

## GENERALIZED COHOMOLOGICAL DIMENSION OF COMPACT METRIC SPACES

Dedicated to professor Yukihiro Kodama on his 60th birthday

By

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### §0. Introduction.

The notion of homological dimension was introduced by P. S. Alexandroff in the later 1920's. The further contribution to the development of homological dimension theory on compact metric spaces was made by L. S. Pontrjagin [2], K. Borsuk [3], M. F. Bokshtein [4], V. G. Boltynanski [5], E. Dyer [6], Y. Kodama [7, 8], V. I. Kuz'minov [9] and others. New achievement of the theory are surveyed in [10]. In this paper we do not consider homological dimension theory out of compact metric spaces. Moreover everywhere in this paper cohomological language is used instead of homological one, and therefore a dual notion of cohomological dimension is considered.

A compact metric space  $X$  has *cohomological dimension with respect to an abelian group  $G$  equal or less than  $n$*  (written,  $c\text{-dim}_G X \leq n$ ) iff for an arbitrary closed subset  $A \subset X$  and an arbitrary integer  $k \geq n$ , the inclusion  $A \rightarrow X$  induces an epimorphism of  $k$ -dimensional Čech cohomology groups with coefficients.

ALEXANDROFF THEOREM [11]. *For every finite-dimensional compact metric space, we hold the equation  $c\text{-dim}_Z X = \dim X$ .*

As justifiably L. Rubin remarks in [13], we do not have enough information about infinite-dimensional spaces to determine a theory. In this paper a notion of generalized cohomological dimension  $c\text{-dim}_E$  for some class of spectra  $\delta = \{E\}$  is introduced. For all  $E \in \mathcal{E}$ , those dimensions  $c\text{-dim}_E$  coincide with the covering dimension  $\dim$  for finite dimensional compacta. Namely, they are distinguished in the class of infinite-dimensional compacta.

In §1, the inequalities

$$c\text{-dim}_Z X \leq c\text{-dim}_E X \leq \dim X$$

are considered. By  $Z$  we will denote the Eilenberg-MacLane spectrum  $\{K(Z, n)\}$  as well as the additive group of integers. In the case of sphere spectrum  $S$  one gets the notion of stable cohomotopical dimension. In §2, an example which distinguishes  $c\text{-dim}_Z$  and  $c\text{-dim}_S$  is constructed. Also, in §2, a strongly infinite-dimensional compactum  $X$  with  $c\text{-dim}_Z X=3$  is constructed.

The following theorem is well-known.

**THEOREM [11].** *For a compact metric space  $X$  the following conditions are equivalent:*

- 1)  $\dim X \leq n$ ,
- 2) *for an arbitrary closed subset  $A \subset X$  and any map  $\varphi: A \rightarrow S^n$ , there exists a continuous extension  $\bar{\varphi}: X \rightarrow S^n$  of  $\varphi$ .*

Such a space  $S^n$  is called a *test space* for dimension  $n$ . A kind of test spaces was considered in [14]. In §3, the following statement is proved:

*The space  $S^n \cup_{\alpha} B^{n+k+1}$  is a test space for dimension  $n$  for any  $\alpha \in \pi_{n+k}(S^n)$  and  $k > 0$ .*

Here  $S^n \cup_{\alpha} B^{n+k+1}$  is the space obtained from  $S^n$  by attaching a cell  $B^{n+k+1}$  by a representation  $\partial B^{n+k+1} = S^{n+k} \rightarrow S^n$  of  $\alpha$ .

### §1. Definition of cohomological dimension by spectra.

The *suspension* of a space  $X$  is denoted by  $\Sigma X$ . A *spectrum*  $E$  [15] is a sequence of CW-complexes  $\{E_n: n \in \mathbb{Z}\}$  (with base points) and embeddings  $\epsilon_n: \Sigma E_n \rightarrow E_{n+1}$ . Given a spectrum  $E$  defines a *generalized cohomology theory* by the formula:

$$E^n(X) = \lim [\Sigma^k X, E_{n+k}].$$

Here  $[X, Y]$  is the set of all homotopy classes of maps (preserving the base points) from  $X$  to  $Y$ . There is a group structure on the set  $[X, Y]$  in the case of  $X = \Sigma Z$  for some space  $Z$ . Let  $\Omega X$  be the *loop space* on  $X$ . We define the space

$$\Omega^{\infty} E_{n+\infty} = \lim \{ \Omega^k E_{n+k}; \Omega^k \epsilon_n \},$$

where  $\Omega^k X = \Omega(\Omega \dots (\Omega X) \dots)$  is the  $k$ -iterated loop space and  $\tilde{\epsilon}_n: E_n \rightarrow \Omega E_{n+1}$  is the *adjoint map* of  $\epsilon_n$ . Then we have

$$E^n(X) = [X, \Omega^{\infty} E_{n+\infty}].$$

It can be shown that any generalized cohomology theory which satisfies

some axioms can be obtained by the above way from a suitable spectrum  $\mathbf{E}$  [15].

DEFINITION 1. A compact metric space  $X$  has cohomological dimension with respect to a spectrum  $\mathbf{E}$  equal or less than  $n$  (written,  $c\text{-dim}_{\mathbf{E}} X \leq n$ ) if for an arbitrary closed subset  $A \subset X$ , the inclusion  $A \rightarrow X$  induces an epimorphism of cohomology groups  $E^m$  for  $m \geq n$ .

In other words,  $c\text{-dim}_{\mathbf{E}} X \leq n$  iff for an arbitrary closed subset  $A \subset X$  and any map  $\varphi: A \rightarrow \Omega^\infty E_{m+\infty}$ , there exists a continuous extension  $\bar{\varphi}: X \rightarrow \Omega^\infty E_{m+\infty}$  for  $m \geq n$ .

If we choose  $\mathbf{G} = \{K(G, n)$  for  $n \geq 0$  and a singleton for  $n < 0\}$  as a spectrum  $\mathbf{E}$ , we get the notion of the ordinal cohomological dimension with respect to  $G$  as the coefficient group. Not every dimension with respect to  $G$  as the coefficient group. Every generalized cohomology theory can not define a proper dimension theory because it is periodic. This is a reason why we use some restriction on a class of spectra.

