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LATTICE-VALUED REPRESENTATION OF THE CUT-ELIMINATION THEOREM[1)]

By

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In 1934 G. Gentzen [1] presented the first order classical and intuitionistic predicate calculi *LK* and *LJ*, and expressed and proved his Hauptsatz or the cut-elimination theorem for them. In 1953 G. Takeuti [11] announced the fact that his fundamental conjecture or the cut-elimination theorem for his *GLC* implies finitistically the consistency of analysis, where *GLC* is a simple type theory formulated analogously to *LK*. From that time on he has proved successively but constructively that the fundamental conjecture is true for many subsystems of *GLC*.

In 1967 M. Takahashi [9] gave a general affirmative solution to Takeuti's fundamental conjecture by means of non-constructive methods (see also [10]). Takahashi's proof based on a result of K. Schütte [7] and previously W. Tait [8] had proved the cut-elimination theorem for second order predicate logic. In 1971 G.Y. Girard [2], for the intuitionistic *GLC*, gave a syntactical cut-elimination procedure and proved the finiteness of the procedure by use of non-constructive arguments but by no use of the law of excluded middle.

Gentzen [1] says his Hauptsatz had been found originally for the natural intuitionistic calculus *NJ*, that is a first order intuitionistic system of natural deductins given in [1], but he did not discourse in detail. In 1965 D. Prawitz [5] formulated the Hauptsatz or his normal form theorem for *NJ*[2)] (and for a classical natural deduction system admitting no disjunctions nor existential quantifications). There are several studies of the normal form theorem for higher order natural deduction systems: Prawitz [6], P. Martin-Löf [3], [4], and so on.

In this paper, as our Main Theorem, we shall give a semi-algebraic representation of the cut-elimination theorem. No concrete cut-elimination procedure

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1) This paper was read at a Symposium, 27–29 November 1989, to commemorate the 80th birthday of Prof. Katuzi Ono, Professor Emeritus of Nagoya University.

2) The author attended Prof. Ono's lecture concerning his formulation of the normal form theorem for *NJ* at a Logic Symposium, 15–18 October 1966, Chiba. At that time, none of the participants knew the Prawitz formulation.

will be given in this paper. It is to be wished that we give a purely algebraic representation and to prove, by means of it, we prove the finiteness of a concrete cut-elimination procedure.

§ 1. Formulas, terms and formal systems

1.1. In this paper, the formulas and terms of the simple type thpe theory will be used in the limited forms specified in the following.

1.11. A free variable of type ρ is a *term of type* ρ, for every non-negative integer ρ.

1.12. If s is a term of type $\rho+1$ and t is a term of type ρ, then $s(t)$ is a *formula*.

1.13. If A is a formula, then $\neg A$ is a *formula*. If A and B are formulas, then $A\wedge B$, $A\vee B$ and $A\supset B$ are *formulas*.

1.14. If $F(a)$ is a formula, a is a free variable of type ρ and x is a bound variable of type ρ, then $\forall xF(x)$ and $\exists xF(x)$ are *formulas* and $\lambda xF(x)$ is a *term of type* $\rho+1$.

1.15. The only *terms* and *formulas* are those given by 1.11-1.14.

STIPULATION. Hereafter, when we use a metamathematical expression of the form $\forall xF(x)$, $\exists xF(x)$ or $\lambda xF(x)$, then $F(t)$ means the result of substituting a term t for those occurrences of x in $F(x)$ which occur in none of the scopes of $\forall x$, $\exists x$ or λx with the same x in the inside of $F(x)$.

1.2. As *formal systems* expressing the *classical* or the *intuitionistic* simple type theory we use sequential calculi similar to Gentzen's first order predicate calculi *LK* and *LJ*.

1.21. Every *uppermost sequent* of *formal proof* has the form $D\rightarrow D$, where D is an arbitrary formula.

1.22. The set of our *rules of inference* consists of those of *LK* and the following additional rules 1.221 and 1.222.

1.221. Introduction of

$\forall$ in antecedent:

$$\frac{F(t),\ \Gamma\rightarrow\Theta}{\forall xF(x),\ \Gamma\rightarrow\Theta},$$

$\exists$ in succedent:

$$\frac{\Gamma\rightarrow\Theta,\ F(t)}{\Gamma\rightarrow\Theta,\ \exists xF(x)},$$

$\exists$ in antecedent:

$$\frac{F(a),\ \Gamma\rightarrow\Theta}{\exists xF(x),\ \Gamma\rightarrow\Theta},$$

$\forall$ in succedent:

$$\frac{\Gamma\rightarrow\Theta,\ F(a)}{\Gamma\rightarrow\Theta,\ \forall xF(x)},$$

λ in antecedent: λ in succedent:

$$\frac{F(t),\ \Gamma\rightarrow\Theta}{\{\lambda xF(x)\}(t),\ \Gamma\rightarrow\Theta},\qquad \frac{\Gamma\rightarrow\Theta,\ F(t)}{\Gamma\rightarrow\Theta,\ \{\lambda xF(x)\}(t)}.$$

In the above respective rules, t or a is a term or free variable of type ρ, respectively, if x is a bound variable of type ρ, but in particular the free variable a must not occur in the conclusion of the inference.

