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ON THE ANTI-SELF-DUALITY OF THE YANG-MILLS CONNECTION OVER HIGHER DIMENSIONAL KAEHLERIAN MANIFOLD

By

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1. Introduction.

Let M be a Kaehler manifold of complex dimension $n\geqq 2$, with a Kaehler form Φ, where Φ is locally expressed by $\Phi=\sqrt{-1}\sum g_{\alpha\bar{\beta}}dz^{\alpha}\wedge d\bar{z}^{\beta}$ and a Kaehler metric $g=\sum g_{\alpha\bar{\beta}}dz^{\alpha}\otimes d\bar{z}^{\beta}$. A connection A on a principal fibre bundle $\boldsymbol{P}$ over M with the structure group G is said to be *Yang-Mills* when it gives a critical point of the Yang-Mills functional. It satisfies the Yang-Mills equation $d_A*F_A=0$ for the curvature F_A. Thus with the Bianchi identity $d_AF_A=0$ Yang-Mills connection is a connection whose curvature is harmonic with respect to the covariant derivative d_A.

When M has complex dimension 2, i.e., Kaehler surface, the Hodge $*$ operator determines a decomposition

$$\Lambda^2T^*M=\Lambda_+^2\oplus\Lambda_-^2$$

of the space of 2-forms, where $\Lambda_{\pm}^2$ denotes the eigenspace subbundle of $*$ of eigenvalue ± 1. Thus $*^2=id$ implies that the adjoint bundle $\mathfrak{g}_P=\boldsymbol{P}\times_{Ad}\mathfrak{g}$ valued 2-form $F_A=dA+(1/2)[A\wedge A]$ splits into $F^+=(1/2)(F_A+*F_A)$ and $F^-=(1/2)(F_A-*F_A)$, which are called the *self-dual* part and the *anti-self-dual* part of F_A respectively, where $\mathfrak{g}$ denotes the Lie algebra of G. Thus a connection A on a principal fibre bundle $\boldsymbol{P}$ over a Kaehler surface M being *Yang-Mills* is equivalent to $d_AF^+=0$ or $d_AF^-=0$.

But for a higher dimensional Kaehler manifold these formulae give us no meaning. Thus instead of using Hodge $*$ operator let us introduce another operator $\#$, which is defined in section 2 such as $\#=*^{-1}\circ L^{(n-2)}/(n-2)!$, where L means the multiplication by Φ. Then a connection A on a principal fibre bundle $\boldsymbol{P}$ over higher dimensional Kaehler manifold M being *Yang-Mills* is equivalent to $d_A\#F_A=0$ (cf. Proposition 3.1 (ii)).

Also let us define an operator $\tilde{\#}$ such that $\tilde{\#}$ is equal to $\#$ on $F^{2,0}+F^{0,2}+$

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$F_0^{1,1}$, and $\tilde{\#}=\#/(n-1)$ on $F^0\otimes\Phi$, where $F^{p,q}$ is the (p, q)-component and $F_0^{1,1}$ means the primitive (1, 1) form and F^0 is 0-form. Then we can consider the *self-duality* and *anti-self-duality* of F_A in the sense of $\tilde{\#}F^+=F^+$ and $\tilde{\#}F^-=-F^-$, where the self-dual part is $F^+=F^{2,0}+F^{0,2}+F^0\otimes\Phi$ and the anti-self-dual part F^- is a form of type (1, 1) orthogonal to Kaehler form Φ, that is, $F_0^{1,1}$.

Then our anti-self-dual connection minimizes the Yang-Mills functional, and then is a Yang-Mills connection (cf. Theorem 4.2).

Now we can state main theorems which give the curvature form conditions for a Yang-Mills connection to be anti-self-dual, and which generalize some results of M. Itoh for Kaehler surfaces [3].