A spectrum  $\mathbf{E}$  is called *connected* [15] if for every  $n$ ,

$$\lim_{k \rightarrow \infty} \pi_{i+k}(E_{n+k}) = 0 \quad \text{for every } i < n.$$

DEFINITION 2. A connected spectrum  $\mathbf{E}$  is called *perfectly connected* for every  $n \geq 0$ , the  $(n+1)$ -skeleton  $E_n^{(n+1)}$  of the CW-complex  $E_n$  is a  $n$ -dimensional sphere and for every  $n < 0$ ,  $E_n$  is the singleton.

EXAMPLES. 1) The sphere spectrum  $\mathbf{S} = \{S^n$  for  $n \geq 0$  and the singleton for  $n < 0\}$  is perfectly connected.

2) The Eilenberg-MacLane spectrum  $\mathbf{Z} = \{K(\mathbb{Z}, n)$  for  $n \geq 0$  and the singleton for  $n < 0\}$  is perfectly connected.

3) The Eilenberg-MacLane spectrum  $\mathbf{Z}_2 = \{K(\mathbb{Z}_2, n)$  for  $n \geq 0$  and the singleton  $n < 0\}$  is connected but not perfectly connected.

Following K. Kuratowski, by  $X\tau Y$ , we denote the condition: *every map  $f: A \rightarrow Y$  from a closed subset  $A$  of  $X$  to  $Y$  admits a continuous extension  $\bar{f}: X \rightarrow Y$* . Then we note the following equivalences.

1.  $X\tau S^n \iff \dim X \leq n$ .
2.  $X\tau K(G, n) \iff c\text{-dim}_G X \leq n$ .
3.  $\bigwedge_{m \geq n} X\tau \Omega^\infty E_{m+\infty} \iff c\text{-dim}_{\mathbf{E}} X \leq n$ .

Moreover, by  $X\tau^k Y$ , we denote the following condition: *for every closed subset  $A \subset X$  and every map  $f: \Sigma^k A \rightarrow Y$ , there is a continuous extension  $\bar{f}: \Sigma^k X \rightarrow Y$  of  $f$* .

Let  $\varepsilon_n: E_n \rightarrow \Omega E_{n+1}$  be the adjoint map of  $\varepsilon_n$  in a spectrum  $\mathbf{E}$ . A spectrum  $\mathbf{E}$  is called  $\Omega$ -spectrum if for every  $n$ , the map  $\varepsilon_n$  is a weak homotopy equivalence. Let  $\mathbf{E}$  be an  $\Omega$ -spectrum. Then it is easy to see that the condition  $X\tau E_m$  implies the condition  $X\tau^k E_{m+k}$ . Moreover the implication  $X\tau E_m \Rightarrow X\tau E_{m+1}$  can be proved by S. Ferry's method [17, Appendix]. On the other hand, there is a weak homotopy equivalence  $w_n: E_n \rightarrow \Omega^\infty E_{n+\infty}$ . Since both  $E_n$  and  $\Omega^\infty E_{n+\infty}$  are absolute neighborhood extensors for the class of compact metric spaces, the map  $w_n$  induces an equivalence  $[X, \Omega^\infty E_{n+\infty}] \rightarrow [X, E_n]$  for every compactum  $X$ . Thereby the equivalence 3 can be simplified for an  $\Omega$ -spectrum  $\mathbf{E}$ :  $c\text{-dim}_{\mathbf{E}} X \leq n \Leftrightarrow X\tau E_n$ .

We remark that every spectrum  $\mathbf{E}$  is weak homotopy equivalent to an  $\Omega$ -spectrum. Consequently, every generalized cohomology theory can be defined by some  $\Omega$ -spectrum.

LEMMA 1. *Suppose that a spectrum  $\mathbf{E}$  be perfectly connected. Then  $\dim X \geq c\text{-dim}_{\mathbf{E}} X$ .*

PROOF. The case of  $\dim X = \infty$  is trivial. Hence, suppose that  $\dim X = m < \infty$ . By the definition, it suffices to verify the condition  $X\tau^k E_{n+k}$  for all  $k$  and  $n \geq m$ . Let  $A$  be an arbitrary closed subset of  $X$  and let  $f: \Sigma^k A \rightarrow E_{n+k}$  be a map. Since  $\dim \Sigma^k A \leq m+k$ , there exists a map  $g: \Sigma^k A \rightarrow E_{n+k}$  such that  $g(\Sigma^k A) \subset E_{n+k}^{(m+k)}$  and  $g$  is homotopic to  $f$  (written,  $g \simeq f$ ). By the perfect connectedness of  $\mathbf{E}$ , we have

$$E_{n+k}^{(m+k)} = (E_{n+k}^{(n+k+1)})^{(m+k)} = \begin{cases} \text{singleton} & \text{if } n > m \\ S^{m+k} & \text{if } n = m. \end{cases}$$

In both cases there exists a continuous extension  $\bar{g}: \Sigma^k X \rightarrow E_{n+k}^{(m+k)}$ . Hence there exists a continuous extension  $\bar{f}: \Sigma^k X \rightarrow E_{n+k}^{(m+k)}$  of the initial map  $f$ .

Let  $C(X, Y)$  denote the space of all maps from  $X$  to  $Y$ . There exists the natural embedding  $j$  of  $Y$  into  $C(X, Y)$  given by

$$j(y)(x) = (x, y).$$

Then the embedding  $j$  induces an embedding  $j': Y \rightarrow C(X, X \wedge Y)$ . In the case of  $X = S^k$  the embedding  $j'$  induces the natural embedding  $i: Y \rightarrow \Omega^k \Sigma^k Y$ .

LEMMA 2. *Let  $i: S^n \rightarrow \Omega^k \Sigma^k S^n$  be the natural embedding. Then for every map  $f: X \rightarrow \Omega^k \Sigma^k S^n$  of an  $(n+1)$ -dimensional compactum (=compact metric space)  $X$ , there exists a homotopy  $H: X \times I \rightarrow \Omega^k \Sigma^k S^n$  such that*

(i)  $H_0 = f$ ,

- (ii)  $\text{Im } H_1 \subset i(S^n)$ ,
- (iii)  $f|f^{-1}(i(S^n)) = H_1|f^{-1}(i(S^n))$ .

PROOF. It is a consequence of the fact that the embedding  $i$  induces an isomorphism of the  $l$ -dimensional homotopy groups for  $l \leq n$ .

