

## INNER DERIVATIONS OF HIGHER ORDERS

By

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**Summary.** We define inner derivations of higher order of a ring  $R$  and we prove that they correspond to the inner automorphisms of a suitable ring. Moreover, we prove that any higher derivation of  $R$  is inner if and only if any usual derivation of  $R$  is inner.

### I.

Let  $R$  be a ring with identity and let  $S$  be a segment of  $N = \{0, 1, 2, \dots\}$ , that is,  $S = N$  or  $S = \{0, 1, \dots, s\}$  for some  $s \geq 0$ .

A family  $d = (d_n)_{n \in S}$  of mappings  $d_n : R \rightarrow R$  is called a *derivation of order  $s$  of  $R$*  (where  $s = \sup S \leq \infty$ ) if the following properties are satisfied :

- (1)  $d_n(a+b) = d_n(a) + d_n(b)$ ,
- (2)  $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ ,
- (3)  $d_0 = id_R$ .

The set of derivations of order  $s$  of  $R$ , denoted by  $D_s(R)$ , is the group under the multiplication  $*$  defined by the formula

$$(d * d')_n = \sum_{i+j=n} d_i \circ d'_j,$$

where  $d, d' \in D_s(R)$  and  $n \in S$  ([1], [5], [7]).

It is easy to prove the following two lemmas.

LEMMA 1.1. *Let  $a \in R$ ,  $d_0 = id_R$ , and*

$$d_n(x) = a^n x - a^{n-1} x a = a^{n-1} (ax - xa)$$

for  $n \geq 1$ ,  $x \in R$ . Then  $d = (d_n)_{n \in S}$  belongs to  $D_s(R)$ .

LEMMA 1.2. *Let  $d \in D_s(R)$ ,  $k \in S \setminus \{0\}$  and let  $\delta = (\delta_n)_{n \in S}$  be the family of mappings from  $R$  to  $R$  defined by*

$$\delta_n = \begin{cases} 0, & \text{if } k \nmid n, \\ d_r, & \text{if } n = rk. \end{cases}$$

Then  $\delta \in D_s(R)$ .

The derivation  $d$  from Lemma 1.1 will be denoted by  $[a, 1]$  and the derivation  $\delta$  from Lemma 1.2, for  $d=[a, 1]$ , will be denoted by  $[a, k]$ . Therefore, for  $a \in R$ ,  $k \in S \setminus \{0\}$ ,  $x \in R$ ,  $n \in S$ :

$$[a, k]_n(x) = \begin{cases} x & , \quad \text{if } n=0, \\ 0 & , \quad \text{if } k \nmid n, \\ a^r x - a^{r-1} x a, & \text{if } n \neq 0 \text{ and } n=kr. \end{cases}$$

Let  $\mathbf{a} = (a_n)_{n \in S}$  be a sequence in  $R$ . Denote by  $\Delta(\mathbf{a})$  the element in  $D_s(R)$  defined by

$$\Delta(\mathbf{a})_n = ([a_1, 1] * [a_2, 2] * \cdots * [a_n, n])_n.$$

For example

$$\Delta(\mathbf{a})_1(x) = a_1 x - x a_1$$

$$\Delta(\mathbf{a})_2(x) = a_1^2 x - a_1 x a_1 + a_2 x - x a_2$$

$$\Delta(\mathbf{a})_3(x) = a_1^3 x - a_1^2 x a_1 + a_1 a_2 x + x a_2 a_1 - a_1 x a_2 - a_2 x a_1 + a_3 x - x a_3$$

$$\Delta(\mathbf{a})_4(x) = a_1^4 x - a_1^3 x a_1 + a_1^2 x a_2 - a_2 x a_2 + a_1^2 a_2 x - a_1^2 x a_2 - a_1 a_2 x a_1$$

$$+ a_1 x a_2 a_1 + a_1 a_3 x - a_1 x a_3 - a_3 x a_1 + x a_3 a_1 + a_4 x - x a_4.$$

DEFINITION 1.3. Let  $d \in D_s(R)$ . If there exists a sequence  $\mathbf{a} = (a_n)_{n \in S}$  of elements of  $R$  such that  $d = \Delta(\mathbf{a})$  then  $d$  is called an *inner derivation of order  $s$  of  $R$* .

## II.

Denote by  $T$  the additive group of the product of  $s+1$  copies of  $R$ . The element  $(a_n)_{n \in S}$  will be always denoted by  $\mathbf{a}$ . We define a multiplication on  $T$  as follows:

$$\mathbf{a}\mathbf{b} = \mathbf{c}, \quad \text{where } c_n = \sum_{i+j=n} a_i b_j.$$

$T$  is a ring with identity  $(1, 0, 0, \dots)$  ([7], [8]). Notice that an element  $\mathbf{a}$  is invertible in  $T$  iff  $a_0$  is invertible in  $R$ .