THEOREM A. *Let M be a complex n-dimensional compact Kaehler manifold with the sum of any two distinct eigenvalues of the Ricci tensor is positive. Let A be an irreducible Yang-Mills connection. If* $[F^{2,0}\wedge F^{0,2}]=0$, *then A is anti-self-dual.*

REMARK. M. Itoh [3] obtained the above result for a compact Kaehler surface with positive scalar curvature.

With another commutative curvature condition we also have the following.

THEOREM B. *Let M be a compact Kaehler manifold with the same condition as in Theorem A. If* $[F^{2,0}\wedge F^{1,1}]=0$ *and* $[F^0\wedge F^{2,0}]=0$, *then A is anti-self-dual.*

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2. Self-duality and anti-self-duality.

Let M be an n-dimensional compact complex manifold with a Kaehler metric g. Let Φ be its Kaehler form. When M is a compact Kaehler surface, the Hodge $*$ operator is involutive. Thus naturally we can consider self-dual 2 form (or anti-self-dual 2 form). But in a higher dimensional manifold it gives us no meaning. However H. J. Kim [4] defined the involutive operator # as follows.

We denote by $A'=\sum A^p$ the exterior algebra of all smooth real valued forms on M. Now let us define the Lipschitz operator L by $L\phi=\phi\wedge\Phi$, $\phi\in A'$ and the operator $\Lambda: A'\to A'$ which is the adjoint of L. Then it is well known that $*$, L, and Λ satisfy the following relations

(2.1) $$\Lambda=L^*=*^{-1}\circ L\circ*, \quad (\Lambda L-L\Lambda)|_{A^k}=n-k, \quad \Lambda(\Phi)=n.$$

(2.2) $$*^2|_{A^k}=(-1)^{k(n-k)}.$$

(2.3) $$*(\Phi^k/k!)=\Phi^{n-k}/(n-k)!, \quad k=0, 1, \cdots n.$$

We denote also by $A^{p,q}$ the space of C^∞-(p, q) forms on M and by $A_0^{p,q}$ the space of primitive (p, q) forms, that is,

$$A_0^{p,q}=\{\alpha\in A^{p,q}|\Lambda\alpha=0\}.$$

Then

LEMMA 2.1 (R. O. Well [7]). *Let* $k=p+q$.
(i) *if* $k\geqq n$, *then* $A_0^{p,q}=0$.
(ii) *if* $k\geqq n$, *then* $A_0^{p,q}=\{\alpha\in A^{p,q}|L^{n-k+1}\alpha=0\}$
$=\{\alpha\in A^{p,q}|C_{p,q}*L^{(n-k)}\alpha/(n-k)!=\alpha\}$,
where $C_{p,q}=(-1)^{pq}(\sqrt{-1})^{p^2-q^2}$.

The space A^2 of 2-forms is decomposed as

$$A^2=A^{2,0}+A^{0,2}+A_0^{1,1}+A_\Phi^{1,1}$$

where $A_\Phi^{1,1}$ denotes the space of (1, 1) type propotional to Φ. And let us now consider the operator $\#$ which is defined by H. J. Kim:

$$\#: A^2 \xrightarrow{L^{(n-2)}/(n-2)!} A^{2(n-1)} \xrightarrow{*^{-1}=*} A^2, \quad \text{i.e.,}\ \#=*^{-1}\circ L^{(n-2)}/(n-2)!$$

Then we have the following from the definition of $\#$ and Lemma 2.1.

LEMMA 2.2. (i) $A_0^{1,1}=\{\alpha\in A^2|\#\alpha=-\alpha\}$,
(ii) $A^{2,0}+A^{0,2}=\{\alpha\in A^2|\#\alpha=\alpha\}$,
(iii) $A_\Phi^{1,1}=\{\alpha\in A^2|\#\alpha=(n-1)\alpha\}$.

