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ON LORENTZ MANIFOLDS WITH ABUNDANT ISOMETRIES

By

Hiroo MATSUDA

0. Introduction.

Let M be an n-dimensional Lorentz manifold with metric $\langle , \rangle$ of signature $(-, +, \cdots, +)$. Then there is no r-dimensional isometry group whose isotropy subgroup at every point is compact for $n(n-1)/2+1<r\leqq n(n+1)2$ (c.f., [5], Proposition). In [6], we determined n-dimensional Lorentz manifolds M which admit an $n(n-1)/2+1$-dimensional isometry group with compact isotropy subgroup at every point for $n\geqq 4$.

The first purpose of this note is to determine simply connected M admitting an $n(n-1)/2$-dimensional isometry group with compact isotropy subgroup at every point for $n\geqq 4$ (see §2). We will prove the following Theorem A.

THEOREM A. *Let $(M, \langle , \rangle)$ be a simply connected n-dimensional Lorentz manifold admitting a connected $n(n-1)/2$-dimensional isometry group with compact isotropy subgroup at every point in M $(n\geqq 4)$. Then M is isometric to the warped product manifold $(I\times N, -dt^2+\phi(t)ds_N^2)$ where I is an open interval and N is the simply connected $(n-1)$-dimensional Riemannian manifold with metric ds_N^2 of constant curvature and $\phi(t)$ is a positive function on I.*

For isometry groups whose dimension are less than $n(n-1)/2$, we will have the following proposition in §1.

PROPOSITION 1.1. *If $n\geqq 6$, there is no r-dimensional isometry group with compact isotropy subgroup at every point for $(n-1)(n-2)/2+3\leqq r\leqq n(n-1)/2-1$.*

In view of Proposition 1.1, it is natural to ask which Lorentz manifold of dimension n admits an $(n-1)(n-2)/2+2$-dimensional isometry group with compact isotropy subgroup. The second purpose of this note is to determine simply connected manifold M admitting an isometry group of dimension $(n-1)(n-2)/2+2$ with compact isotropy subgroup at every point (see §3). We will prove the following Theorem B.

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THEOREM B. *Let* $(M, \langle\ ,\ \rangle)$ *be a simply connected n-dimensional Lorentz manifold adimitting a connected* $(n-1)(n-2)/2+2$*-dimensional isomery group with compact isotropy subgroup at every point* $(n\geqq 6)$. *Then* $(M, \langle\ ,\ \rangle)$ *must be one of the following*:

(1) $(\boldsymbol{L}^2\times V^{n-1}, ds_L^2+ds_V^2)$;

(2) $(\boldsymbol{L}^2\times \boldsymbol{E}^{n-1}, -dt^2+ds^2+\exp(-2c_0t-2c_1s)ds_E^2)$ *(c_0 and c_1 are some constants such that $c_0\neq 0$* or *$c_1\neq 0$)*;

(3) $(U^2\times V^{n-2}, ds_0^2+ds_V^2)$;

(4) $(U^2\times \boldsymbol{E}^{n-2}, ds_0^2+f^2ds_E^2)$ *($f=y^{-c_2}$, c_2 is a non-zero constant)*;

(5) $(U^2\times V^{n-2}, ds_\kappa^2/\alpha^2+ds_V^2)$ *(α is a non-zero constant)*;

(6) $(U^2\times \boldsymbol{E}^{n-2}, ds_\kappa^2/\beta^2+h^2ds_E^2)$ *($h=(\beta y)^{-c_3}$, c_3 and β are non-zero constants)*;

If $n=9$, then the following additional case is possible:

(7) $(\boldsymbol{R}\times \boldsymbol{E}^8, -dt^2+\exp(-2c_4t)ds_E^2)$ *($c_4>0$: a constant)*.

Here $(\boldsymbol{L}^2, ds_L^2)$ is the 2-dimensional Minkowski space, $(\boldsymbol{E}^m, ds_E^2)$ the m-dimensional Euclidean space and (V^{n-2}, ds_V^2) the simply connected $(n-2)$-dimensional Riemannian space of constant curvature. Further, (U^2, ds_κ^2) is the upper half-space $U^2=\{(x, y); y>0\}$ with metric $-2dxdy/y^2$ (when $\kappa=0$) $\kappa(dx^2-dy^2)/y^2$ (when $\kappa=1$ or -1).

REMARK 0.1. The space (6) with $c_3=1$ is the upper half-space $U^n=\{(x_1, \cdots, x_n); x_n>0\}$ with constant curvature 1 or -1 according to $\kappa=1$ or -1 respectively. The space (7) is isometric to the 9-dimensional upper-half space with constant curvature c_4^2 by the transformation

$$\boldsymbol{R}\times \boldsymbol{E}^8\ni(t, x_1, \cdots, x_8)\longrightarrow(x_1, \cdots, x_8, e^{c_4t}/c_4)\in U^9.$$

For these spaces, see [4] and [8].

The space (4) with $c_2=1$ is the upper half-space with constant curvature 0.

Throughout this note, we shall be in C^∞-category and manifolds shall be connected, unless otherwise stated.

1. Preliminaries.

Let $(M, \langle\ ,\ \rangle)$ be an n-dimensional Lorentz manifold with metric $\langle\ ,\ \rangle$ of signature $(-, +, \cdots, +)$. Let G be a connected isometry group of $(M, \langle\ ,\ \rangle)$, H_0 the isotropy subgroup of G at a point $o\in M$ and $G(o)$ the G-orbit of o. Then the linear isotropy subgroup $\tilde{H}_0=\{dh; h\in H_0\}$ acting on T_oM is a closed subgroup of $O(1, n-1)=\{A\in GL(n, \boldsymbol{R}); {}^tASA=S\}$, where S is the matrix

$$\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

If H_o is compact, $\tilde{H}_o$ is conjugate to a subgroup of $O(1)\times O(n-1)$ (c.f., [10, p. 335]).

LEMMA 1.2. *If* $\dim H_o=(n-1)(n-2)/2$ *and* H_o *is compact, then* $\dim G(o)\leqq 1$ *or* $\geqq n-1$ *for* $n\geqq 3$.

PROOF. Since $\tilde{H}_o$ is compact and of dimension $(n-1)(n-2)/2=\dim(O(1)\times O(n-1))$, $\tilde{H}_o$ contains the connected component $1\times SO(n-1)$ of $O(1)\times O(n-1)$. Thus T_oM is naturally decomposed into the direct sum of 1-dimendional and $(n-1)$-dimensional subspaces which are $\tilde{H}_o$-invariant and irreducible. On the other hand, $T_o(G(o))$ is also $\tilde{H}_o$-invariant. Therefore we have $\dim T_o(G(o))\leqq 1$ or $\geqq n-1$.