For any  $k \in S$ , let  $\pi_k$  denote the  $k$ -th projection from  $T$  to  $R$ . If  $a \in R$  then  $j_k(a)$ ,  $p_k(a)$  and  $q_k(a)$  (where  $k \in S$ ,  $l \in S \setminus \{0\}$ ) denote the elements of  $T$  defined by the following conditions:

$$\pi_n j_k(a) = \begin{cases} 0, & \text{for } n \neq k, \\ a, & \text{for } n = k, \end{cases} \quad \pi_n p_l(a) = \begin{cases} 0, & \text{if } l \nmid n, \\ a^r, & \text{if } n = rl, \end{cases}$$

$$\pi_n q_l(a) = \begin{cases} 1, & \text{for } n=0, \\ 0, & \text{for } n \geq 1, n \neq l, \\ a, & \text{for } n=l. \end{cases}$$

Let  $T_k$  (for  $k \in S \setminus \{0\}$ ) denote the set of elements  $\mathbf{a}$  in  $T$  such that  $a_0=1$  and  $a_i=0$  for  $i=1, 2, \dots, k$ , and let  $T_0$  be the set of elements  $\mathbf{a}$  in  $T$  such that  $a_0=1$ .

Observe that  $q_k(a) = 1_T + j_k(a)$ , and every element in  $T_k$  is of the form  $1 + j_{n+1}(1)\mathbf{a}$ , for some  $\mathbf{a} \in T$ .

It is easy to verify the following

LEMMA 2.1. *Let  $k \in S, a \in R$ .*

- (1) *If  $\mathbf{a}, \mathbf{b} \in T_k$  then  $\mathbf{ab}, \mathbf{a}^{-1} \in T_k$ .*
- (2)  *$p_k(a)^{-1} = q_k(-a)$ .*
- (3) *If  $\mathbf{b} \in T_0$  then  $\mathbf{b}p_k(-b_k) = \mathbf{a}$ , where  $a_n = b_n$  for  $n=0, 1, \dots, k-1$ , and  $a_k=0$ .*

Now we prove two lemmas.

LEMMA 2.2. *Let  $\mathbf{b} \in T_0$ . Then there exists an element  $\mathbf{a}$  in  $T_0$  such that  $\mathbf{b}p_1(a_1)p_2(a_2) \dots p_k(a_k) \in T_k$ , for any  $k \in S \setminus \{0\}$ .*

PROOF. Let  $a_1 = -b_1$ . Then, by Lemma 2.1(3), we have  $\mathbf{b}p_1(a_1) \in T_1$ . Suppose that elements  $a_1, \dots, a_n$  satisfy the condition

$$\mathbf{v}^{(k)} = \mathbf{b}p_1(a_1) \dots p_k(a_k) \in T_k$$

for  $k=1, 2, \dots, n$ .

Put  $a_{n+1} = -\pi_{n+1}(\mathbf{v}^{(n)})$ . Then

$$\begin{aligned} \mathbf{v}^{(n+1)} &= \mathbf{v}^{(n)}p_{n+1}(a_{n+1}) \\ &= \mathbf{b}p_1(a_1) \dots p_{n+1}(a_{n+1}) \in T_{n+1} \end{aligned}$$

by Lemma 2.1(3).

LEMMA 2.3. *Let  $\mathbf{a} \in T_0$ . Then there exists  $\mathbf{b} \in T_0$  such that*

$$p_1(a_1)p_2(a_2) \dots p_k(a_k)\mathbf{b} \in T_k$$

for any  $k \in S \setminus \{0\}$ .

PROOF. Put  $b_0=1$  and  $b_n = \pi_n(\mathbf{u}^{(n)})$ , for  $n \geq 1$ , where  $\mathbf{u}^{(n)} = q_n(-a_n) \dots q_1(-a_1)$ .

Then  $b_n = \pi_n(\mathbf{u}^{(k)})$  for any  $n \in S \setminus \{0\}$  and  $k \geq n$ . In fact, if  $k \geq n$  then

$$\begin{aligned}
\pi_n(\mathbf{u}^{(k+1)}) &= \pi_n(\mathbf{u}^{(k)} + j_{k+1}(-a_{k+1})\mathbf{u}^{(k)}) \\
&= \pi_n(\mathbf{u}^{(k)}) + \pi_n(j_{k+1}(-a_{k+1})\mathbf{u}^{(k)}) \\
&= \pi_n(\mathbf{u}^{(k)}).
\end{aligned}$$

Therefore, if  $\mathbf{b}=(b_n)_{n \in S}$  then  $\pi_i(\mathbf{b}-\mathbf{u}^{(k)})=0$  for  $i=0, 1, \dots, k$ . So  $\mathbf{b}=\mathbf{u}^{(k)} + j_{k+1}(1)\mathbf{v}^{(k)}$ , for some  $\mathbf{v}^{(k)} \in T$ , and, by Lemma 2.1, we have

$$\begin{aligned}
p_1(a_1)p_2(a_2) \cdots p_k(a_k)\mathbf{b} &= p_1(a_1) \cdots p_k(a_k)(q_k(-a_k) \cdots q_1(-a_1) + j_{k+1}(1)\mathbf{v}^{(k)}) \\
&= 1_T + j_{n+1}(1)\mathbf{c},
\end{aligned}$$

for some  $\mathbf{c} \in T$ . This completes the proof.