THEOREM 1. *Let  $E$  be a perfectly connected spectrum and let  $X$  be a finite-dimensional compactum. Then  $\dim X = c\text{-dim}_E X$ .*

PROOF. In view of Lemma 1, it is sufficient to prove the inequality  $\dim X \leq c\text{-dim}_E X$ . Assume the contrary:  $\dim X = m + 1$  and  $c\text{-dim}_E X \leq m$ . Let  $A$  be a closed subset and let  $f: A \rightarrow E_m^{(m+1)} = S^m$  be a map. By the inequality  $c\text{-dim}_E X \leq m$ , there exists a map  $\bar{f}: X \rightarrow \Omega^\infty E_{m+\infty}$  such that  $\bar{f}|A = f$ . Then there exists an integer  $k \geq 0$  such that  $\bar{f}(X) \subset \Omega^k E_{m+k}$ . Then the map  $\bar{f}$  associates a map  $\varphi: X \times S^k \rightarrow E_{m+k}$ . Since  $\dim X = m + 1$ , there exists a map  $\psi: X \times S^k \rightarrow E_{m+k}^{(m+k+1)} = S^{m+k} = \Sigma^k E_m^{(m+1)} = \Sigma^k S^m$  such that

$$\varphi \simeq \psi \text{ maps from } (X \times S^k, A \times S^k) \text{ to } (\Sigma^k S^m, S^m).$$

The map  $\psi$  associates a map  $\xi: X \rightarrow \Omega^k \Sigma^k S^m$  such that

$$\xi \simeq \bar{f} \text{ in } \Omega^k E_{m+k} \text{ and } \xi|A \simeq \bar{f}|A = f \text{ in } S^m.$$

Moreover, by Lemma 2, we have a map  $\eta: X \rightarrow S^m$  such that

$$\eta \simeq \xi \text{ rel. } A.$$

Hence the map  $f$  has a continuous extension  $\tilde{f}: X \rightarrow S^m$ . Therefore  $\dim X \leq m$ . But it is a contradiction.

THEOREM 2. *Let  $E$  be a perfectly connected  $\Omega$ -spectrum. Then  $c\text{-dim}_Z X \leq c\text{-dim}_E X$  for any compactum  $X$ .*

The proof of Theorem 2 needs the following notion.

DEFINITION 3. *Let  $L$  be simplicial complex. A map  $f: X \rightarrow |L|$  is called an  $E_n$ -modification of  $L$  if*

- 1) *for each simplex  $\sigma \in L$ , the space  $f^{-1}(|\sigma|)$  is homotopy equivalent to a finite product of  $E_n$  and  $f^{-1}(|\sigma|) \in \text{ANR}(\text{Comp})$ ,*
- 2) *for each simplex  $\sigma \in L$ , the inclusion  $f^{-1}(|\partial\sigma|) \hookrightarrow f^{-1}(|\sigma|)$  induces an epimorphism*

$$H^n(f^{-1}(|\sigma|); \mathbf{Z}) \longrightarrow H^n(f^{-1}(|\partial\sigma|); \mathbf{Z}).$$

Here we denote by  $|L|$  a geometric realization of a simplicial complex  $L$ , and by  $\partial\sigma$  the boundary of a simplex  $\sigma$ . The one-point space is regarded as a

0-multiple product of  $E_n$ .

LEMMA 3. Let  $E$  be a perfectly connected  $\Omega$ -spectrum and let  $n$  be an integer  $>1$ . Then every simplicial complex  $L$  has an  $E_n$ -modification.

PROOF OF THEOREM 2. Suppose that  $c\text{-dim}_E X=1$ . It is easy to see that there exists a retraction  $r: E_1 \rightarrow E_1^{(2)}=S^1$ . This implies the condition  $X\tau S^1$  which is equivalent to  $c\text{-dim}_Z X \leq 1$ .

Assume that  $c\text{-dim}_E X=n>1$ . Clearly, the condition  $X\tau E_n$  implies the condition  $X\tau(E_n \times \dots \times E_n)$ . Let  $A \subset X$  be an arbitrary closed subset of  $X$  and let  $\varphi: A \rightarrow K(Z, n)$  be an arbitrary map. Then there exists a compact polyhedron  $L$  and maps  $q: X \rightarrow L, \psi: q(A) \rightarrow K(Z, n)$  such that  $\varphi \simeq \psi \circ q|_A$ . Without loss of generality one can regard  $q(A)$  as a subpolyhedron of  $L$ . Choose a sufficiently small triangulation  $\tau$  of  $L$ . By Lemma 2, we have an  $E_n$ -modification  $f: Y \rightarrow |\tau|=L$ . Then by the condition 2) of  $E_n$ -modifications, there is a continuous extension  $g: Y \rightarrow K(Z, n)$  of  $\psi \circ f$ . By the condition 1) of  $E_n$ -modifications, one can construct a map  $\eta: X \rightarrow Y$  with the following properties: if  $q(x) \in |\sigma|$ , then  $f \circ \eta(x) \in |\sigma|$  for each simplex  $\sigma \in \tau$ . Then

$$q|_A \simeq f \circ \eta|_A \text{ in } q(A).$$

Hence we have that

$$g \circ \eta|_A = \psi \circ f \circ \eta|_A \simeq \psi \circ q|_A \simeq \varphi.$$

Therefore there is a continuous extension  $\bar{\varphi}: X \rightarrow K(Z, n)$  of  $\varphi$ . It follows that  $c\text{-dim}_Z X \leq n$ .

LEMMA 4. For every map  $\varphi: S^n \rightarrow \Omega X$  the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} S^n & \xrightarrow{-1} & S^n \\ \varphi \downarrow & & \downarrow \phi \\ \Omega X & \xrightarrow{-1} & \Omega X \end{array}$$

PROOF. Let us consider the standard  $n$ -dimensional sphere  $S^n = \{X \in E^{n+1}: |x|=1\}$  in the  $(n+1)$ -dimensional Euclidean space. Let  $s: S^n \rightarrow S^n$  be the symmetry in  $E^{n+1}$  with respect to the hyperplane  $E^n \ni \{0\}$ . Then the map  $s$  has degree  $=-1$ . A map  $\varphi: S^n \rightarrow \Omega X$  associates with the adjoint map  $\phi: \Sigma S^n \rightarrow X$ . We regard that the suspension  $\Sigma S^n$  is naturally realized in  $E^{n+1} \times E^1 = E^{n+2}$  and  $\phi$  carries the base-point meridian of  $\Sigma S^n$  into the base point of  $X$ .