PROOF OF PROPOSITION 1.1. Let G be a connected isometry group of dimension r. Assume that $(n-1)(n-2)/2+3\leqq r\leqq n(n-1)/2-1$. Then, $\dim H_o= \dim G-\dim(G/H_o)=\dim G-\dim G(o)\geqq(n-2)(n-3)/2+1$. Since H_o is compact, we can regard $\tilde{H}_o$ as a subgroup of $O(1)\times O(n-1)$. If $n-1\neq 4$, there is no k-dimensional subgroup of $O(n-1)$ for $(n-2)(n-3)/2<k<(n-1)(n-2)/2$. Therefore $\dim H_o=(n-1)(n-2)/2$ so that we have $3\leqq\dim G(o)\leqq n-2$. This contradicts Lemma 1.2.

REMARK 1.3. There exist 5-dimensional Lorentz manifolds M admitting a $9(=(5-1)(5-2)/2+3)$-dimensional isometry group G with compact isotropy subgroup. For example, let M be a product manifold $\boldsymbol{R}\times\boldsymbol{C}^2$ with metric $-dt^2+ds_E^2$ and $G=\boldsymbol{R}\times G'$ where ds_E^2 is the Euclidean metric of $\boldsymbol{C}^2$ and G' is the matrix group consisting of all matrices of the form

$$\begin{bmatrix} A & \tau \\ 0 & 1 \end{bmatrix}, \text{ where } A\in U(2),\ \tau\in\boldsymbol{C}^2.$$

Then $\dim G=9$ and the isotropy subgroup at the origin is $U(2)$ which is compact.

2. The case where dim $G=n(n-1)/2$.

Let G be a connected isometry group of dimension $n(n-1)/2$ with compact isotropy subgroup H_x at every point $x\in M$. Then $\tilde{H}_x$ is conjugate to a subgroup of $O(1)\times O(n-1)$, so that we have $\dim H_x \leqq (n-1)(n-2)/2$. On the other hand, $\dim H_x \geqq \dim G - \dim M = (n-1)(n-2)/2-1$. Thus we have $\dim H_x = (n-1)(n-2)/2$ or $(n-1)(n-2)/2-1$. For $n-1\neq 4$, $O(n-1)$ contains no proper closed subgroup of dimension $>(n-2)(n-3)/2$ other than $SO(n-1)$ (c.f., [2, p. 48]). Thus, when $n-1\neq 4$, $\dim H_x=(n-1)(n-2)/2$. For $n-1=4$, $O(n-1)$ contains no subgroups of dimension $5=(5-1)(5-2)/2-1$ (c.f., [1, p. 347]). Thus, for $n\geqq 4$, we have $\dim H_x=(n-1)(n-2)/2$, so $\tilde{H}_x$ contains the connected component $1\times SO(n-1)$ of $O(1)\times O(n-1)$. Therefore, T_xM is naturally decomposed into the direct sum of 1-dimensional and $(n-1)$-dimensional subspaces which are $\tilde{H}_x$-invariant and irreducible. On the other hand, $T_x(G(x))$ is $\tilde{H}_x$-invariant and of dimension $n-1$. Thus we have irreducible decomposision $T_1(x)+T_x(G(x))$ of T_xM by the linear isotropy representation of H_x on T_xM. Since H_x is compact, the restriction η of the metric of M to $T_x(G(x))$ is positive definite, zero or negative definite by the Schur's lemma. Since $n-1\geqq 3$, η must be positive definite. Therefore we have

LEMMA 2.1. *Each orbit $G(x)$ $(x\in M)$ is a spacelike hypersurface.*

Since $\tilde{H}_x$ contains $1\times SO(n-1)$, we have $\langle T_1(x), T_x(G(x))\rangle=0$ so that $T_1(x)$ is timelike. Let $\xi(x)$ be a unit timelike vector belonging to $T_1(x)$.

LEMMA 2.2. *If M is time-orientable, then the vector field $\xi(p):=dg(\xi(x))$ $(p=gx, g\in G)$ is well-defined on $G(x)$ and G-invariant and it is extended to the vector field on M.*

PROOF. The first part of this Lemma is proved by the same method as the proof of Lemma 2 in [6]. Since M is time orientable, there exists a unit timelike vector field ζ on M. Then we can extend ξ on M so as to be $\langle\xi, \zeta\rangle<0$.

From now on, we assume that M is time-orientable. We note that G acts effectively on $G(x)$. In fact, if $g\in G$ acts on $G(x)$ trivially, we have $dg|T_xG(x)=id.$ and $dg(\xi(x))=\xi(x)$, so that $dg=id.$ on $T_xM=\boldsymbol{R}\{\xi(x)\}+T_xG(x)$. Therefore $g=id.$ on M. Furthermore we note that each G-orbit $G(x)$ is isometric to $\boldsymbol{E}^{n-1}$, S^{n-1}, $\boldsymbol{P}^{n-1}$ or $\boldsymbol{H}^{n-1}$, because the $(n-1)$-dimensional Riemannian manifold $G(x)$ admits an isometry group G of maximum dimension $n(n-1)/2$.

LEMMA 2.3. *Each integral curve of ξ is a geodesic.*

PROOF. Let X be an arbitrary fixed non-zero vector in T_xM such that $\langle\xi(x), X\rangle=0$. Since $\tilde{H}_x$ contains $1\times SO(n-1)$ and $n-1\geqq 3$, there exists $h\in H_x$

such that $dh(X)=-X$ and $dh(\xi(x))=\xi(x)$. We have $\langle\nabla_\xi\xi, X\rangle=\langle dh(\nabla_\xi\xi), dh(X)\rangle=-\langle\nabla_\xi\xi, X\rangle$ so that we have $\langle\nabla_\xi\xi, X\rangle=0$. Since X is an arbitrary vector orthogonal to ξ and $\langle\nabla_\xi\xi, \xi\rangle=(1/2)\,\xi\langle\xi, \xi\rangle=0$, we have $\nabla_\xi\xi=0$. Thus each integral curve of ξ is a geodesic.

LEMMA 2.4. *$\nabla_X\xi=\lambda(\pi(X))X$ for any X such that $\langle X, \xi\rangle=0$ where π is the natural projection of the tangent bundle: $TM\to M$ and λ is a function on M which is constant on each G-orbit.*

The proof of Lemma 2.4 is similar to that of Lemma 8 in [6].

LEMMA 2.5. *The 1-form ω defined by $\omega(X)=\langle X, \xi\rangle$ is closed.*

PROOF. The 1-form ω is G-invariant and so $d\omega$ is G-invariant (especially, H_x-invariant). Since $\widetilde{H}_x$ contains $1\times SO(n-1)$ and the linear isotropy representation of H_x on $T_x(G(x))$ is irreducible, we have $d\omega=0$.

PROOF OF THEOREM A. M is time-orientable, because M is simply connected. Since ω is a closed 1-form from Lemma 2.5, there exists a smooth function $f: M\to \boldsymbol{R}$ such that $df=\omega$. Let $\gamma_p(t)$ be an integral curve of ξ such that $\gamma_p(0)=p$. Then we can see $f(\gamma_p(t))=-t+f(p)$. We may assume that $f(M)$ is some open interval containing $0\in\boldsymbol{R}$. Let N be a connected component of $f^{-1}(0)$. Then we have $N=G(o)$ for some $o\in N$. For each $x\in N$, let I_x be the domain of γ_x. Since ξ is G-invariant on $N=G(o)$, for any $p, q\in N$, we have $I_p=I_q$ which is denoted by I. Then the Theorem A will follow immediately from the next Lemma 2.6 and Lemma 2.7.