### III.

If  $d \in D_s(R)$  then  $\exp(d)$  will denote the ring automorphism of  $T$  defined as follows:

$$\exp(d)(\mathbf{a}) = \mathbf{b}, \quad \text{where } b_n = \sum_{i+j=n} d_i(a_j) \quad ([5], [7], [8]).$$

In [7] Ribenboim showed that the mapping  $\exp$  is a group isomorphism from  $D_s(R)$  to the group  $B_s(R)$  of such automorphisms  $h: T \rightarrow T$  that  $h(j_1(1))=j_1(1)$ ,  $\pi_0 h j_0 = id_R$ . If  $h \in B_s(R)$  then the derivation  $d=(d_n)_{n \in S}$ , where  $d_n(x)=\pi_n h j_0(x)$  for  $x \in R$ , satisfies the condition  $h=\exp(d)$  ([7]).

For any  $\mathbf{a} \in T_0$  denote by  $\langle \mathbf{a} \rangle$  the inner automorphism of  $T$  defined by  $\langle \mathbf{a} \rangle(x) = \mathbf{a}^{-1} x \mathbf{a}$ . Observe that  $\langle \mathbf{a} \rangle$  belongs to  $B_s(R)$ .

LEMMA 3.1. (1) If  $a \in R$ ,  $k \in S \setminus \{0\}$  then  $\exp([a, k]) = \langle q_k(-a) \rangle$ .

(2) Let  $\mathbf{a} \in T_k$ . If  $d=(d_n)_{n \in S}$  is an element of  $D_s(R)$  such that  $\exp(d) = \langle \mathbf{a} \rangle$ , then  $d_1=d_2=\dots=d_k=0$ .

PROOF. (1) If  $d \in D_s(R)$  satisfies  $\exp(d) = \langle q_k(-a) \rangle$  then

$$\begin{aligned}
d_n(x) &= \pi_n \langle q_k(-a) \rangle j_0(x) \\
&= \pi_n q_k(-a)^{-1} j_0(x) q_k(-a) \\
&= \pi_n p_k(a) j_0(x) (1_T + j_k(-a)), \quad \text{for } n \in S.
\end{aligned}$$

Hence  $d_n(x)=0$  if  $k \nmid n$ , and  $d_n(x)=a^r x - a^{r-1} x a$  if  $n=kr$ . Therefore  $d=[a, k]$ .

(2) It follows from Lemma 2.1(1) since  $d_n = \pi_n \langle \mathbf{a} \rangle j_0$ .

Now we are ready to prove the following

THEOREM 3.2. Let  $d \in D_s(R)$ . Then  $d$  is inner iff there exists  $\mathbf{b} \in T_0$  such

that  $\exp(d)=\langle \mathbf{b} \rangle$ .

PROOF. Let  $d=\Delta(\mathbf{a})$ , where  $\mathbf{a} \in T_0$ , and let  $\mathbf{b}$  be as in Lemma 2.3. Moreover, let  $\delta=(\delta_n)_{n \in S}$  be the unique derivation satisfying  $\exp(\delta)=\langle \mathbf{b} \rangle$ . We show that  $\delta=d$ .

Let  $n \in S \setminus \{0\}$ . It follows from Lemmas 2.3, 2.1 that

$$\mathbf{b}=q_n(-a_n) \cdots q_1(-a_1)v^{(n)},$$

where  $v^{(n)}$  is an element of  $T_n$ .

Therefore, if  $F=\exp^{-1}$  then

$$\delta=F\langle \mathbf{b} \rangle=F\langle v^{(n)} \rangle * F\langle q_1(-a_1) \rangle * \cdots * F\langle q_n(-a_n) \rangle,$$

and, by Lemma 3.1,

$$\delta_n=(\lceil a_1, 1 \rceil * \cdots * \lceil a_n, n \rceil)_n=d_n.$$

Conversely, let  $\mathbf{b} \in T_0$ ,  $d=\exp^{-1}(\langle \mathbf{b} \rangle)$  and let  $\mathbf{a}$  be such as in Lemma 2.2. We show that  $d=\Delta(\mathbf{a})$ .