1.222. Rule for "extensionality":

$$\frac{s(a),\ \Gamma\rightarrow\Theta,\ t(a)\quad t(a),\ \Gamma\rightarrow\Theta,\ s(a)}{F(s),\ \Gamma\rightarrow\Theta,\ F(t)},$$

where a is a free variable of type ρ for an arbitrary non-negative integer ρ, but it must not occur in the conclusion, and s and t are terms of type $\rho+1$.

We may study the formal systems not containing the rule of extensionality. If that rule were omitted, it would be sufficient to omit only the condition 2.28 from the assumption of the Main Theorem (2.3), but in the proof of the Main Theorem a distinct trivial additional technique would be required.

1.23. Our *classical* formal system for symple type theory is the system which has been stated above, and *intuitionistic* one is that satisfying the additional restriction:

Only sequents whose succedent consists of one formula or is empty are admitted in every inference.

§ 2. Main Theorem

2.1. Let $\mathcal{L}$ be a *complete Boolean algebra* when we study the *classical* formal system, and $\mathcal{L}$ be a *relatively pseudo-complemented complete lattice* when we study the *intuitionistic* formal system.

2.11. A relatively pseudo-complemented lattice is a lattice which has a new operation $\alpha*\beta$ besides the lattice operations $\alpha\cap\beta$ and $\alpha\cup\beta$, and which satisfies the following two conditions:

1) $\alpha\cap(\alpha*\beta)\leqq\beta$,

2) $\alpha\cap\gamma\leqq\beta$ implies $\gamma\leqq\alpha*\beta$,

where $\alpha*\beta$ is called the *pseudo-complement* of α *relative to* β. Because of the completeness, $\mathcal{L}$ has the least element 0. The element $\alpha*0$ is called the *pseudo-complement* of α, and denoted α^*.

2.12. A Boolean algebra is a relatively pseudo-complemented lattice, because both of the above conditions 1) and 2) are satisfied by putting

$$\alpha*\beta=\alpha^c\cup\beta,$$

where α^c is the Boolean complement of α, and $\alpha^*=\alpha^c$ at the time.

2.13. A relatively pseudo-complemented lattice has the greatest element 1 and is distributive. Accordingly, a relatively pseudo-complemented lattice in which the least element exists and $\alpha\cup\alpha^*=1$ holds, for any element α, is a Boolean algebra.

2.14. A complete lattice is relatively pseudo-complemented, if and only if it satisfies the condition

$$\alpha\cap\sup_\lambda\beta_\lambda\leqq\sup_\lambda(\alpha\cap\beta_\lambda).$$

2.2. Let

$\boldsymbol{F}$ be the set of formulas,

$\boldsymbol{T}_\rho$ be the set of terms of type ρ,

FV_ρ be the set of free variables of type ρ,

and

BV_ρ be the set of bound variables of type ρ,

where $\rho=0, 1, 2, \cdots$ and $\boldsymbol{T}_0=FV_0$. We shall consider those mappings

$$m: \boldsymbol{F}\longrightarrow\mathcal{L} \quad\text{and}\quad M: \boldsymbol{F}\longrightarrow\mathcal{L}$$

which satisfy the following conditions 2.20-2.28.

2.20. $m(A)\leqq M(A)$.

2.21. $m(A\wedge B)\leqq m(A)\cap m(B)$, $\quad M(A)\cap M(B)\leqq M(A\wedge B)$.

2.22. $m(A\vee B)\leqq m(A)\cup m(B)$, $\quad M(A)\cup M(B)\leqq M(A\vee B)$.

2.23. $m(\forall xF(x))\leqq\inf\{\ m(F(t))\mid t\in\boldsymbol{T}_\rho\ \}$,

$$\inf\{\ M(F(t))\mid t\in\boldsymbol{T}_\rho\ \}\leqq M(\forall xF(x)),$$

where $x\in BV_\rho$.