LEMMA 2.6. *The map $F: I\times N\to M$ defined by*

$$F(t, x)=\operatorname{Exp} t\xi(x)=\gamma_x(t)$$

is a diffeomorphism.

LEMMA 2.7. *The map $F: (I\times N, -dt^2+\phi(t)ds_N^2)\to(M, \langle\ ,\ \rangle)$ is an isometry, where the metric ds_N^2 on N induced from $\langle\ ,\ \rangle$ and $\phi(t)=\exp 2(\int_0^t\lambda(s)ds)$.*

The proof of Lemmas 2.6 and 2.7 is similar to that of Lemmas 5 and 9 in [6].

3. The case where dim $G=(n-1)(n-2)/2+2$.

We assume that $\dim G=(n-1)(n-2)/2+2$ and H_x is compact for every point $x\in M$.

PROPOSITION 3.1. *G acts transitively on M for $n\geqq 4$ and $n\neq 5$.*

PROOF. Assume that G does not act transitively on M. Then $\dim G(o)\leqq n-1$ for some $o\in M$. Hence $\dim H_o\geqq\dim G-(n-1)=(n-2)(n-3)/2+1$. By the same method as in the proof of Proposition 1.1, we can see that $\dim H_o=(n-1)(n-2)/2$. Hence $\dim G(o)=2$ which contradicts the Lemma 1.2.

REMARK 3.2. In the Proposition 3.1, we cannot remove the condition that the isotropy subgroup at every point is compact. In fact, let M be the Lorentz manifold $\boldsymbol{R}\times N$ with metric $dt^2+ds_N^2$, where (N, ds_N^2) is the $(n-1)$-dimensional de-Sitter space and G be the group $\boldsymbol{R}\times G'$ where G' is the matrix group of the form

$$\begin{bmatrix} (1+a^2+|\chi|^2)/(2a) & \chi & (1-a^2+|\chi|^2)/(2a) \\ (1/a)A^t\chi & A & (1/a)A^t\chi \\ (1-a^2-|\chi|^2)/(2a) & -\chi & (1+a^2-|\chi|^2)/(2a) \end{bmatrix} \quad \begin{array}{l} a>0,\ \chi\in\boldsymbol{R}^{n-2}, \\ A\in SO(n-2). \end{array}$$

G' is the connected subgroup of the proper Lorentz group $SO^+(1, n-1)$ acting on N (c.f. [7]). Then G is an $(n-1)(n-2)/2+2$-dimensional isometry group which has noncompact isotropy subgroups and does not act on M transitively (see § 4).

REMARK 3.3. There exists a 5-dimensional Lorentz manifold M aditting an $8(=(5-1)(5-2)/2+2)$-dimensional isometry group G with compact isotropy subgroup such that G does not acts transitively on M. In fact, take the space in Remark 1.3 as M and set $G=1\times G'$ (G' is the same as in Remark 1.3). Then G is not transitive on M.

From now on, we assume $n\geqq 6$. Set $H=H_o$ for some $o\in M$. By Proposition 3.1, we have $\dim H=(n-2)(n-3)/2$. Since H is compact and connected, $\tilde{H}$ is conjugate to a subgroup of $SO(1)\times SO(n-1)$ so that we can regard $\tilde{H}$ as an $(n-2)(n-3)/2$-dimensional subgroup of $SO(n-1)$. In the case $n-1\neq 8$, a $(n-2)(n-3)/2$-dimensional subgroup $\tilde{H}$ of $SO(n-1)$ leaves one and only one 1-dimensional subspace of $\boldsymbol{R}^{n-1}$ invariant. In the case $n-1=8$, we have either $\tilde{H}=SO(7)$ (which leaves one and only one 1-dimensional subspace of $\boldsymbol{R}^8$ invariant) or $\tilde{H}=Spin(7)$ with spin representation (see Kobayoshi [2, p. 49]).

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. By the use of an $Ad(H)$-invariant positive definite inner product on $\mathfrak{g}$ whose existence is guaranteed by the compactness of H, we have a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ (direct sum) of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{m}]\subset\mathfrak{m}$. Let $\pi: G\to G/H$ be the natural projection. We identify the tangent space T_oM and $\mathfrak{m}$ by $d\pi$. The Lorentz inner product on T_oM induces the Lorentz inner product $\langle\ ,\ \rangle_{\mathfrak{m}}$ on $\mathfrak{m}$ so that $d\pi: \mathfrak{m}\to T_oM$ is a linear isometry. Then the linear isotropy group $\tilde{H}$ acting on T_oM corresponds to $Ad(H)$ on $\mathfrak{m}$ by means of $d\pi$. We note that the inner product $\langle\ ,\ \rangle_{\mathfrak{m}}$ is $Ad(H)$-invariant. We define the Lorentz inner product B on $\mathfrak{g}$ so that

$$B(\mathfrak{h}, \mathfrak{m})=0, \quad B|_{\mathfrak{m}}=\langle\ ,\ \rangle_{\mathfrak{m}}$$

and $B|_{\mathfrak{h}}$ is positive definite. We extend B to the G-left invariant Lorentz metric on G which is denoted by the same letter B. Then (G, B) is a Lorentz manifold and $\pi: G\to G/H=M$ is the semi-Riemannian submersion (for the definition of the semi-Riemannian submersion, see O'Neill [9, p. 212]).

The structure of $\mathfrak{g}$ *for* $n-1\neq 8$. We assume $n-1\neq 8$. Since $Ad(H)$ is compact and $\dim Ad(H)=(n-2)(n-3)/2$, $Ad(H)$ acts on $\mathfrak{m}$ as $I_2\times SO(n-2)$. Then $\mathfrak{m}$ decomposes naturally into 2-dimensional subspace $\mathfrak{m}_2$ and $(n-2)$-dimensional subspace $\mathfrak{m}_1$ such that $Ad(H)|_{\mathfrak{m}_2}=id.$ and $Ad(H)|_{\mathfrak{m}_1}=SO(n-2)$. Using Schur's lemma, we have that $\mathfrak{m}_1$ is spacelike. Furthermore, we have $\langle\mathfrak{m}_1, \mathfrak{m}_2\rangle_{\mathfrak{m}}=0$ so that $\mathfrak{m}_2$ is timelike. Thus we have a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}_1+\mathfrak{m}_2$ such that

$$[\mathfrak{h}, \mathfrak{m}_1]\subset\mathfrak{m}_1, \quad [\mathfrak{h}, \mathfrak{m}_2]=\{0\}.$$

LEMMA 3.4. $[\mathfrak{m}_2, \mathfrak{m}_1]$ *is either* $\{0\}$ *or* $\mathfrak{m}_1$. *More precisely, there exists a linear map* $L: \mathfrak{m}_2\to \boldsymbol{R}$ *such that* $[A, X]=L(A)X$ *for any* $A\in\mathfrak{m}_2$ *and any* $X\in\mathfrak{m}_1$. *Here* L *is either zero or onto map.*