The map  $(-1) \circ \varphi: S^n \rightarrow \Omega X$  induces a map  $\phi \circ \tau$ , where  $\tau = id_{E^{n+1}} \times (-1)$ :

$E^{n+1} \times E^1 \rightarrow E^{n+1} \times E^1$  is the symmetry. The map  $\varphi \circ (-1)$  induces a map  $\phi \circ (s \times id_{E^1})$ . The maps  $\tau$  and  $s \times id_{E^1}$  are homotopic because the hyperplane  $E^{n+1} \times \{0\}$  can be continuously removed in  $E^{n+2}$  to the hyperplane  $E_0^n \times E^1$ . Thus the maps  $(-1) \circ \varphi$  and  $\varphi \circ (-1)$  are carried into the same homotopy class by the adjoint correspondence  $[S^n, \Omega X] \rightarrow [\Sigma S^n, X]$ . Thereby  $(-1) \circ \varphi \simeq \varphi \circ (-1)$ .

We need the following results from the infinite-dimensional manifold theory.

**THEOREM A** [19, 20]. *Let  $A$  be a compactum and let  $M$  be an  $l_2$ -manifold. Suppose that  $F: A \times I \rightarrow M$  is a homotopy such that each of  $F_0$  and  $F_1$  is an embedding of  $A$  into  $M$  as a deficient set. Then there exists a homeomorphism  $h: M \rightarrow M$  such that  $h|_{F_0(A)} = F_1 \circ (F_0|_A)^{-1}$ .*

Recall that  $A$  is a *deficient set* in an  $l_2$ -manifold  $M$  if there is a homeomorphism  $w: M \rightarrow M \times l_2$  such that  $w(A) \subset M \times \{0\}$ .

**THEOREM B.** *Let  $f: M \rightarrow M$  be a homotopy equivalence of an  $l_2$ -manifold  $M$ . Then the map  $f$  is homotopic to a homeomorphism.*

A proof of Theorem B can be derived from a proof of [21], Theorem 7.3.

**DEFINITION 3.** *An embedding  $X \hookrightarrow Y$  is called a symmetric embedding if every homeomorphism  $h: X \rightarrow X$  can be extended to a homeomorphism  $\bar{h}: Y \rightarrow Y$ .*

**LEMMA 5.** *Let  $E_n$  be a countable locally finite CW-complex and let  $i: S^n \rightarrow E_n$  be an embedding, where  $n > 1$ . Suppose that  $g: E_n \rightarrow \Omega X$  is a weak homotopy equivalence. Then the embedding  $i' = i \times id_{\{0\}}: S^n \times \{0\} \rightarrow E_n \times l_2$  is symmetric.*

**PROOF.** Let  $\phi: S^n \rightarrow S^n$  be a homeomorphism. Suppose that  $\deg \phi = 1$ . In this case there exists a homotopy  $F: S^n \times I \rightarrow E_n \times l_2 = M$  between  $i'$  and  $i' \circ \phi$ . Since  $M$  is a  $l_2$ -manifold, by Theorem A, there exists a homeomorphism  $h: M \rightarrow M$  such that  $h|_{i'(S^n)} = i' \circ \phi \circ i'^{-1}$ .

Next, suppose that  $\deg \phi = -1$ . Let consider the diagram:

$$\begin{array}{ccc}
 \Omega X & \xrightarrow{-1} & \Omega X \\
 g \uparrow & & \uparrow g \\
 E_n & \xrightarrow{\quad f \quad} & E_n \\
 i \uparrow & & \uparrow i \\
 S^n & \xrightarrow{-1} & S^n
 \end{array}$$

Since  $g$  is a weak homotopy equivalence and  $E_n$  is a CW-complex, there exists a map  $f: E_n \rightarrow E_n$  such that  $(-1) \circ g \simeq g \circ f$ . It is easy to check that  $f$  is a weak

homotopy equivalence. Hence  $f$  is a homotopy equivalence. Hence, by Theorem B, there exists a homotopy  $H: M \times I \rightarrow M$  such that  $H_0 = f \times id_I$  and  $H_1$  is a homeomorphism. On the other hand, by Lemma 4,  $(-1) \circ g \circ i \simeq g \circ i \circ \phi$ . Since  $g$  is a weak homotopy equivalence and  $S^n$  is a polyhedron,  $f \circ i \simeq i \circ \phi$ . Hence  $H_1 \circ i' \simeq i' \circ \phi$ . Therefore, by Theorem B, there exists a homeomorphism  $h: M \rightarrow M$  such that  $h|_{i' \circ \phi(S^n)} = (H_1 \circ i') \circ (i' \circ \phi)^{-1}$ . Then the homeomorphism  $\Psi = h^{-1} \circ H_1: M \rightarrow M$  is an extension of  $\phi$ .

**PROPOSITION.** For every symmetric embedding  $j: S^n \rightarrow E$ , there exists a functor  $\Phi: \mathbf{Smp} \rightarrow \mathbf{Hom}_{\mathbf{Top}}$  from the category  $\mathbf{Smp}$  of simplicial complexes to the category  $\mathbf{Hom}_{\mathbf{Top}}$  of maps between topological spaces which carries monomorphisms into monomorphisms and satisfying the following conditions for each simplicial complex  $K$ :

- 1) the image  $\text{Im } \Phi(K)$  is a geometric realization  $|K|$  of  $K$ ,
- 2) for each simplex  $\sigma \in K$ , the space  $\Phi(K)^{-1}(|\sigma|)$  is homotopy equivalent to a finite product  $E \times \dots \times E$ .