2.24. $m(\exists xF(x))\leqq\sup\{\ m(F(t))\mid t\in\boldsymbol{T}_\rho\ \}$,

$$\sup\{\ M(F(t))\mid t\in\boldsymbol{T}_\rho\ \}\leqq M(\exists xF(x)),$$

where $x\in BV_\rho$.

2.25. $m(A\supset B)\leqq M(A)*m(B)$, $\quad m(A)*M(B)\leqq M(A\supset B)$.

2.26. $m(\neg A)\leqq M(A)^*$, $\quad m(A)^*\leqq M(\neg A)$.

2.27. $m(\{\lambda xF(x)\}(t))\leqq m(F(t))$, $\quad M(F(t))\leqq M(\{\lambda xF(x)\}(t))$,

where $x\in BV_\rho$ and $t\in\boldsymbol{T}_\rho$.

2.28. $\delta(t_1, t_2)\cap m(F(t_1))\leqq M(F(t_2))$,

where $t_1, t_2\in\boldsymbol{T}_\rho$, $\rho>0$, and $\delta(t_1, t_2)$ is defined by the following

DEFINITION 2.1. $\delta(t_1, t_2)$

$$=\inf\{ [m(t_1(r)) * M(t_2(r))]\cap[m(t_2(r)) * M(t_1(r))] \mid r\in \boldsymbol{T}_{\rho-1} \}$$

2.3. MAIN THEOREM. *Let*

$$m: \boldsymbol{F} \longrightarrow \mathcal{L} \quad \textit{and} \quad M: \boldsymbol{F} \longrightarrow \mathcal{L}$$

be mappings satisfying the conditions 2.20–2.28, *then*

$$m(A) \leqq M(B),$$

for any provable sequent $A \rightarrow B$.

§3. Proof of the Main Theorem

In this section, let a complete lattice $\mathcal{L}$ and mappings m and M be fixed arbitrarily, provided that $\mathcal{L}$ is Boolean or relatively pseudo-complemented according to our formal system for simple type theory is classical or intuitionistic, and that m and M satisfy the conditions 2.20–2.28.

DEFINITION 3.1. $I(A)=\{ \alpha \mid m(A)\leqq\alpha\leqq M(A) \}$.

DEFINITION 3.2 (Recursive definition of D_ρ, $D(s)$ and $\delta(\xi_1, \xi_2)$).

1) When $s\in \boldsymbol{T}_0$, then $D(s)=\{s\}$.

2) D_ρ is the set-theoretical union of all of the sets $D(s)$ for the terms s of type ρ, i.e.

$$D_\rho=\cup\{ D(s) \mid s\in \boldsymbol{T}_\rho \}.$$

3) Let $\xi_1, \xi_2 \in D_\rho$. When $\rho=0$, then

$$\delta(\xi_1, \xi_2)=1, \quad \text{if } \xi_1=\xi_2;$$

and

$$\delta(\xi_1, \xi_2)=0, \quad \text{if } \xi_1\neq\xi_2.$$

When $\rho>0$, then

$$\delta(\xi_1, \xi_2)=\inf\{ [\xi_1(\zeta) * \xi_2(\zeta)]\cap[\xi_2(\zeta) * \xi_1(\zeta)] \mid \zeta\in D_{\rho-1} \}.$$

4) When $s\in \boldsymbol{T}_{\rho+1}$, then

$$D(s)=\{ \phi \mid \phi: D_\rho \longrightarrow \mathcal{L},$$
$$(\xi)(t)[\xi\in D(t),\ t\in \boldsymbol{T}_\rho \Rightarrow \phi(\xi)\in I(s(t))],$$
$$(\xi_1)(\xi_2)[\xi_1, \xi_2\in D_\rho \Rightarrow \delta(\xi_1, \xi_2)\cap\phi(\xi_1)\leqq\phi(\xi_2)] \}.$$

COROLLARY 1.

1) $\delta(\xi, \xi)=1$.

2) $\delta(\xi_1, \xi_2)=\delta(\xi_2, \xi_1)$.

3) $\delta(\xi_1, \xi_2)\cap\delta(\xi_2, \xi_3)\leqq\delta(\xi_1, \xi_3)$.

COROLLARY 2. *Because of* $\boldsymbol{T}_0=FV_0$, *for* $t_1, t_2\in\boldsymbol{T}_0$, $\delta(t_1, t_2)$ *has been defined in Definition* 3.2, *and the condition* 2.28

$$\delta(t_1, t_2)\cap m(F(t_1))\leqq M(F(t_2))$$

is satisfied also in the case.