PROOF. For any fixed $A\in\mathfrak{m}_2$, we define a linear map $f_A: \mathfrak{m}_1\to\mathfrak{g}$ by $f_A(X)=[A, X]$ $(X\in\mathfrak{m}_1)$. Let p_0, p_1 and p_2 be orthogonal projection from $\mathfrak{g}$ to $\mathfrak{h}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively. Since $\mathfrak{h}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are $Ad(H)$-invariant and $Ad(h)f_A=f_A Ad(h)$ for any $h\in H$, we have

$$(*) \qquad p_i f_A Ad(h)=Ad(h)p_i f_A \quad \text{for any} \quad h\in H (i=0, 1, 2).$$

Step 1. We claim $[\mathfrak{m}_2, \mathfrak{m}_1]\subset\mathfrak{h}+\mathfrak{m}_1$. Since $Ker(p_2 f_A)$ is $Ad(H)$-invariant by $(*)$ and the adjoint representation of H on $\mathfrak{m}_1$ is irreducible, we have $Ker(p_2 f_A)=\{0\}$ or $\mathfrak{m}_1$. Suppose $Ker(p_2 f_A)=\{0\}$ for some $A\in\mathfrak{m}_2$. Then $p_2 f_A: \mathfrak{m}_1\to\mathfrak{m}_2$ is injective so that $\dim Im(p_2 f_A)=n-2>2=\dim\mathfrak{m}_2$. Hence we have $Ker(p_2 f_A)=\mathfrak{m}_1$

for any $A\in\mathfrak{m}_2$, that is, $[\mathfrak{m}_2, \mathfrak{m}_1]\subset\mathfrak{h}+\mathfrak{m}_1$.

Step 2. We claim $[\mathfrak{m}_2, \mathfrak{m}_1]\subset\mathfrak{m}_1$. By the same procedure as that of Step 1, we have $Ker(p_0 f_A)=\{0\}$ or $\mathfrak{m}_1$. Suppose $Ker(p_0 f_A)=\{0\}$ for some $A\in\mathfrak{m}_2$. Then $\dim p_0 f_A(\mathfrak{m}_1)=n-2$. We can verify easily that $p_0 f_A(\mathfrak{m}_1)$ is ideal in $\mathfrak{h}$. On the other hand, there is no ideal of dimension $n-2$ in $\mathfrak{h}=\mathfrak{so}(n-2)$. Hence we have $Ker(p_0 f_A)=\mathfrak{m}_1$ for any $A\in\mathfrak{m}_2$, that is $[\mathfrak{m}_2, \mathfrak{m}_1]\subset\mathfrak{m}_1$.

Step 3. By the above discussion, f_A is a linear map from $\mathfrak{m}_1$ into itself and commutes with the action of $Ad(H)=SO(n-2)$ on $\mathfrak{m}_1$. Hence there exists linear map $L: \mathfrak{m}_2\to\boldsymbol{R}$ such that $[A, X]=L(A)X$ $(A\in\mathfrak{m}_2, X\in\mathfrak{m}_1)$.

LEMMA 3.5. $[\mathfrak{m}_1, \mathfrak{m}_1]\subset\mathfrak{h}$.

PROOF. Let p_0, p_1 and p_2 be maps as in the proof of Lemma 3.4. Given orthonormal vectors X and Y in $\mathfrak{m}_1$, there exists $h\in H$ such that $Ad(h)=id.$ on $\mathfrak{m}_2$ and $Ad(h)X=-X$, $Ad(h)Y=Y$ (for, $n-2\geqq 4$). Then we have

$$p_2[X, Y]=Ad(h)p_2[X, Y]=p_2[Ad(h)X, Ad(h)Y]$$
$$=-p_2[X, Y]$$

which implies $p_2[X, Y]=0$. Hence $p_2[\mathfrak{m}_1, \mathfrak{m}_1]=\{0\}$. Let express $p_1[X, Y]$ as $aX+bY+cZ$, where Z is a unit vector orthogonal to X and Y. Since $n-2\geqq 4$, there exists $h'\in H$ such that $Ad(h')=id.$ on $\mathfrak{m}_2$ and $Ad(h')X=-X$, $Ad(h')Y=-Y$, $Ad(h')Z=-Z$. The equality $Ad(h')p_1[X, Y]=p_1 Ad(h')[X, Y]$ implies $p_1[X, Y]=0$. Thus we have $[\mathfrak{m}_1, \mathfrak{m}_1]\subset\mathfrak{h}$.

From the same method as in Kobayashi and Nagano [3, p. 212], we have

LEMMA 3.6. $[\mathfrak{m}_2, \mathfrak{m}_2]\subset\mathfrak{m}_2$.

From Lemma 3.6, there exists a basis $\{e_0, e_1\}$ of $\mathfrak{m}_2$ such that $B(e_0, e_0)=-1$, $B(e_1, e_1)=1$ and $B(e_0, e_1)=0$, and there exist constants a and b such that $[e_0, e_1]=ae_0+be_1$. Then there are the following four possibilities:

CASE I: $[e_0, e_1]$ is a zero vector (i.e., $\mathfrak{m}_2$ is commutative);

CASE II: $[e_0, e_1]$ is a non-zero null vector (i.e., $a\neq 0$, $b=\delta a$, where $\delta^2=1$);

CASE III: $[e_0, e_1]$ is a spacelike vector (i.e., $b^2-a^2=\alpha^2$, $\alpha>0$);

CASE IV: $[e_0, e_1]$ is a timelike vector (i.e., $b^2-a^2=-\alpha^2$, $\alpha>0$).

There exists a basis f_0, f_1 such that

in case II,

$$B(f_0, f_0)=0=B(f_1, f_1), \quad B(f_0, f_1)=-1, \quad [f_0, f_1]=f_1,$$

in case III,

$$B(f_0, f_0)=-1, \quad B(f_1, f_1)=1, \quad B(f_0, f_1)=0 \quad \text{and} \quad [f_0, f_1]=\alpha f,$$

in case IV,

$$B(f_0, f_0)=1, \quad B(f_1, f_1)=-1, \quad B(f_0, f_1)=0 \quad \text{and} \quad [f_0, f_1]=\alpha f_1.$$

In case I, we denote e_0 and e_1 by f_0 and f_1 respectively. Hereafter, in any cases, we consider f_0 and f_1 instead of e_0 and e_1. Furthermore, in any cases, we denote $L(f_0)$ and $L(f_1)$ by c_0 and c_1 respectively where L is the linear map in Lemma 3.4.

LEMMA 3.7. *In cases II, III, and IV, we have* $c_1=0$.

PROOF. Let X be a non-zero vector belonging to $\mathfrak{m}_1$. By the Jacobi's identity

$$[f_0, [f_1, X]]=[[f_0, f_1], X]+[f_1, [f_0, X]],$$

we have $c_0c_1X=\beta c_1X+c_0c_1X$ ($\beta=1$ or α) so that we have $c_1=0$.