**PROOF.** The full subcategory of  $\mathbf{Smp}$  consisting of simplexes of dimension  $\leq m$  is denoted by  $\mathbf{Smp}_m$ . We define  $\Phi$  by the induction on  $m \geq n$ . For  $m = n$ , we define

$$\Phi(K) = id_{|K|}.$$

Let  $\sigma_{n+1}$  be an  $(n+1)$ -dimensional simplex. Fix a homeomorphism  $h: |\sigma_{n+1}| \rightarrow S^n$ . The diagram

$$\begin{array}{ccc} |\sigma_{n+1}| & \xrightarrow{j \circ h} & E \\ & \searrow c & \downarrow \phi \\ & & pt \end{array}$$

means the morphism  $j \circ h \xrightarrow{(id, \phi)} c$  in the category  $\mathbf{Hom}_{\mathbf{Top}}$ . Here a morphism of maps

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{\beta} & Y' \end{array}$$

in  $\mathbf{Hom}_{\mathbf{Top}}$  is denoted by

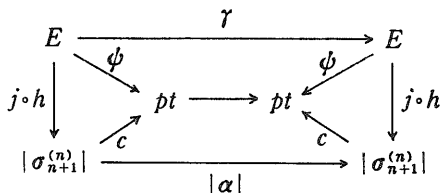
$$(\alpha, \beta): f \longrightarrow g \quad \text{or} \quad f \xrightarrow{(\alpha, \beta)} g.$$



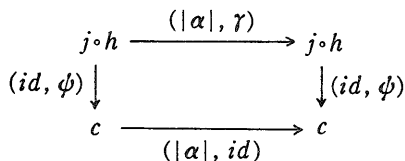
Let  $M: \mathbf{Hom}_{\mathbf{Top}} \rightarrow \mathbf{Top}$  be the *mapping cylinder functor*. It is easy to see that  $M(c)$  is naturally homeomorphic to  $|\sigma_{n+1}|$ . We define

$$\Phi(\sigma_{n+1}) = M(id, \phi).$$

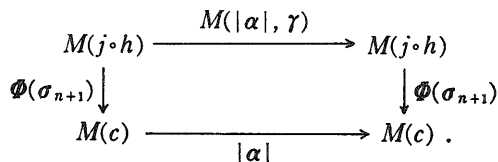
Let  $\alpha: \sigma_{n+1} \rightarrow \sigma_{n+1}$  be an isomorphism in  $\mathbf{Smp}$ . Since  $j$  is a symmetric embedding, there exists a commutative diagram:



In the language of the category  $\mathbf{Hom}_{\mathbf{Top}}$  it means the following commutative diagram:



Hence the functor  $M$  gives the following commutative diagram:



Therefore we define

$$\Phi(\alpha) = (M(|\alpha|, \gamma), |\alpha|).$$

For an arbitrary  $(n+1)$ -dimensional simplicial complex  $K$ , the map  $\Phi(K)$  is constructed by pasting maps  $\Phi(\sigma)$  for all  $\sigma \in K$ .

Suppose that the functor  $\Phi$  is defined on  $\mathbf{Smp}_m$ , where  $m > n$ . Now we define  $\Phi(\sigma_{m+1})$  for an  $(m+1)$ -simplex  $\sigma_{m+1}$ . Let  $F = \{\sigma_{m+1} \rightarrow \sigma_m\}$  be the family of all surjective simplicial maps. By  $\dot{F}$  we denote the family  $\{g | \sigma_{m+1}^{(m)}: \sigma_{m+1}^{(m)} \rightarrow \sigma_m | g \in F\}$ . By  $\Delta f$  we denote the diagonal product

$$\Phi_1(\sigma_{m+1}^{(m)}) \xrightarrow{\Delta f} \pi\{\Phi_1(\sigma_m) | f \in \Phi_1(\dot{F})\},$$

here by  $\Phi_1(L)$  and  $\Phi_2(L)$  we denote the domain and range of the map  $\Phi(L)$ , respectively. For each map  $\alpha: L \rightarrow K$ ,  $\Phi(\alpha)$  means the following commutative diagram:

$$\begin{array}{ccc} \Phi_1(L) & \xrightarrow{\Phi_1(\alpha)} & \Phi_1(K) \\ \Phi(L) \downarrow & & \downarrow \Phi(K) \\ \Phi_2(L) & \xrightarrow{\Phi_2(\alpha)} & \Phi_2(K) . \end{array}$$

The diagram

$$\begin{array}{ccc} \Phi_1(\sigma_{m+1}^{(m)}) & \xrightarrow{\Delta f} & \pi\Phi_1(\sigma_m) \\ \Phi(\sigma_{m+1}^{(m)}) \downarrow & & \downarrow \\ |\sigma_{m+1}^{(m)}| & \xrightarrow{c} & pt \end{array}$$

induces the map

$$\Phi(\sigma_{m+1}): M(\Delta f) \longrightarrow M(c) = |\sigma_{m+1}| .$$

Any isomorphism  $\alpha: \sigma_{m+1} \rightarrow \sigma_m$  induces a bijective correspondence  $A: \Phi_1(\dot{F}) \rightarrow \Phi_1(\dot{F})$ . The correspondence  $A$  induces a homeomorphism

$$a: \pi\{\Phi_1(\sigma_m) | f \in \Phi_1(\dot{F})\} \rightarrow \pi\{\Phi_1(\sigma_m) | f \in \Phi_1(\dot{F})\}$$

which commutes the following diagram:

$$\begin{array}{ccc} \pi\Phi_1(\sigma_m) & \xrightarrow{a} & \pi\Phi_1(\sigma_m) \\ \Delta f \downarrow & & \downarrow \Delta f \\ \Phi_1(\sigma_{m+1}^{(m)}) & \xrightarrow{\Phi_1(\alpha')} & \Phi_1(\sigma_{m+1}^{(m)}) \end{array}$$

where  $\alpha' = \alpha | \sigma_{m+1}^{(m)}$ . A commutative diagram

$$\begin{array}{ccc} \Delta f & \xrightarrow{(\Phi_1(\alpha'), a)} & \Delta f \\ \downarrow & & \downarrow \\ c & \xrightarrow{(|\alpha|, id)} & c \end{array}$$

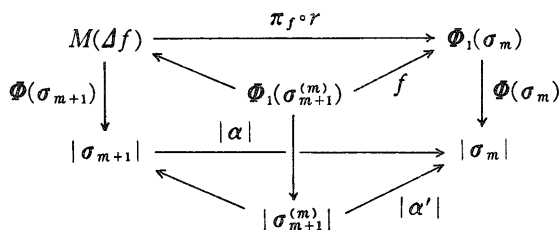
induces the diagram

$$\begin{array}{ccc} M(\Delta f) & \xrightarrow{M(\Phi_1(\alpha'), a)} & M(\Delta f) \\ \Phi(\sigma_{m+1}) \downarrow & & \downarrow \Phi(\sigma_{m+1}) \\ |\sigma_{m+1}| & \xrightarrow{|\alpha|} & |\sigma_{m+1}| . \end{array}$$