LEMMA 3.1. *Let us assume*

$$(\xi_1)(\xi_2)[\xi_1\in D(t_1), \xi_2\in D(t_2)\Rightarrow\delta(\xi_1, \xi_2)\leqq\delta(t_1, t_2)]$$

for all $t_1, t_2\in\boldsymbol{T}_\rho$, *and* ρ *be a fixed non-negative integer. If* s *is a term of type* $\rho+1$ *and* $\phi: D_\rho\longrightarrow\mathcal{L}$ *is the mapping defined by putting*

$$\phi(\xi)=\sup\{\ \delta(\eta, \xi)\cap m(s(r))\mid\eta\in D(r), r\in\boldsymbol{T}_\rho\ \}$$

for all $\xi\in D_\rho$, *then* $\phi\in D(s)$, *i.e.*:

1) $\xi\in D(t), t\in\boldsymbol{T}_\rho\Rightarrow\phi(\xi)\in I(s(t))$;

2) $\xi_1, \xi_2\in D_\rho\Rightarrow\delta(\xi_1, \xi_2)\cap\phi(\xi_1)\leqq\phi(\xi_2)$.

PROOF.

1) Let $\xi\in D(t)$ and $t\in\boldsymbol{T}_\rho$. If $\eta\in D(r)$ and $r\in\boldsymbol{T}_\rho$, then $\delta(\eta, \xi)\leqq\delta(r, t)$ by the assumption, accordingly

$$\delta(\eta, \xi)\cap m(s(r))\leqq\delta(r, t)\cap m(s(r))\leqq M(s(t));$$

hence

$$\phi(\xi)\leqq M(s(t)),$$

by the definition of ϕ. On the other hand,

$$m(s(t))=\delta(\xi, \xi)\cap m(s(t))\leqq\phi(\xi)$$

by the definition of ϕ; hence

$$m(s(t))\leqq\phi(\xi)\leqq M(s(t)),$$

i.e.

$$\phi(\xi)\in I(s(t)).$$

2) If $\xi_1, \xi_2\in D_\rho$, then

$$\begin{aligned}\delta(\xi_1, \xi_2)\cap\phi(\xi_1)&=\delta(\xi_1, \xi_2)\cap\sup\{\ \delta(\eta, \xi_1)\cap m(s(r))\mid\eta\in D(r), r\in\boldsymbol{T}_\rho\ \}\\&=\sup\{\ \delta(\eta, \xi_1)\cap\delta(\xi_1, \xi_2)\cap m(s(r))\mid\eta\in D(r), r\in\boldsymbol{T}_\rho\ \}\\&\leqq\sup\{\ \delta(\eta, \xi_2)\cap m(s(r))\mid\eta\in D(r), r\in\boldsymbol{T}_\rho\ \}\\&=\phi(\xi_2).\end{aligned}$$

LEMMA 3.2. *Let ρ be an integer and $D(r)$ be not empty for all $r\in \boldsymbol{T}_{\rho}$. Then*

$$(\xi_1)(\xi_2)[\xi_1\in D(t_1),\ \xi_2\in D(t_2)\Rightarrow \delta(\xi_1,\ \xi_2)\leqq\delta(t_1,\ t_2)],$$

for all $t_1,\ t_2\in \boldsymbol{T}_{\rho+1}$.

PROOF. Let $t_1,\ t_2\in \boldsymbol{T}_{\rho+1},\ r\in \boldsymbol{T}_{\rho},\ \xi_1\in D(t_1),\ \xi_2\in D(t_2)$ and $\zeta\in D(r)$. Then

$$m(t_1(r))\leqq\xi_1(\zeta)\leqq M(t_1(r))$$

and

$$m(t_2(r))\leqq\xi_2(\zeta)\leqq M(t_2(r)),$$

accordingly

$$\xi_1(\zeta)*\xi_2(\zeta)\leqq m(t_1(r))*M(t_2(r))$$

and

$$\xi_2(\zeta)*\xi_1(\zeta)\leqq m(t_2(r))*M(t_1(r));$$

hence

$$\delta(\xi_1,\ \xi_2)\leqq\delta(t_1,\ t_2).$$

THEOREM 3.1. *Let $s,\ t_1,\ t_2\in \boldsymbol{T}_{\rho}$ and ρ be an arbitrary non-negative integer.*

1) *$D(s)$ is not empty.*

2) $\xi_1\in D(t_1),\ \xi_2\in D(t_2)\Rightarrow\delta(\xi_1,\ \xi_2)\leqq\delta(t_1,\ t_2)$.

PROOF. By mathematical induction on ρ, in the induction step of which Lemmas 3.1 and 3.2 are used.