Determination of M for $n-1\neq 8$. Since M is simply connected, H is connected so that $Ad(H)$ acts on $\mathfrak{m}_2$ as the identity transformation. Therefore we have

LEMMA 3.8. *For each $f_u\in\mathfrak{m}_2$ ($u=0, 1$), the vector field ξ_u defined by*

$$\xi_u(p) := dgd\pi(f_u(e)) \qquad (p=g(o),\ g\in G)$$

is well-defined on M and G-invariant where e is the identity in G.

We have the following formulas (**) according to the above each case I$\sim$IV:

CASE I. $\nabla_{\xi_u}\xi_v=0,\ \nabla_X\xi_u=-c_uX$ ($u, v=0, 1$);

CASE II. $\nabla_{\xi_0}\xi_0=-\xi_0,\ \nabla_{\xi_0}\xi_1=\xi_1,\ \nabla_{\xi_1}\xi_0=0,$

(**) $\qquad \nabla_{\xi_1}\xi_1=0, \quad \nabla_X\xi_0=-c_0X, \quad \nabla_X\xi_1=0;$

CASES III and IV. $\nabla_{\xi_0}\xi_0=0,\ \nabla_{\xi_0}\xi_1=0,\ \nabla_{\xi_1}\xi_0=-\alpha\xi_1,$

$\qquad \nabla_{\xi_1}\xi_1=-\alpha\xi_0, \quad \nabla_X\xi_0=-c_0X, \quad \nabla_X\xi_1=0.$

Here X is any vector field orthogonal to ξ_0 and ξ_1 and ∇ is the Levi-Civita connection of the Lorentz metric $\langle\ ,\ \rangle$ on M.

By the G-invariance of ξ_u and the above formulas, we have

LEMMA 3.9. (1) *In the cases I and II, the integral curve of ξ_1 is a complete geodesic.*

(2) *In the cases I, III and IV, the integral curve of* ξ_0 *is a complete geodesic.*

By the similar way as the proof of Lemma 2.5, we have

LEMMA 3.10. (1) *In the cases I, III and IV, the* 1*-form* ω_0 *on* M *defined by*

$$\omega_0(X):=\langle X, \xi_0\rangle$$

is G-invariant and closed.

(2) *In the cases I and II, the* 1*-form* ω_1 *on* M *defined by*

$$\omega_1(X):=\langle X, \xi_1\rangle$$

is G-invariant and closed.

Now we will determine $G/H=M$ in each cases I, II, III and IV.

CASE I. Lemma 3.10 implies that there exist smooth functions ψ_0 and ψ_1 such that $d\psi_u=\omega_u$ $(u=0, 1)$. Since ξ_0 and ξ_1 are G-invariant, there exist 1-parameter groups of transformation ϕ_t^0 and ϕ_s^1 generated by ξ_0 and ξ_1 respectively. We can verify easily that for $p\in M$,

$$(\sharp)\qquad \begin{cases}\psi_0(\phi_t^0(p))=-t+\psi_0(p), & \psi_0(\phi_s^1(p))=\psi_0(p),\\ \psi_1(\phi_t^0(p))=\psi_1(p), & \psi_1(\phi_s^1(p))=s+\psi_1(p).\end{cases}$$

Let M_1^0 be a connected component of $M_1=\{p\in M;\psi_0(p)=\psi_1(p)=0\}$. Then M_1^0 is a connected $(n-2)$-dimensional closed submanifold of M. Furthermore M_1^0 is spacelike, because ξ_0 and ξ_1 are orthogonal to M_1.

LEMMA 3.11. *The map* $F: \boldsymbol{R}\times\boldsymbol{R}\times M_1^0\to M$ *defined by*

$$F(t, s, x)=\phi_t^0(\phi_s^1(x))$$

is a diffeomorphism, and $M_1=M_1^0$ *is simply connected.*

PROOF. Suppose that $F(t, s, x)=F(t', s', x')$. Then, from $(\sharp)$, we have $t=t'$ and $s=s'$. Therefore we have $\phi_t^0(\phi_s^1(x))=\phi_t^0(\phi_t^1(x'))$ so that we have $x=x'$. Thus F is injective. It is clear that F is smooth. Setting $N=F(\boldsymbol{R}\times\boldsymbol{R}\times M_1^0)$, then N is open in M. It remains to be shown that N is closed in M. Suppose that $\{F(t_k, s_k, x_k)=p_k\}$ is a sequence converging some point q in M. Since $t_k=-\psi_0(p_k)$ and $s_k=\psi_1(p_k)$, we have $t_k\to t_0:=-\psi_0(q)$ and $s_k\to s_0:=\psi_1(q)$ as $k\to\infty$. Since $x_k=\phi_{-s_k}^1(\phi_{-t_k}^0(p_k))$ converges $x_0:=\phi_{-s_0}^1(\phi_{-t_0}^0(q))$ as $k\to\infty$ and M_1^0 is closed, x_0 belongs to M_1^0 so that $q=\phi_{t_0}^0(\phi_{s_0}^1(x_0))$ belongs to N. Thus N is closed. Thus we have $N=F(\boldsymbol{R}\times\boldsymbol{R}\times M_1^0)$.

REMARK 3.12. For each $(a, b)\in \boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b):=\{p\in M; \psi_0(p)=a, \psi_1(p)=b\}$ is a simply connected $(n-2)$-dimensional spacelike submanifold of M.

LEMMA 3.13. *For each* $(a, b)\in \boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b)$ *is congruent to* $M_1=M_1(0, 0)$ *in* M.

PROOF. Since G acts on M transitively, for some point p in $M_1(a, b)$ there exists $g\in G$ such that $g(o)=p (o\in M_1)$. Then we have $g(M_1)\subset M_1(a, b)$. In fact, for each point $q\in g(M_1)$, there exists a smooth curve $\tilde{c}: [0, 1]\to g(M_1)$ such that $\tilde{c}(0)=p$ and $\tilde{c}(1)=q$. Put $c:=g^{-1}\tilde{c}$. Then c is a smooth curve in M_1, so we have $\psi_0(c(s))=0=\psi_1(c(s))$ for any $s\in[0, 1]$. Therefore we have

$$(d\psi_u/ds)(\tilde{c}(s))=\langle\xi_u(\tilde{c}(s)), \dot{\tilde{c}}(s)\rangle=\langle dg\xi_u(c(s)), dg\dot{c}(s)\rangle$$
$$=\langle\xi_u(c(s)), \dot{c}(s)\rangle=(d\psi_u/ds)(c(s))=0 \qquad (u=0, 1).$$

Thus we have $\psi_0(q)=a$ and $\psi_1(q)=b$ so that we have $g(M_1)\subset M_1(a, b)$. Since $g(M_1)$ is open and closed in $M_1(a, b)$, we have $g(M_1)=M_1(a, b)$.

LEMMA 3.14. M_1 *is a homogeneous Riemannian manifold.*

PROOF. For any $p, q\in M_1$, there exists $g\in G$ such that $g(p)=q$. By the same method as in the proof of Lemma 3.13, we can see that $g|_{M_1}$ is an isometric transformation of M_1.