We define

$$\Phi(\alpha) = (M(\Phi_1(\alpha'), a), |\alpha|): \Phi(\sigma_{m+1}) \longrightarrow \Phi(\sigma_{m+1}) .$$

For an arbitrary surjective morphism  $\alpha: \sigma_{m+1} \rightarrow \sigma_m$ , there is a commutative diagram:



where  $r$  is the deformation retraction of the mapping cylinder  $M(\Delta f)$  onto the image  $\pi\Phi_1(\sigma_m)$  and  $\pi_f: \pi\Phi_1(\sigma_m) \rightarrow \Phi_1(\sigma_m)$  is the projection on the factor  $\Phi_1(\sigma_m)$  with an index  $f = \Phi_1(\alpha')$ . Hence we define

$$\Phi(\alpha) = (\pi_f \circ r, |\alpha|): \Phi(\sigma_{m+1}) \longrightarrow \Phi(\sigma_m).$$

For any morphism  $\alpha: \sigma_{m+1} \rightarrow \sigma_k$ , where  $k < m$ , we define

$$\Phi(\alpha) = \Phi(\beta) \circ \Phi(\alpha'): \Phi(\sigma_{m+1}) \longrightarrow \Phi(\sigma_k),$$

where  $\alpha = \beta \circ \alpha'$  and  $\alpha': \sigma_{m+1} \rightarrow \sigma_m$  is a surjective simplicial map. It is easy to check that  $\Phi(\alpha)$  does not depend on a decomposition of  $\alpha$  into  $\beta \circ \alpha'$ .

For an arbitrary  $(m+1)$ -dimensional simplicial complex  $K$ , the map  $\Phi(K)$  is defined by pasting corresponding maps  $\Phi(\sigma)$  for all  $\sigma \in K$ .

REMARK 1. If  $j: S^n \rightarrow E$  induces an epimorphism of  $n$ -dimensional cohomology groups, then the following statement is true:

3) the inclusion  $\Phi(\sigma)^{-1}(|\partial\sigma|) \hookrightarrow \Phi(\sigma)^{-1}(|\sigma|)$  induces an epimorphism of  $n$ -dimensional cohomology groups.

PROOF OF LEMMA 3. We regard that  $E_n$  is a locally finite countable CW-complex for every  $n$ . By Lemma 5, the inclusion  $E_n^{(n+1)} \times \{0\} \hookrightarrow E_n \times l_2$  is a symmetric embedding. In view of Remark 1,  $\Phi(L)$  is an  $E_n$ -modification of  $L$ .

REMARK 2. In the case that  $E_n$  is not a countable CW-complex one must consider, in Lemma 5,  $Y$ -manifold instead of an  $l_2$ -manifold with some suitable  $Y$ , for example  $Y = \mathbb{R}^\infty$ .

**§ 2. Cohomological dimension of strongly infinite-dimensional compacta.**

A space  $X$  is *strongly infinite-dimensional* if it contains an essential family  $\{(A_i, B_i) | i=1, 2, \dots\}$ : this means that each  $(A_i, B_i)$  is a disjoint pair of closed subsets in  $X$  and if  $C_i$  is a closed subset of  $X$  separating  $A_i$  and  $B_i$ ,  $i=1, 2, \dots$ , then  $\bigcap_{i=1}^\infty C_i \neq \emptyset$ . In other words, there exists an essential map  $f: X \rightarrow I^\infty$  from

$X$  onto the Hilbert cube  $I^\infty$ : this means that for each  $n$ , the relative map

$$w_n \circ f : (X, f^{-1}(w_n^{-1}(\partial I^n))) \longrightarrow (I^n, \partial I^n)$$

is essential, where  $w_n : I^\infty \rightarrow I^n = \underbrace{I \times \dots \times I}_{n \text{ times}}$  is the projection.

DEFINITION 4. Let  $f : X \rightarrow L$  be a map and let  $L$  be a polyhedron with a triangulation  $\tau$ . We shall write  $c\text{-dim}_Z(f, \tau) \leq n$  if the following condition is satisfied: for every subpolyhedron  $A \subset Y$  with respect to  $\tau$ , the image of the homomorphism  $(f|_{f^{-1}(A)})^* : \check{H}^n(A; Z) \rightarrow \check{H}^n(f^{-1}(A); Z)$  is contained in the image of the homomorphism  $j^* : \check{H}^n(X; Z) \rightarrow \check{H}^n(f^{-1}(A); Z)$ , where  $j : f^{-1}(A) \hookrightarrow X$  is the inclusion.

LEMMA 6 [10]. Let  $X \subset I^\infty$  be a compactum in the Hilbert cube  $I^\infty$  and suppose that  $X = \varprojlim \{X_i, p_i^{i+1}\}$ , where each  $X_i \subset I^\infty$  is a compact polyhedron with a fixed triangulation  $\tau_i$ , such that  $\varinjlim \text{mesh}(\tau_i) = 0$  and  $\varinjlim p_i^\infty = id_X$ . Assume that  $c\text{-dim}_Z(p_i^{i+1}, \tau_i) \leq n$  for infinitely many  $i$ . Then also  $c\text{-dim}_Z X \leq n$ .

The following is actually proved in [10, 12]:

LEMMA 7. For any prime number  $p$ , a finite simplicial complex  $K$  and given finite nonzero elements  $\alpha_i \in K_{\mathbb{C}}^*(|K|, |L_i|; Z_p)$ , where  $L_i$  is a subcomplex of  $K$ , there exists a compact polyhedron  $M$  and a map  $f : M \rightarrow |K|$  such that

- 1)  $c\text{-dim}_Z(f, K) \leq 3$ ,
- 2)  $f^*(\alpha_i) \neq 0$  for all  $i$ .

THEOREM 3. There exists a strongly infinite-dimensional compactum  $X$  with  $c\text{-dim}_Z X = 3$ .

PROOF. Let  $S = \{I^i, w_i^{i+1}\}$  be the standard inverse sequence for  $I^\infty$ . We shall construct an inverse sequence  $S' = \{X_i, q_i^{i+1}\}$ , where each  $X_i$  is a compact polyhedron with a triangulation  $\tau_i$ , and a morphism between inverse sequences  $\{f_i\} : S' \rightarrow S$ .

Define  $X_1 = I$  and  $f_1 = id_1$ .