Let  $n \in S \setminus \{0\}$ . It follows from Lemmas 2.2, 2.1 that

$$\mathbf{b}=v^{(n)}q_n(-a_n) \cdots q_1(-a_1),$$

where  $v^{(n)} \in T_n$ , and hence

$$d=F\langle \mathbf{b} \rangle=F\langle q_1(-a_1) \rangle * \cdots * F\langle q_n(-a_n) \rangle * F\langle v^{(n)} \rangle,$$

where  $F=\exp^{-1}$ .

Therefore, by Lemma 3.1, we have

$$d_n=(\lceil a_1, 1 \rceil * \cdots * \lceil a_n, n \rceil)_n \quad \text{i.e.} \quad d=\Delta(\mathbf{a}).$$

COROLLARY 3.3. *The set of inner derivations of order  $s$  of  $R$  is a normal subgroup of  $D_s(R)$ .*

#### IV.

Recall that the usual (classical) derivation of  $R$  is the additive mapping  $\delta: R \rightarrow R$  such that  $\delta(ab)=\delta(a)b+a\delta(b)$ , for all elements  $a, b \in R$ . The set of usual derivations of  $R$  corresponds bijectively, in the natural way, to the set  $D_1(R)$ . Evidently a usual derivation is inner iff there exists an element  $a \in R$  such that  $\delta(x)=ax-xa$  for any  $x \in R$ .

It is easy to see that

LEMMA 4.1. *Let  $d, d' \in D_s(R)$ . If  $d_i=d'_i$  for  $i=0, 1, \dots, n < s$  then  $d_{n+1}=d'_{n+1}$*

is a usual derivation.

Now we can prove

**THEOREM 4.2.** *If every classical derivation of  $R$  is inner then so is every derivation of order  $s$  of  $R$ .*

**PROOF.** Let  $d \in D_s(R)$ . We must construct an element  $\mathbf{a} \in T$  such that  $d = \Delta(\mathbf{a})$ .

Since  $d_1$  is a classical derivation then there exists  $a_1 \in R$  such that  $d_1(x) = a_1x - xa_1$ , for any  $x \in R$ . So we have  $d_1 = [a_1, 1]$ .

Let  $d'_2 = [a_1, 1]_2$ . Then  $(1_R, d_1, d'_2)$  and  $(1_R, d_1, d_2)$  are derivations of order 2 and hence, by Lemma 4.1, there exists  $a_2 \in R$  such that  $d_2(x) = d'_2(x) + a_2x - xa_2$  for any  $x \in R$ . Therefore,

$$\begin{aligned} d_2 &= d'_2 + [a_2, 2]_2 = [a_1, 1]_2 + [a_2, 2]_2 \\ &= ([a_1, 1] * [a_2, 2])_2. \end{aligned}$$

Next let  $d'_3 = ([a_1, 1] * [a_2, 2])_3$ . Since  $(1_R, d_1, d_2, d'_3)$ ,  $(1_R, d_1, d_2, d_3)$  are derivations of order 3 then, by Lemma 4.1,  $d_3(x) = d'_3(x) + a_3x - xa_3$  for some  $a_3 \in R$ . So we have

$$\begin{aligned} d_3 &= d'_3 + [a_3, 3]_3 \\ &= ([a_1, 1] * [a_2, 2])_3 + [a_3, 3]_3 \\ &= ([a_1, 1] * [a_2, 2] * [a_3, 3])_3 \end{aligned}$$

and so on.

The assumption of the above theorem is satisfied for a large class of rings (see for example [3], [4], [2]).

## V.

We end this paper with the following three remarks.

**REMARK 5.1.** Let  $a \in R$ . If  $d = [a, 1]^{-1}$  then  $d_n(x) = xa^n - axa^{n-1}$ , for  $n \geq 1$ ,  $x \in R$ .

**REMARK 5.2.** Let  $a \in R$ . Let  $d = (d_n)_{n \in S}$  be the family of mappings from  $R$  to  $R$  defined by

$$\begin{aligned} d_0(x) &= x \\ d_1(x) &= ax - xa \end{aligned}$$

$$d_n(x) = a^n x + x(-a)^n + 2 \sum_{k=1}^{n-1} a^{n-k} x(-a)^k, \quad \text{for } n \geq 2.$$

Then  $d \in D_s(R)$  (in general) but  $\delta = (2d_n)_{n \in S}$  is an inner derivation of order  $s$  of  $R$ . Namely,  $\delta = [a, 1] * [-a, 1]^{-1}$ .

REMARK 5.3. Let  $d \in D_s(R)$ . Suppose that there exists an element  $a \in R$  such that  $d_n = a^{n-1} d_1$  for any  $n \in S \setminus \{0\}$ . If the set  $d_1(R)$  contains a regular element then  $d = [a, 1]$ .

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