Set $G_1:=\{g\in G; gM_1=M_1\}$. Then G_1 is a Lie subgroup of G. We can verify that H is included in G_1 by the same discussion as in the proof of Lemma 3.13. Furthermore, G_1 acts on M_1 effectively. Thus $\dim G_1=\dim M_1+\dim H=(n-1)(n-2)/2$. Therefore the simply connected $(n-2)$-dimensional Riemannian manifold M_1 admitting an isometry group G_1 of maximum dimension $(n-1)(n-2)/2$ is isometric to S^{n-2}, $\boldsymbol{H}^{n-2}$ or $\boldsymbol{E}^{n-2}$.

LEMMA 3.15. *The map*

$$F: (\boldsymbol{R}\times\boldsymbol{R}\times M_1, -dt^2+ds^2+\exp(-2c_0t-2c_1s)ds^2_{M_1}) \longrightarrow (M, \langle\ ,\ \rangle)$$

is an isometry where $ds^2_{M_1}$ *is the metric of* M_1.

PROOF. Let $(V, \Phi=(t_2, \cdots, t_{n-1}))$ be a local coordinate around a point p in M_1. Then $(\boldsymbol{R}\times\boldsymbol{R}\times V, id\times\Phi=(t, s, t_2, \cdots, t_{n-1}))$ is a local coordinate around (a, b, p) in $\boldsymbol{R}\times\boldsymbol{R}\times M_1$. Put $\tilde{V}:=F(\boldsymbol{R}\times\boldsymbol{R}\times M_1)$ and define $\tilde{\Phi}: \tilde{V}\to\boldsymbol{R}^n$ by $(id\times\Phi)\circ F^{-1}$. Then $(\tilde{V}, \tilde{\Phi}=(x_0, x_1, \cdots, x_{n-1}))$ is a local coordinate around $\tilde{p}=F(a, b, p)$ in M. Since $[\xi_0, \xi_1]=0$, we can see $dF(\partial/\partial t)=\partial/\partial x_0=\xi_0$ and $dF(\partial/\partial s)=\partial/\partial x_1=\xi_1$. Furthermore we have $dF(\partial/\partial t_j)=\partial/\partial x_j$ $(j=2, \cdots, n-1)$. We can also see that $\langle\partial/\partial x_u, \partial/\partial x_j\rangle=0$ $(u=0, 1)$. In fact

$$\langle\partial/\partial x_u, \partial/\partial x_j\rangle=\langle\xi_u, dF(\partial/\partial t_j)\rangle=(\partial/\partial t_j)(\phi_u(F(t, s, x))$$
$$=\begin{cases}(\partial/\partial t_j)(-t)=0 & (u=0)\\ (\partial/\partial t_j)(s)=0 & (u=1)\end{cases}.$$

Since $\nabla_X\xi_u=-c_uX$ $(u=0, 1)$ for any X orthogonal to ξ_0 and ξ_1, we have

$$\partial/\partial t\langle\partial/\partial t_i, \partial/\partial t_j\rangle=-2c_0\langle\partial/\partial t_i, \partial/\partial t_j\rangle$$

and

$$\partial/\partial s\langle\partial/\partial t_i, \partial/\partial t_j\rangle=-2c_1\langle\partial/\partial t_i, \partial/\partial t_j\rangle$$

so that we have

$$\langle\partial/\partial t_i, \partial/\partial t_j\rangle=\exp(-2c_0t-2c_1s)g_{ij}(t_2, \cdots, t_{n-1}).$$

Thus we have

$$F^*\langle\ ,\ \rangle=-dt^2+ds^2+\exp(-2c_0t-2c_1s)ds^2_{M_1}.$$

LEMMA 3.16. *If M_1 is $\boldsymbol{H}^{n-2}$ or S^{n-2}, then $c_0=c_1=0$, i.e., the metric of $\boldsymbol{R}\times\boldsymbol{R}\times M_1$ is a product metric.*

PROOF. Since, for each $(a, b)\in\boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b)$ is isometric to M_1 by Lemma 3.13, the scalar curvature $S(a, b)$ of $M_1(a, b)$ coincides with the scalar curvature $S(0, 0)$ of M_1 which is non-zero. On the other hand, we have $S(a, b)=\exp(-2c_0a-2c_1b)\times S(0, 0)$ by Lemma 3.15. Since a and b are arbitrary, we have $c_0=c_1=0$.

We notice that, in the case $M_1=\boldsymbol{E}^{n-2}$, there are two cases (1) $c_0=c_1=0$ and (2) $c_0\neq 0$ or $c_1\neq 0$.

Summing up, in the case I, $(M, \langle\ ,\ \rangle)$ must be one of the following:

(i) $(\boldsymbol{L}^2\times M_1, ds^2_L+ds^2_{M_1})$ where $(\boldsymbol{L}^2, ds^2_L)$ is the 2-dimensional Minkowski space and $(M_1, ds^2_{M_1})$ is a simply connected $(n-2)$-dimensional Riemannian manifold of constant curvature;

(ii) $(\boldsymbol{R}^2\times\boldsymbol{E}^{n-2}, -dt^2+ds^2+\exp(-2c_0t-2c_1s)ds^2_E)$ where $c_0\neq 0$ or $c_1\neq 0$.

CASE II. Since ω_1 is closed, there exists a smooth function $\phi_1: M\rightarrow\boldsymbol{R}$ such that $d\phi_1=\omega_1$. Define the vector field η on M by $\eta(p):=\exp(-\phi_1(p))\xi_0(p)$ $(p\in M)$.

LEMMA 3.17. *The 1-form $\tilde{\omega}_0$ defined by $\tilde{\omega}_0(X):=\langle\eta, X\rangle$ is closed so that there exists smooth function $\tilde{\phi}_0: M\rightarrow\boldsymbol{R}$ such that $d\tilde{\phi}_0=\tilde{\omega}_0$.*

PROOF. Since $d\tilde{\omega}_0(X, Y)=\langle\nabla_X\eta, Y\rangle-\langle\nabla_Y\eta, X\rangle$ for any vector fields X and Y, we can verify that $\tilde{\omega}_0$ is closed by formulas (**).

Since ξ_0 is G-invariant, there exists the 1-parameter group of transformations ϕ_s^0 generated by ξ_0. Let $c_p(t)$ be the integral curve of ξ_1 through a point $p\in M$. From the G-invariance of ξ_1, $c_p(t)$ is defined for any $t\in\boldsymbol{R}$. Define the vector field ζ on M by $\zeta(q)=\exp(\psi_1(q))\xi_1(q)$ $(q\in M)$. Let ϕ_t^1 be the 1-parameter group of transformations generated by ζ. Then we have $\phi_t^1(p)=c_p(\exp(\psi_1(p))t)$ so that ϕ_t^1 is complete. Noting that $[\xi_0, \zeta]=0$, we have $\phi_s^0\phi_t^1=\phi_t^1\phi_s^0$. We can verify the following:

$$\tilde{\psi}_0(\phi_s^0(p))=\tilde{\psi}_0(p),\quad \tilde{\psi}_0(\phi_t^1(p))=-t+\tilde{\psi}_0(p),$$

$$\psi_1(\phi_s^0(p))=-s+\psi_1(p),\quad \psi_1(\phi_t^1(p))=\psi_1(p)\quad \text{for } p\in M.$$

Let M_1^0 be a connected component of $M_1:=\{p\in M;\ \tilde{\psi}_0(p)=\psi_1(p)=0\}$. Then M_1^0 is an $(n-1)$-dimensional closed submanifold of M. Furthermore M_1^0 is spacelike, because ξ_0 and ξ_1 are orthogonal to M_1^0.