Suppose that the following part of the sequences  $S'$  is constructed:  $X_1 \xleftarrow{q_1^2} X_2 \xleftarrow{q_2^3} \dots \xleftarrow{q_{n-1}^n} X_n$ , and a family  $\{f_i : X_i \rightarrow I^i\}_{i=1,2,\dots,n}$  is defined such that

- 1)  $c\text{-dim}_Z(q_i^{i+1}, \tau_i) \leq 3$ ,
- 2)  $(q_i^n)^* \circ f_i^*(\mu_i) \neq 0$  for  $i=1, 2, \dots, n$ , where  $\mu_i \in K_{\mathbb{C}}^*(I^i, \partial I^i; Z_p)$  is a generator,
- 3)  $X_i \subset I_1^\infty \times \dots \times I_i^\infty \times \{0\} \times \{0\} \times \dots \subset \prod_{i=1}^\infty I_i^\infty = I^\infty$ , and
- 4)  $\text{mesh}(\tau_i) < 1/i$  with respect to a fixed metric on  $I^\infty$ .

Lemma 7 for  $\tau_n$  and  $(q_i^n)^* \circ f_i^*(\mu_i) \in K_{\mathcal{C}}^*(X_n, (q_i^n)^{-1}(f_i^{-1}(\partial I^i)); \mathbf{Z}_p)$ ,  $i=1, 2, \dots, n$ , gives us a compact polyhedron  $Y_n$  and a map  $g_n: Y_n \rightarrow |\tau_n| = X_n$  with the properties:  $c\text{-dim}_{\mathbf{Z}}(g_n, \tau_n) \leq 3$  and  $g_n^* \circ (q_i^n)^* \circ f_i^*(\mu_i) \neq 0$  for all  $i=1, 2, \dots, n$ . Define

$$X_{n+1} = Y_n \times I, f_{n+1} = (f_n \circ g_n) \times id_I, \text{ and } q_n^{n+1} = g_n \circ v_n,$$

where  $v_n: Y_n \times I \rightarrow Y_n$  is the projection on the first factor. Embed  $X_{n+1}$  into  $I_1^\infty \times \dots \times I_{n+1}^\infty \times \{0\} \times \dots$  as the graph of  $q_n^{n+1}$ . Then it is easy to see that  $c\text{-dim}_{\mathbf{Z}}(q_n^{n+1}, \tau_n) \leq 3$ .

We choose a triangulation  $\tau_{n+1}$  of  $X_{n+1}$  with  $mesh(\tau_{n+1}) < 1/n + 1$ .

Künneth formula implies that  $f_{n+1}^*(\mu_n) \neq 0$ . Since  $v_n^*$  is an isomorphism,  $(q_i^{n+1})^* \circ f_i^*(\mu_i) = v_n^* \circ g_n^* \circ (q_i^n)^* \circ f_i^*(\mu_i) \neq 0$ ,  $i=1, 2, \dots, n$ .

Define  $X = \varprojlim S' \subset I^\infty$  and  $f = \varprojlim \{f_i\}: X \rightarrow I^\infty$ . For each  $n \geq 1$ , the homomorphism  $(w_n^\infty \circ f)^*: K_{\mathcal{C}}^*(I^n, \partial I^n; \mathbf{Z}_p) \rightarrow K_{\mathcal{C}}^*(X, (w_n^\infty \circ f)^{-1}(\partial I^n); \mathbf{Z}_p)$  is not trivial because  $(w_n^\infty \circ f)^*(\mu_n) = (q_n^n)^* \circ f_n^*(\mu_n) \neq 0$ . Hence the map  $f$  is essential, and thereby  $X$  is strongly infinite-dimensional. On the other hand, by Lemma 6, we have that  $c\text{-dim}_{\mathbf{Z}} X \leq 3$ . The converse inequality  $c\text{-dim}_{\mathbf{Z}} X \geq 3$  is trivial by the construction of Lemma 7.

**THEOREM 4.** *There exists a compactum  $X$  with the dimensions  $c\text{-dim}_{\mathbf{Z}} X = 3$  and  $c\text{-dim}_{\mathbf{S}} X = \infty$ .*

**PROOF.** In [10, 12], for any  $n$ , a compactum  $X_n$  with  $c\text{-dim}_{\mathbf{Z}} X_n = 3$  and a map  $f_n: X_n \rightarrow S^n$  with  $f_n^* \neq 0$ , where  $f_n^*: \tilde{K}_{\mathcal{C}}^*(S^n; \mathbf{Z}_p) \rightarrow \tilde{K}_{\mathcal{C}}^*(X_n; \mathbf{Z}_p)$  is induced by  $f_n$ . Then the suspension map  $\Sigma^k f_n$  is essential for every  $k$ . Hence  $\pi_s^n(X_n) \neq 0$ . Then  $c\text{-dim}_{\mathbf{S}} X_n \geq n$ . Therefore  $X = \alpha(\bigoplus_n X_n)$  is desired compactum, where  $\alpha(\bigoplus_n X_n)$  is the one point compactification of the topological sum  $\bigoplus_n X_n$ .

**PROBLEM 1.** Does there exist a strongly infinite-dimensional compactum  $X$  with finite stable cohomotopy dimension  $c\text{-dim}_{\mathbf{S}} X < \infty$ ?

**PROBLEM 2 (S. Nowak).** Is it true that  $\dim X = c\text{-dim}_{\mathbf{S}} X$  for every compactum  $X$ ?

**§ 3. The dimension  $\text{dim}_\alpha$ .**

**LEMMA 8.** *Let  $X$  be a compactum and let  $K$  be a CW-complex. Then the condition  $X\tau K$  implies the condition  $X\tau \Sigma K$ .*

**PROOF.** Let  $A \subset X$  be a closed subset and let  $\varphi: A \rightarrow \Sigma K$  be a map. By the definition,  $\Sigma K = \text{con}_1 K \cup \text{con}_2 K$ , where  $\text{con}_1 K \cap \text{con}_2 K = K$  and each  $\text{con}_i K$ ,

$i=1, 2$ , is a cone over  $K$ . Denote  $A_i = \varphi^{-1}(\text{con}_i K)$ ,  $i=1, 2$ . Then there is a map  $f: A \rightarrow \mathbb{R}$  with the properties:

$$f(A_1) \subset \mathbb{R}^+, f(A_2) \subset \mathbb{R}^-, \text{ and } f^{-1}(0) = A_1 \cap A_2.$$