THEOREM 3.2.

1) $\alpha\in I(A),\ \beta\in I(B)\Rightarrow\alpha\cap\beta\in I(A\wedge B),\quad \alpha\cup\beta\in I(A\vee B),\quad \alpha*\beta\in I(A\supset B)$.

2) $\alpha\in I(A)\Rightarrow\alpha^*\in I(\neg A)$.

3) $(\xi)(t)[\xi\in D(t),\ t\in \boldsymbol{T}_{\rho}\Rightarrow\phi(\xi)\in I(F(t))]$

$$\Rightarrow \inf\{\ \phi(\xi)\mid\xi\in D_{\rho}\ \}\in I(\forall xF(x)),$$
$$\sup\{\ \phi(\xi)\mid\xi\in D_{\rho}\ \}\in I(\exists xF(x)).$$

4) $(\xi)(t)[\xi\in D(t),\ t\in \boldsymbol{T}_{\rho}\Rightarrow\phi(\xi)\in I(F(t))]$,

$$(\xi_1)(\xi_2)[\xi_1,\ \xi_2\in D_{\rho}\Rightarrow\delta(\xi_1,\ \xi_2)\cap\phi(\xi_1)\leqq\phi(\xi_2)]\ \Rightarrow\ \phi\in D(\lambda xF(x)).$$

In 3) *and* 4), *x is a bound variable of type ρ.*

PROOF. By the conditions 2.21–2.27 and Definitions 3.1 and 3.2.

DEFINITION 3.3. We introduce a new symbol, which is denoted by $\bar{\xi}$, for every element ξ of every D_{ρ} and we assume that the new symbols are different with one another. If $\xi\in D_{\rho}$, then $\bar{\xi}$ is called an *$\mathcal{L}$-constant of type ρ*. An *$\mathcal{L}$-term of type ρ* or an *$\mathcal{L}$-formula* is the result of replacing all variables by $\mathcal{L}$-

constants of the respective types throughout a term of type ρ or a formula, respectively.

DEFINITION 3.4 (Recursive definition of $[\![s]\!]$ for an $\mathcal{L}$-term s and $[\![\mathcal{A}]\!]$ for an $\mathcal{L}$-formula $\mathcal{A}$).

1) $[\![\bar{\xi}]\!]=\xi$

2) $[\![s(t)]\!]=[\![s]\!]([\![t]\!])$.

3) $[\![\mathcal{A}\wedge\mathcal{B}]\!]=[\![\mathcal{A}]\!]\cap[\![\mathcal{B}]\!]$, $[\![\mathcal{A}\vee\mathcal{B}]\!]=[\![\mathcal{A}]\!]\cup[\![\mathcal{B}]\!]$,

$[\![\mathcal{A}\supset\mathcal{B}]\!]=[\![\mathcal{A}]\!]*[\![\mathcal{B}]\!]$, $[\![\neg\mathcal{A}]\!]=[\![\mathcal{A}]\!]^*$.

4) $[\![\forall x\mathcal{F}(x)]\!]=\inf\{\ [\![\mathcal{F}(\bar{\xi})]\!]\mid\xi\in D_\rho\ \}$, $[\![\exists x\mathcal{F}(x)]\!]=\sup\{\ [\![\mathcal{F}(\bar{\xi})]\!]\mid\xi\in D_\rho\ \}$,

where x is a bound variable of type ρ.

5) If x is a bound variable of type ρ. Then $[\lambda x\mathcal{F}(x)]$ is the mapping

$$[\![\lambda x\mathcal{F}(x)]\!]:D_\rho\longrightarrow\mathcal{L}$$

defined by putting

$$[\![\lambda x\mathcal{F}(x)]\!](\xi)=[\![\mathcal{F}(\bar{\xi})]\!]$$

for all $\xi\in D_\rho$.

COROLLARY 1.

1) *Let $f(a, b, \cdots)$ or $F(a, b, \cdots)$ be a term or a formula, respectively, and $a, b, \cdots$ be all free variables contained in it. If $\xi\in D(s)$, $\eta\in D(t)$, $\cdots$, and if $f(\bar{\xi}, \bar{\eta}, \cdots)$ or $F(\bar{\xi}, \bar{\eta}, \cdots)$ is an $\mathcal{L}$-term or an $\mathcal{L}$-formula, then*

$$[\![f(\bar{\xi}, \bar{\eta}, \cdots)]\!]\in D(f(s, t, \cdots))\quad or\quad [\![F(\bar{\xi}, \bar{\eta}, \cdots)]\!]\in I(F(s, t, \cdots)),$$

respectively.