LEMMA 3.18. *The map $F: \boldsymbol{R}\times\boldsymbol{R}\times M_1^0\to M$ defined by*

$$F(t, s, x)=\phi_t^1\phi_s^0(x)\quad \text{for}\quad (t, s, x)\in\boldsymbol{R}\times\boldsymbol{R}\times M_1^0$$

is a diffeomorphism, and $M_1=M_1^0$ is simply connected.

The proof is similar to that of Lemma 3.11.

REMARK 3.19. For each $(a, b)\in\boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b):=\{p\in M;\ \tilde{\psi}_0(p)=a,\ \psi_1(p)=b\}$ is a simply connected $(n-2)$-dimensional spacelike submanifold of M.

The following two Lemma 3.20 and 3.21 are proved by the same method as in Lemma 3.13 and 3.14 respectively.

LEMMA 3.20. *For each $(a, b)\in\boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b)$ is congruent to M_1 in M.*

LEMMA 3.21. *M_1 is a homogeneous Riemannian manifold.*

Set $G_1:=\{g\in G;\ g(M_1)=M_1\}$. Then we also have that G_1 is a closed Lie subgroup of G and includes H. G_1 acts effectively on M_1 so that M_1 is S^{n-2}, $\boldsymbol{H}^{n-2}$ or $\boldsymbol{E}^{n-2}$.

LEMMA 3.22. *The map $F: (\boldsymbol{R}\times\boldsymbol{R}\times M_1,\ -2\exp(-s)dtds+\exp(-2c_0s)ds_{M_1}^2)\to(M, \langle\ ,\ \rangle)$ is an isometry.*

PROOF. As in the proof of Lemma 3.15, we take a local coordinate $(V, \Phi=(t_2, \cdots, t_{n-2}))$ around a point p in M_1 and a local coordinate $(\tilde{V}, \tilde{\Phi}=(x_0, x_1, \cdots, x_{n-1}))$ around a point $F(a, b, p)$ in M. Then we can see $dF(\partial/\partial t)=\partial/\partial x_0=\exp(-s)\xi_1$, $dF(\partial/\partial s)=\partial/\partial x_1=\xi_0$ and $dF(\partial/\partial t_i)=\partial/\partial x_i$ $(i=2, \cdots, n-1)$ at $(t, s, p) \in \boldsymbol{R}\times\boldsymbol{R}\times M_1$. Furthermore, we can see

$$\partial/\partial s\langle\partial/\partial t_i, \partial/\partial t_j\rangle=-2c_0\langle\partial/\partial t_i, \partial/\partial t_j\rangle$$

and

$$\partial/\partial t\langle\partial/\partial t_i, \partial/\partial t_j\rangle=0 \quad (i, j=2, \cdots, n-1)$$

so that we have

$$\langle\partial/\partial t_i, \partial/\partial t_j\rangle=\exp(-2c_0 s)g_{ij}(t_2, \cdots, t_{n-1}).$$

Uhus we have

$$F^*\langle\ ,\ \rangle=-2\exp(-s)dtds+\exp(-2c_0 s)ds_{M_1}^2.$$

We also have the following Lemma 3.23 by the same method as in the case I.

LEMMA 3.23. *If M_1 is S^{n-2} or $\boldsymbol{H}^{n-2}$, then $c_0=0$.*

We note that the space $(\boldsymbol{R}\times\boldsymbol{R}, -2\exp(-s)dtds)$ is isometric to the upper half-space $U^2=\{(x, y); y>0\}$ with flat metric $-2dxdy/y^2$ by the transformation $(t, s)\to(x, y)=(t, \exp(s))$.

Thus, in case II, M must be one of the following:

(iii) $(U^2\times M_1, -2dxdy/y^2+ds_{M_1}^2)$ where $(M_1, ds_{M_1}^2)$ is a simply connected $(n-2)$-dimensional Riemannian manifold of constant curvature;

(iv) $(U^2\times \boldsymbol{E}^{n-2}, -2dxdy/y^2+(1/y)^{2c_0}ds_E^2)$.

REMARK 3.24. When $c_0=1$, the space (iv) is the n-dimensional upper half-space $U^2=\{(x_1, \cdots, x_n): x_n>0\}$ with flat metric

$$(1/x_n^2)(-2dx_{n-1}dx_n+dx_1^2+\cdots+dx_{n-2}^2).$$

CASE III and IV. Since ω_0 is closed by Lemma 3.10, there exists a smooth function $\phi_0: M\to\boldsymbol{R}$ with $d\phi_0=\omega_0$. Put $\eta(p)=\exp(-\kappa\alpha\phi_0(p))\xi_1(p)$ where $\kappa=\langle\xi_1, \xi_1\rangle$ (i.e., $\kappa=1, -1$ in the cases III, IV respectively). Define a 1-form $\tilde{\omega}_1$ by $\tilde{\omega}_1(X)=\langle X, \eta\rangle$. Then we have the following Lemma by the same method as in Lemma 3.17.

LEMMA 3.25. *$\tilde{\omega}_1$ is a closed 1-form so that there exists a smooth function $\tilde{\phi}_1: M\to\boldsymbol{R}$ with $d\tilde{\phi}_1=\tilde{\omega}_1$.*

Since ξ_0 is G-invariant, there exists the 1-parameter group of transformations ϕ_t^0 generated by ξ_0. Let $c_p(s)$ be an integral curve of ξ_1 through a point $p \in M$. Then, for each point $p \in M$, $c_p(t)$ is defined for any $t \in \boldsymbol{R}$, because of the G-invariance of ξ_1. Define the vector field ζ on M by $\zeta(p)=\exp(\kappa\alpha\psi_0(p))\xi_1(p)$ ($p \in M$). Let ϕ_s^1 be the 1-parameter group of transformations generated by ζ. Then we have $\phi_s^1(p)=c_p(\exp(\kappa\alpha\psi_0(p))s)$ so that ϕ_s^1 is complete. Noting $[\xi_0, \zeta]=0$, we have $\phi_t^0\phi_s^1=\phi_s^1\phi_t^0$. We can verify the following:

$$\psi_0(\phi_t^0(p))-\kappa t+\psi_0(p), \quad \psi_0(\phi_s^1(p))=\psi_0(p)$$
$$\tilde{\psi}_1(\phi_t^0(p))=\tilde{\psi}_1(p), \quad \tilde{\psi}_1(\phi_s^1(p))=\kappa s+\tilde{\psi}_1(p).$$

Let M_1^0 be a connected component of $M_1:=\{p \in M; \psi_0(p)=\tilde{\psi}_1(p)=0\}$. Then by the same procedure as in the case II, we have Lemmas 3.26, 3.27, 3.29, 3.30 and Remark 3.28.