Let  $\tilde{f}: X \rightarrow \mathbb{R}$  be a continuous extension of  $f$ . Define  $F_1 = \tilde{f}^{-1}(\mathbb{R}^+)$  and  $F_2 = \tilde{f}^{-1}(\mathbb{R}^-)$ . From the condition  $X\tau K$ , there exists a map  $\phi: F_1 \cap F_2 \rightarrow K$  such that  $\phi|_{A_1 \cap A_2} = \varphi|_{A_1 \cap A_2}$ . Since  $\text{con}_i K$  is an absolute extensor for compacta, there exists a continuous extension  $\phi_i: F_i \rightarrow \text{con}_i K$  of  $\phi \cup \varphi|_{A_i}$ ,  $i=1, 2$ . Then the union  $\phi_1 \cup \phi_2$  is a desired extension of  $\varphi$  over  $X$ . Thus,  $X\tau \Sigma K$ .

Let  $\alpha \in \pi_{n+k}(S^n)$  is an element of the  $(n+k)$ -dimensional homotopy group of the  $n$ -sphere. Define

$$B_\alpha = S^n \cup_f D^{n+k+1},$$

as a complex obtained from the  $n$ -sphere  $S^n$  by attaching an  $(n+k+1)$ -cell  $D^{n+k+1}$  using an attaching map  $f: \partial D^{n+k+1} \rightarrow S^n$  from the homotopy class  $\alpha$ . It is easy to see that if  $f \simeq f': \partial D^{n+k+1} \rightarrow S^n$ , the spaces  $S^n \cup_f D^{n+k+1}$  and  $S^n \cup_{f'} D^{n+k+1}$  are homotopy equivalent. Moreover, it can be seen that the spaces  $B_{\Sigma\alpha}$  and  $\Sigma(B_\alpha)$  are homotopy equivalent.

DEFINITION 5. The dimension  $\dim_\alpha X$  of a compactum  $X$  is defined as follows:

$$\dim_\alpha X \leq m \iff \begin{cases} \dim X \leq m & \text{if } m < n \\ X\tau \Sigma^{m-n} B_\alpha & \text{if } m \geq n. \end{cases}$$

In view of Lemma 8, the definition is correct.

THEOREM 5.  $\dim_\alpha X = \dim X$  for every compactum  $X$ .

PROOF. The implication  $\dim X \leq m \Rightarrow \dim_\alpha X \leq m$  is trivial.

Suppose that  $\dim_\alpha X \leq m$  and  $m \geq n$ . It is sufficient to prove the inequality  $\dim X \leq m$ . Let  $\varphi: A \rightarrow S^m$  be an arbitrary map from a closed subset  $A$  of  $X$  to the  $m$ -sphere  $S^m$ . By the condition  $\dim_\alpha X \leq m$ , there exists a continuous extension  $\varphi_1: X \rightarrow S^m \cup_{\Sigma^{m-n} f} D^{m+k+1}$  of  $\varphi$ , where  $f \in \alpha \in \pi_{n+k}(S^n)$ . Choose an  $(m+k+1)$ -cell  $B^{m+k+1} \subset \text{Int } D^{m+k+1}$  and define a retraction

$$r_1: S^m \cup_{\Sigma^{m-n} f} D^{m+k+1} - \text{Int } B^{m+k+1} \longrightarrow S^m.$$

Denote  $A_1 = \varphi_1^{-1}(\partial B^{m+k+1})$ . From the condition  $X\tau \Sigma^{m-n+k} B_\alpha$ ,  $\varphi_1|_{A_1}$  has a continuous extension

$$\varphi_2: \varphi_1^{-1}(B^{m+k+1}) \longrightarrow S^{m+k} \cup_{\Sigma^{m-n+k}} D^{m+2k+1}$$

Choose an  $(m+2k+1)$ -cell  $B^{m+2k+1} \subset \text{Int } D^{m+2k+1}$  and define a retraction  $r_2$  and

$A_2$  and so on.

By the construction,  $r_i|\partial B^{m+i k+1} \simeq \Sigma^{m-n+(i-1)k} f$ ,  $i=1, 2, \dots$ . By the *nilpotency theorem* for the ring  $\pi_*^S[22]$ , there exists a number  $l$  such that  $\xi=r_1 \circ r_2 \circ \dots \circ r_l|\partial B^{m+l k+1}$  is null-homotopic. Consequently  $\xi \circ \varphi_l|A_l$  has a continuous extension

$$\eta : \varphi_l^{-1}(B^{m+l k+1}) \longrightarrow S^m .$$

The finite stratification  $Z_l \subset Z_{l-1} \subset \dots \subset Z_1 \subset Z_0 = X$  is defined by  $Z_i = \varphi_i^{-1}(B^{m+i k+1})$  for  $i=1, 2, \dots, l$ . Define

$$\begin{aligned} \bar{\varphi}|Z_0 - Z_1 &= r_1 \circ \varphi_1, \\ \bar{\varphi}|Z_1 - Z_2 &= r_1 \circ r_2 \circ \varphi_2, \\ &\dots\dots\dots \\ \bar{\varphi}|Z_{l-1} - Z_l &= r_1 \circ r_2 \circ \dots \circ r_l \circ \varphi_l, \\ \bar{\varphi}|Z_l &= \eta . \end{aligned}$$

Then it is easy to see that  $\bar{\varphi} : X \rightarrow S^m$  is well-defined and continuous, and  $\bar{\varphi}$  is an extension of  $\varphi$ . Thus,  $\dim X \leq m$ .

REMARK 3. The sequence of spaces

$$E_m = \begin{cases} pt & \text{if } m < 0 \\ S^m & \text{if } 0 \leq m < n \\ \Sigma^{m-n} B_\alpha & \text{if } m \geq n \end{cases}$$

defines a spectrum  $E(\alpha)$ . Then Theorems 1 and 5 imply the inequality  $c\text{-dim}_{E(\alpha)} X \leq \dim_\alpha X$  for every compactum  $X$ .

PROBLEM 3. Let  $K$  be a compact  $(n-1)$ -connected polyhedron with  $\pi_n(K) = Z$ . Then one can define the dimension  $\dim_K$  as the above. Is it true that  $\dim_K X = \dim X$  for every compactum  $X$ ?

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