2) *If $\bar{\xi}_1$ and $\bar{\xi}_2$ are $\mathcal{L}$-constants of one and the same type and if $f(\bar{\xi}_1)$ or $\mathcal{F}(\bar{\xi}_1)$ is an $\mathcal{L}$-term or an $\mathcal{L}$-formula, respectively, then*

$$\delta(\xi_1, \xi_2)\leqq\delta([\![f(\bar{\xi}_1)]\!], [\![f(\bar{\xi}_2)]\!])$$

or

$$\delta(\xi_1, \xi_2)\cap[\![\mathcal{F}(\bar{\xi}_1)]\!]\leqq[\![\mathcal{F}(\bar{\xi}_2)]\!].$$

PROOF. By mathematical induction on the number of the occurrences of logical symbols (including λ) in the term or the formula, in the induction step of which we use Theorem 3.2 and relations

$$\delta(\eta_1, \eta_2)\cap\delta(\zeta_1, \zeta_2)\cap\eta_1(\zeta_1)\leqq\eta_2(\zeta_2),$$

$$(\alpha_1*\alpha_2)\cap(\beta_1*\beta_2)\cap(\alpha_1\cap\beta_1)\leqq\alpha_2\cap\beta_2,$$

$$(\alpha_1*\alpha_2)\cap(\beta_1*\beta_2)\cap(\alpha_1\cup\beta_1)\leqq\alpha_2\cup\beta_2,$$

$$(\alpha_2 * \alpha_1) \cap (\beta_1 * \beta_2) \cap (\alpha_1 * \beta_1) \leqq \alpha_2 * \beta_2,$$

$$\alpha_2 * \alpha_1 \leqq (\alpha_1^*) * (\alpha_2^*),$$

$$\inf\{ \phi_1(\xi) * \phi_2(\xi) \mid \xi \in D_\rho \} \cap \inf\{ \phi_1(\xi) \mid \xi \in D_\rho \} \leqq \inf\{ \phi_2(\xi) \mid \xi \in D_\rho \},$$

$$\inf\{ \phi_1(\xi) * \phi_2(\xi) \mid \xi \in D_\rho \} \cap \sup\{ \phi_1(\xi) \mid \xi \in D_\rho \} \leqq \sup\{ \phi_2(\xi) \mid \xi \in D_\rho \}.$$

COROLLARY 2. *If* $\xi_1, \xi_2 \in D_\rho$ *and* $\rho > 0$, *then*

1) $[\![\bar{\xi}_1 = \bar{\xi}_2]\!] = \delta(\xi_1, \xi_2)$,

2) $[\![\bar{\xi}_1 = \bar{\xi}_2]\!] \cap [\![\mathcal{F}(\bar{\xi}_1)]\!] \leqq [\![\mathcal{F}(\bar{\xi}_2)]\!]$,

where $a = b$ *means the formula*

$$\forall x[(a(x) \supset b(x)) \wedge (b(x) \supset a(x))].$$

THEOREM 3.3. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* $\mathcal{L}$*-formulas obtained from formulas* A *and* B, *respectively, by respective substitutions which coincide with one another for the free variables common to* A *and* B. *If the sequent* $A \to B$ *is provable, then*

$$[\![\mathcal{A}]\!] \leqq [\![\mathcal{B}]\!].$$

THEOREM 3.4 (Main Theorem). *If a sequent* $A \to B$ *is provable, then*

$$m(A) \leqq M(B).$$

PROOF. By 1) of Theorem 3.1, we can choose an element ξ of $D(a)$ for every free variable a. Let $\mathcal{A}$ and $\mathcal{B}$ be the $\mathcal{L}$-formulas obtained by substituting $\bar{\xi}$ for every free variable a in A and B, respectively. By Corollary 1 of Definition 3.4, we have

$$[\![\mathcal{A}]\!] \in I(A) \quad \text{and} \quad [\![\mathcal{B}]\!] \in I(B).$$

Accordingly, if $A \to B$ is provable, then by Theorem 3.3

$$m(A) \leqq [\![\mathcal{A}]\!] \leqq [\![\mathcal{B}]\!] \leqq M(B).$$

§ 4. Proof of Cut-Elimitation Theorem by use of the Main Theorem

4.1. The complete lattice $\mathcal{L}$.

DEFINITION 4.1. For a sequent $\Gamma \to \Theta$, $M(\Gamma \to \Theta)$ means the set of all sequents $\Delta \to \Lambda$ having the property that $\Gamma, \Delta \to \Theta, \Lambda$ is provable without use of cuts, i. e.

$$M(\Gamma\to\Theta)=\{\ \Delta\to\Lambda \mid \Gamma,\ \Delta\to\Theta,\ \Lambda \text{ is cut-free provable }\}.$$

However, in the *intuitionistic* case, let the succedent Λ of any sequent belonging to $M(\Gamma\to\Theta)$ be always *empty*.

