal{G}^{<\omega}$ such that $F\cap S\neq\emptyset$ for every $S\in\mathcal{A}$ and*

$$F\subset\operatorname{int}(\bigcup\mathcal{A})\subset\bigcup\mathcal{A}\subset U.$$

PROOF. By the normality of X, there exists an open set V such that $F\subset V\subset\overline{V}\subset U$. It follows from Lemma 1 that there exists $\mathcal{B}\in\mathcal{G}^{<\omega}$ such that $\overline{V}\subset\bigcup\mathcal{B}\subset U$. Then $\mathcal{A}=\{S\in\mathcal{B} : S\cap F\neq\emptyset\}$ satisfies the required conditions.

LEMMA 3. *Let X be a compact space and $\mathcal{G}$ a closed subbase for X. Then for every $S\in\mathcal{G}$ and any closed sets E, $F\subset X$ with $E\cap F=\emptyset$ there exists $\mathcal{A}\in\mathcal{G}^{<\omega}$ such that $S=\bigcup\mathcal{A}$ and either $T\cap E=\emptyset$ or $T\cap F=\emptyset$ for each $T\in\mathcal{A}$.*

PROOF. Because $E\cap F=\emptyset$, there exists an open set U such that $E\subset U\subset\overline{U}\subset X\setminus F$. Thus, by Lemma 1, there exists $\mathcal{A}_1$, $\mathcal{A}_2\in\mathcal{G}^{<\omega}$ such that

$$\overline{U}\subset\bigcup\mathcal{A}_1\subset X\setminus F$$

and

$$S\setminus U\subset\bigcup\mathcal{A}_2\subset X\setminus E.$$

Then $\mathcal{A}=\{T\cap S:\ T\in\mathcal{A}_1\cup\mathcal{A}_2\}$ satisfies the required conditions.

3. Results

Let X be a space and A a closed subset of X. By X MOD A we denote the quotient space of X obtained by identifying A to a point and by $\pi: X\to X$ MOD A the quotient map.

THEOREM 1. *If X is a supercompact space and A is a finite subset of X, then X MOD A is also supercompact.*

PROOF. By induction, it suffices to prove our theorem in case $A=\{p, q\}$. Let $\mathcal{S}$ be a binary closed subbase for X. It follows from Lemma 2 that there exist $\mathcal{P}, \mathcal{Q}\in\mathcal{S}^{<\omega}$ such that

$$p\in\cap\mathcal{P}\cap\mathrm{int}(\cup\mathcal{P});$$

$$q\in\cap\mathcal{Q}\cap\mathrm{int}(\cup\mathcal{Q})$$

and

$$(\cup\mathcal{P})\cap(\cup\mathcal{Q})=\emptyset\,.$$

Let

$$\mathcal{S}_1=\{S\in\mathcal{S}:\ S\cap A\neq\emptyset \text{ and either } S\subset\cup\mathcal{P} \text{ or } S\subset\cup\mathcal{Q}\};$$

$$\mathcal{S}_2=\{S\in\mathcal{S}:\ S\cap A=\emptyset \text{ and either } S\cap\mathcal{P}=\emptyset \text{ or } S\cap\cup\mathcal{Q}=\emptyset\}$$

and

$$\mathcal{S}_0=\mathcal{S}_1\cup\mathcal{S}_2\,.$$

To complete the proof, we only have to check that $\pi(\mathcal{S}_0)$ is a binary closed subbase for X MOD A. Lemma 3 implies that every $S\in\mathcal{S}$ satisfying $S\cap A=\emptyset$ can be represented as an union of finite elements of $\mathcal{S}_2$. It follows that

(A) for every open set $U\supset A$ there exists $\mathcal{A}\in\mathcal{S}_2{}^{<\omega}$ such that $X\setminus U\subset\cup\mathcal{A}\subset X\setminus A$;

(B) for every point $x\in X\setminus A$ and every open set $U\ni x$ with $\bar{U}\cap A=\emptyset$ there exists $\mathcal{A}\in\mathcal{S}_0{}^{<\omega}$ such that $X\setminus U\subset\cup\mathcal{A}\subset X\setminus\{x\}$.

In fact, (A) is obtained immediately. For (B), there exists $\mathcal{P}'\in\mathcal{S}^{<\omega}$ such that

$$p\in\cap\mathcal{P}'\cap\mathrm{int}(\cup\mathcal{P}')$$

and

$$\cup\mathcal{P}'\cap\bar{U}=\emptyset\,.$$

Thus

$$p\in\mathrm{int}(\cup\{P'\cap P:\ P'\in\mathcal{P}' \text{ and } P\in\mathcal{P}\})\,.$$

Clearly, $P'\cap P\in\mathcal{S}_1$ but $P'\cap P\not\ni x$ for all $P'\in\mathcal{P}'$ and all $P\in\mathcal{P}$. That is, there exists $\mathcal{A}_1\in\mathcal{S}_1{}^{<\omega}$ such that

$$p\in \mathrm{int}(\cup\mathcal{A}_1) \quad \text{but} \quad \cup\mathcal{A}_1\not\ni x. \tag{*}$$