LEMMA 3.26. *M_1^0 is a connected $(n-2)$-dimensional spacelike closed submanifold of M.*

LEMMA 3.27. *The map $F: \boldsymbol{R}\times\boldsymbol{R}\times M_1^0 \to M$ defined by*

$$F(t, s, x)=\phi_s^1\phi_t^0(x) \qquad \text{for} \quad (t, s, x) \in \boldsymbol{R}\times\boldsymbol{R}\times M_1^0$$

is a diffeomorphism, and $M_1=M_1^0$ is simply connected.

REMARK 3.28. For each $(a, b) \in \boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b):=\{p \in M; \psi_0(p)=a, \tilde{\psi}_1(p)=b\}$ is a simply connected $(n-2)$-dimensional spacelike submanifold of M.

LEMMA 3.29. *For each $(a, b) \in \boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b)$ is congruent to M_1 in M.*

LEMMA 3.30. *M_1 is a homogeneous Riemannian manifold.*

By the same method as in the case II, M_1 is isometric to S^{n-1}, $\boldsymbol{H}^{n-1}$ or $\boldsymbol{E}^{n-2}$. We also have following Lemmas 3.31 and 3.32.

LEMMA 3.31. *The map*

$$F: (\boldsymbol{R}\times\boldsymbol{R}\times M_1, -\kappa(dt^2-\exp(-2\alpha t)ds^2)+\exp(-2c_0 t)ds_{M_1}^2) \to (M, \langle\ ,\ \rangle)$$

is an isometry.

LEMMA 3.32. *If $M_1=S^{n-2}$ or $\boldsymbol{H}^{n-2}$, then $c_0=0$.*

We note that $(\boldsymbol{R}\times\boldsymbol{R}, -\kappa(dt^2-\exp(-2\alpha t)ds^2)$ is isometric to $(U^2=\{(x, y); y>0\}, ds_\kappa^2=\kappa(dx^2-dy^2)/(\alpha y)^2)$ by the transformation $(t, s) \to (x=s, y=\exp(\alpha t)/\alpha)$.

Thus, in case III, $(M, \langle\ ,\ \rangle)$ must be one of the following:

(v) $(U^2\times M_1,\ ds_{+1}^2/\alpha^2+ds_{M_1}^2)$;

(vi) $(U^2\times \boldsymbol{E}^{n-2},\ ds_{+1}^2/\alpha^2+(1/\alpha y)^{2c/\alpha}ds_E^2)$,

and in case IV, $(M, \langle\ ,\ \rangle)$ must be one of the spaces

(vii) $(U^2\times M_1,\ ds_{-1}^2/\alpha^2+ds_{M_1}^2)$,

(viii) $(U^2\times \boldsymbol{E}^{n-2},\ ds_{-1}^2/\alpha^2+(1/\alpha y)^{2c/\alpha}ds_E^2)$,

where $(M_1, ds_{M_1}^2)$ is a simply connected $(n-2)$-dimensional Riemannian manifold of constant curvature.

The case $n=9$. When $n-1=8$, $\tilde{H}$ is isomorphic to $SO(7)$ or $Spin(7)$ which has a spin representation. When H is isomorphic to $SO(7)$, the argument is the same as in the case $n-1\neq 8$. Therefore it is enough to deal with the case that H is isomorphic of $Spin(7)$.

Since $\tilde{H}$ is conjugate to the subgroup $Spin(7)$ of $SO(8)$, there exists a timelike G-invariant vector field ξ on M with $\langle\xi, \xi\rangle=-1$.

By the same method as the proof of Lemma 2.5, we have

LEMMA 3.33. *The 1-form* ω *defined by* $\omega(X)=\langle\xi, X\rangle$ *is G-invariant and closed so that threre exists a smooth function* $f: M\to \boldsymbol{R}$ *with* $df=\omega$.

The G-invariance of ξ implies the completeness of ξ. There exists the 1-parameter group of transformations ϕ_t generated by ξ. Then we have $f(\phi_t(p))=-t+f(p)$ $(t\in \boldsymbol{R},\ p\in M)$. Put $N=\{p\in M;\ f(p)=0\}$. Then a connected component N^o of N is a connected closed 8-dimensional spacelike hypersurface of M. By the similar way as in the case I, N^o is a homogeneous Riemannian manifold admitting an isometry group $G':=\{g\in G;\ g(N^o)=N^o\}$ of dimension $8(8-1)/2+1=29$ which acts effectively on N^o and includes H. Then, by the theorem in [8], N^o is isometric to $\boldsymbol{E}^8$ and $G'=Spin(7)\boldsymbol{R}^8$ (a semi-direct product). We have $\nabla_X\xi=-cX$ for any X orthogonal to ξ where c is a constant. In fact, $Spin(7)$ acts transitively on $S^7:=\{Z\in T_XM;\ \langle Z, \xi\rangle=0,\ \langle Z, Z\rangle=1\}$ so that the proof is the same as in [6, Lemma 8]. We also have that the map $F: \boldsymbol{R}\times N^o\to M$ defined by $F(t, x)=\phi_t(x)$ for $(t, x)\in \boldsymbol{R}\times N^o$ is a diffeomorphism and the map $F: (\boldsymbol{R}\times N^o,\ -dt^2+\exp(-2c)ds_{N^o}^2\to(M, \langle\ ,\ \rangle)$ is an isometry.

4. Final Comment.

In connection with Remark 3.2, we must correct some parts in the previous paper [6]. There are some ambiguous stataments in [6]. In the Theorem, the statement "whose isotropy subgroup is compact" should be "whose isotropy subgroup *at every point* is compact". The statement "H is compact" that precedes Lemma 1 should be "H is compact *at every point*". We cannot remove the condition that the isotropy subgroup at every is compact, by the following example.

EXAMPLE. Let M be the n-dimensional de-Sitter space $S_1^n=\{(u_0, u_1, \cdots, u_n) \in \boldsymbol{R}^{n+1}; -u_0^2+u_1^2+\cdots+u_n^2=1\}$ and G the matrix group of the form

$$\begin{bmatrix} (1+a^2+|\chi|^2)/(2a) & \chi & (1-a^2+|\chi|^2)/(2a) \\ (1/a)A^t\chi & A & (1/a)A^t\chi \\ (1-a^2-|\chi|^2)/(2a) & -\chi & (1+a^2-|\chi|^2)/(2a) \end{bmatrix} \quad \begin{array}{l} a>0,\ \chi\in \boldsymbol{R}^{n-1} \\ A\in SO(n-1), \end{array}$$

(c.f., Remark 3.2). Then, for every point p in S_1^n such that $u_0+u_n>0$ (resp. <0), the G-orbit of p is $U^+=\{(v_0, \cdots, v_n)\in S_1^n; v_0+v_n>0\}$ (resp. $U^-=\{(v_0, \cdots, v_n) \in S_1^n: v_0+v_n<0\}$) and the isotropy subgroup at p is compact. But, for every point q in S_1^n such that $u_0+u_n=0$, the G-orbit of q is a lightlike hypersurface of S_1^n and the isotropy subgroup at q is non-compact.

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Department of Mathematics,
Kanazawa Medical University,
Uchinada-machi, Ishikawa-ken,
920-02, Japan