DEFINITION 4.2. A set α of sequents is said to be *closed*, if α satisfies the condition

$$\alpha=\cap\{\ M(\Gamma\to\Theta) \mid \alpha\subset M(\Gamma\to\Theta)\ \},$$

where $\cap$ is the set-theoretical intersection operator. And $\mathcal{L}$ is the complete lattice consisting of all of the closed sets α, β, $\cdots$ of sequents, and the ordering of $\mathcal{L}$ is defined by the set-theoretical inclusion.

COROLLARY.

1) *$M(\Gamma\to\Theta)$ is closed, i.e. an element of $\mathcal{L}$.*

2) *When $\Gamma\to\Theta$ is cut-free provable, then $M(\Gamma\to\Theta)$ is the greatest element* 1 *of $\mathcal{L}$ and consists of all sequents (whose succedents are empty in the intuitionistic case).*

3) *$M(\to)$ is the least element* 0 *of $\mathcal{L}$ and consists of all cut-free provable sequents (whose succedents are empty in the intuitionistic case).*

4) *The lattice-theoretical infimum in $\mathcal{L}$ is the set-theoretical intersection.*

5) *The lattice-theoretical supremum $\alpha\cup\beta$ or $\sup_\lambda \alpha_\lambda$ in $\mathcal{L}$ is not the set-theoretical union, in general, but*

$$\alpha\cup\beta=\cap\{\ \gamma \mid \alpha,\ \beta\subset\gamma\ \}$$

and

$$\sup_\lambda \alpha_\lambda=\cap\{\ \gamma \mid (\lambda)[\alpha_\lambda\subset\gamma]\ \}.$$

THEOREM 4.1. *$\mathcal{L}$ is relatively pseudo-complemented, and*

$$\alpha*\beta=\cap\{\ M(\Gamma,\ \Delta\to\Theta,\ \Lambda) \mid \Gamma\to\Theta\in\alpha,\quad \beta\subset M(\Delta\to\Lambda)\ \}.$$

PROOF. We put

$$\gamma_0=\cap\{\ M(\Gamma,\ \Delta\to\Theta,\ \Lambda) \mid \Gamma\to\Theta\in\alpha,\quad \beta\subset M(\Delta\to\Lambda)\ \},$$

and we shall show the following two properties:

1) $\alpha\cap\gamma_0\subset\beta$.

2) If $\alpha\cap\gamma\subset\beta$, then $\gamma\subset\gamma_0$.

Proof of 1). If $\Gamma\to\Theta\in\alpha\cap\gamma_0$ and $\beta\subset M(\Delta\to\Lambda)$, then

$$\Gamma\to\Theta\in M(\Gamma,\ \Delta\to\Theta,\ \Lambda),$$

that is to say Γ, Δ, $\Gamma\to\Theta$, Λ, Θ is cut-free provable, accordingly

$$\Gamma\to\Theta\in M(\Delta\to\Lambda).$$

Therefore, we have $\alpha \cap \gamma_0 \subset \beta$.

Proof of 2). We assume that $\alpha \cap \gamma \subset \beta$, and $\Pi \to \Sigma \in \gamma$, and we shall prove $\Pi \to \Sigma \in \gamma_0$. For the purpose it is sufficient to prove $\Pi \to \Sigma \in M(\Gamma, \Delta \to \Theta, \Lambda)$ under the assumptions $\alpha \cap \gamma \subset \beta$, $\Pi \to \Sigma \in \gamma$, $\Gamma \to \Theta \in \alpha$ and $\beta \subset M(\Delta \to \Lambda)$. By the last four assumptions,

$$\Gamma, \Pi \to \Theta, \Sigma \in \alpha \cap \gamma \subset \beta \subset M(\Delta \to \Lambda),$$

accordingly $\Delta, \Gamma, \Pi \to \Lambda, \Theta, \Sigma$ is cut-free provable; hence

$$\Pi \to \Sigma \in M(\Gamma, \Delta \to \Theta, \Lambda).$$

COROLLARY. $\alpha^* = \cap \{ M(\Gamma \to \Theta) \mid \Gamma \to \Theta \in \alpha \}$.