Similarly, there exists $\mathcal{A}_2\in\mathcal{G}_1^{<\omega}$ such that

$$q\in \mathrm{int}(\cup\mathcal{A}_2) \quad \text{but} \quad \cup\mathcal{A}_2\not\ni x. \tag{**}$$

Moreover, Lemma 1 implies that there exists $\mathcal{A}_3\in\mathcal{G}_2^{<\omega}$ such that

$$X\setminus(U\cup \mathrm{int}(\cup\mathcal{A}_1)\cup \mathrm{int}(\cup\mathcal{A}_2))\subset\cup\mathcal{A}_3\subset X\setminus\{x, p, q\},$$

Thus it follows from (*) and (**) that $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3$ satisfies the required conditions in (B).

It is clear that (A) and (B) imply that $\pi(\mathcal{G}_0)$ is a closed subbase for X MOD A.

Finally, we show that $\pi(\mathcal{G}_0)$ is binary. Let $\mathcal{A}\subset\mathcal{G}_0$ such that $\pi(\mathcal{A})$ is linked. If $\mathcal{A}\subset\mathcal{G}_1$ or $\mathcal{A}\subset\mathcal{G}_2$, then it is trivial to verify that $\cap\pi(\mathcal{A})\neq\emptyset$. If $\mathcal{A}\not\subset\mathcal{G}_1$ and $\mathcal{A}\not\subset\mathcal{G}_2$, then either $S\cap\cup\mathcal{P}=\emptyset$ for every $S\in\mathcal{A}$ or $S\cap\cup\mathcal{D}=\emptyset$ for every $S\in\mathcal{A}$. Thus $\mathcal{A}$ is linked and hence $\cap\mathcal{A}\neq\emptyset$. So we have $\cap\pi(\mathcal{A})\neq\emptyset$.

THEOREM 2. *If X is a supercompact space and A is a closed G_δ-subset of X, then X MOD A is also supercompact.*

PROOF. Let $\mathcal{G}$ be a binary closed subbase for X. Then, by Lemma 2, there exists a sequence $\{\mathcal{B}_n: n=1, 2, \cdots\}$ in $\mathcal{G}^{<\omega}$ such that

(1) for every n and every $B\in\mathcal{B}_n$, $B\cap A\neq\emptyset$;

(2) for every n,

$$A\subset \mathrm{int}(\cup\mathcal{B}_{n+1})\subset\cup\mathcal{B}_{n+1}\subset \mathrm{int}(\cup\mathcal{B}_n);$$

(3) $A=\cap\{\cup\mathcal{B}_n: n=1, 2, \cdots\}$.

Let

$$\mathcal{G}_1=\cup\{\mathcal{B}_n: n=1, 2, \cdots\};$$

$$\mathcal{G}_2=\{S\in\mathcal{G}: S\cap A=\emptyset \text{ and for every pair } E, F\in\mathcal{G}_1,$$
$$\text{if } E\cap F=\emptyset \text{ then either } S\cap E=\emptyset \text{ or } S\cap F=\emptyset\}.$$

Now suppose that $S\in\mathcal{G}$ satisfies $S\cap A=\emptyset$. Then there exists n such that $S\cap(\cup\mathcal{B}_n)=\emptyset$. Let

$$\mathcal{B}=\cup\{\mathcal{B}_i: i=1, \cdots, n-1\}$$

and

$$\{(E^i, F^i): i=1, \cdots, k\}=\{(E, F)\in\mathcal{B}\times\mathcal{B}: E\cap F=\emptyset\}.$$

Then for every $i\leqq k$, Lemma 3 implies that there exist $S_1^i, \cdots, S_{m(i)}^i\in\mathcal{G}$ such that

$$S=S_1^i\cup\cdots\cup S_{m(i)}^i$$

and for each $j=1, \cdots, m(i)$,

$$S_j^i \cap E^i = \emptyset \quad \text{or} \quad S_j^i \cap F^i = \emptyset .$$

Thus

$$S = \cap \{S_1^i \cap \cdots \cap S_{m(i)}^i : i=1, \cdots, k\}$$
$$= \cup \{\cap \{S_{f(i)}^i : i=1, \cdots, k\} : f \in \prod_{i=1}^{k} \{1, \cdots, m(i)\}\}.$$

It is trivial that $\cap \{S_{f(i)}^i : i=1, \cdots, k\} \in \mathcal{G}_2$ for every $f \in \prod_{i=1}^k \{1, \cdots, m(i)\}$. Hence, for every $S \in \mathcal{G}$, if $S \cap A = \emptyset$, then S can be represented as a union of finite elements of $\mathcal{G}_2$. Then, using the same method as Theorem 1, we can show that $\pi(\mathcal{G}_1 \cup \mathcal{G}_2)$ is a binary closed subbase for X MOD A.

Acknowledgement. I would like to express my gratitude to Professor R. Nakagawa for his directions.

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Added in Proof. It can be proved that the converse of Theorem 1 is also true, but the converse of Theorem 2 is not true even in the case that the closed G_δ-set is supercompact.