LEMMA 4.1. *In the classical case, if* $\alpha \subset M(\Delta \to \Lambda)$, *then* $\Delta \to \Lambda \in \alpha^*$. *(When* Λ *is empty, the lemma holds also for the intuitionistic case.)*

PROOF. By the corollary of Theorem 4.1, it is sufficient to prove $\Delta \to \Lambda \in M(\Gamma \to \Theta)$ under assumptions $\Gamma \to \Theta \in \alpha$ and $\alpha \subset M(\Delta \to \Lambda)$. By the assumptions, we have

$$\Gamma \to \Theta \in M(\Delta \to \Lambda),$$

that is to say $\Delta, \Gamma \to \Lambda, \Theta$ is cut-free provable; hence

$$\Delta \to \Lambda \in M(\Gamma \to \Theta).$$

THEOREM 4.2. *In the classical case,* $\mathcal{L}$ *is a Boolean algebra.*

PROOF. We prove $\alpha \cup \alpha^* = 1$. If $\alpha \cup \alpha^* \subset M(\Delta \to \Lambda)$, then by Lemma 4.1

$$\Delta \to \Lambda \in \alpha^* \subset M(\Delta \to \Lambda),$$

accordingly $\Delta, \Delta \to \Lambda, \Lambda$ is cut-free provable; hence

$$M(\Delta \to \Lambda) = 1.$$

Therefore, we have $\alpha \cup \alpha^* = 1$.

4.2. The mappings m and M from $\boldsymbol{F}$ into $\mathcal{L}$.

DEFINITION 4.3.

1) $m(A) = \cap \{ \alpha \mid A \to \in \alpha \}$.
2) $M(A) = M(\to A)$.

COROLLARY.

1) $m(A) \subset M(\Gamma \to \Theta)$, *if and only if* $A, \Gamma \to \Theta$ *is cut-free provable.*

2) *$\Gamma\to\Theta\in M(A)$, if and only if $\Gamma\to\Theta$, A is cut-free provable.*

TEEOREM 4.3. *The above defined mappings m and M satisfy the conditions* 2.20-2.28.

PROOF. As examples, we shall prove only 2.23 and 2.25.

The former in 2.23. According to 1) of the corollary of Definition 4.3, the rule of "introduction of $\forall$ in antecedent" means the fact that

$$m(F(t))\subset M(\Gamma\to\Theta) \text{ implies } m(\forall xF(x))\subset M(\Gamma\to\Theta);$$

hence

$$m(\forall xF(x))\subset m(F(t)),$$

accordingly

$$m(\forall xF(x))\subset \inf_t m(F(t)).$$

The latter in 2.23. According to 2) of the corollary of Definition 4.2, the rule of "introduction of $\forall$ in succedent" means the fact that

$$\Gamma\to\Theta\in M(F(t)) \text{ for all } t\in \boldsymbol{T}_\rho \text{ implies } \Gamma\to\Theta\in M(\forall xF(x));$$

hence

$$\inf_t M(F(t))\subset M(\forall xF(x)).$$

The former in 2.25. The rule of "introduction of $\supset$ in antecedent" means the fact that

$$\Gamma\to\Theta\in M(A) \text{ and } m(B)\subset M(\Delta\to\Lambda) \text{ imply } m(A\supset B)\subset M(\Gamma,\Delta\to\Theta,\Lambda);$$

hence, by Theorem 4.1,

$$m(A\supset B)\subset M(A)*m(B).$$

The latter in 2.25. The rule of "introduction of $\supset$ in succedent" means the fact that

$$A,\Gamma\to\Theta\in M(B) \text{ implies } \Gamma\to\Theta\in M(A\supset B).$$

On the other hand, $\Gamma\to\Theta\in m(A)*M(B)$ implies

$$A,\Gamma\to\Theta\in m(A)\cap(m(A)*M(B))\subset M(B).$$

Therefore

$$\Gamma\to\Theta\in m(A)*M(B) \text{ implies } \Gamma\to\Theta\in M(A\supset B);$$

hence

$$m(A)*M(B)\subset M(A\supset B).$$

4.3. Proof of cut-elimination theorem.

THEOREM 4.4 (Cut-Elimination Theorem). *If a sequent $A\to B$ is provable, then it is cut-free provable.*

PROOF. Let $A \to B$ be a provable sequent. Then, by Main Theorem (2.3) and by help of Definition 4.2, Theorems 4.1, 4.2, 4.3 and 2) of Definition 4.3, we have

$$m(A) \subset M(B) = M(\to B);$$

hence, by 1) of the corollary of Definition 4.3, the sequent $A \to B$ is cut-free provable, q. e. d.

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