Now we define an operator

$$\tilde{\#}=\begin{cases}\# & \text{on } A^{2,0}+A^{0,2}+A_0^{1,1},\\ \#/(n-1) & \text{on } A_\Phi^{1,1}.\end{cases}$$

Then we get $\tilde{\#}^2=id$ which implies that A^2 is decomposed into the self-dual part $A_+^2=A^{2,0}+A^{0,2}+A_\Phi^{1,1}$ and the anti-self-dual part $A_0^{1,1}$. Hence the curvature form F_A also can be splitted into the self-dual part $F^+=F^{2,0}+F^{0,2}+F^0\otimes\Phi$ and the anti-self-dual part $F^-=F_0^{1,1}$, i.e., $\tilde{\#}F^+=F^+$, and $\tilde{\#}F^-=-F^-$.

3. Anti-self-duality of Yang-Mills connection.

Let $\boldsymbol{P}$ be a principal fibre bundle over a compact Kaehler manifold M with a compact semi-simple Lie group G. Let A be a connection on $\boldsymbol{P}$. Then we get:

PROPOSITION 3.1. *The following conditions are equivalent.*

(i) *A is Yang-Mills i.e., $d_A * F_A=0$,*

(ii) $d_A \# F_A=0$,

(iii) $2\bar{\partial}_A F^{2,0}+n\partial_A(F^0 \otimes \Phi)=0$,

(iv) $\partial_A^* F^{2,0}=-ni\partial_A F^0/2(n-1)$.

PROOF.

(i)⇔(ii) It is well known that a connection A being Yang-Mills if and only if the curvature satisfies Yang-Mills equation $d_A * F_A=0$. With $\delta_A \Phi^{n-2}=0$ the Yang-Mills equation $d_A * F_A=0$ implies

$$*d_A \# F_A=\delta_A(F_A \wedge \Phi^{n-2})/(n-2)!=0\,,$$

that is, $d_A \# F_A=0$, where δ_A means the formal adjoint of d_A such that $\delta_A= -*d_A*$.

Conversely $*d_A \# F_A=0$ gives $(\delta_A F_A) \wedge \Phi^{n-2}=0$ because $\delta_A \Phi^{n-2}=0$. Since the nondegeneracy of Φ^{n-2} is invariant by taking an orthonormal dual basis, we can assert that $(\delta_A F_A) \wedge \Phi^{n-2}=0$ implies $\delta_A F_A=0$, that is, $d_A * F_A=0$. From this fact a connection A being Yang-Mills is equivalent to $d_A \# F_A=0$.

(ii)⇔(iii) From Lemma 2.2 it follows that

$$\# F_A=F^{2,0}+F^{0,2}-F_0^{1,1}+(n-1)(F^0 \otimes \Phi)\,.$$

Then by the assumption (ii) we have that

$$0=d_A \# F_A=(\partial_A+\bar{\partial}_A)(F^{2,0}+F^{0,2}-F_0^{1,1}+(n-1)(F^0 \otimes \Phi))\,,$$

from which it follows that

$$\partial_A F^{0,2}-\bar{\partial}_A F_0^{1,1}+(n-1)\bar{\partial}_A(F^0 \otimes \Phi)=0\,, \tag{3.1}$$

$$\bar{\partial}_A F^{2,0}-\partial_A F_0^{1,1}+(n-1)\partial_A(F^0 \otimes \Phi)=0\,. \tag{3.2}$$

On the other hand, the Bianchi identity gives that

$$\partial_A F^{0,2}+\bar{\partial}_A(F^0 \otimes \Phi)+\bar{\partial}_A F_0^{1,1}=0, \quad (\text{resp. } \bar{\partial}_A F^{2,0}+\partial_A(F^0 \otimes \Phi)+\partial_A F_0^{1,1}=0). \tag{3.3}$$

Summing up (3.1) and (3.3), we obtain $2\partial_A F^{0,2}+n\bar{\partial}_A(F^0 \otimes \Phi)=0$.

Conversely, it suffices to show that (3.1) holds since (3.1) and its conjugate part (3.2) is equivalent to $d_A \# F_A = 0$. Thus the left side of (3.1) becomes $-(\partial_A F^{0,2} + \partial_A F_0^{1,1} + \bar{\partial}_A(F^0 \otimes \Phi))$ because of the assumption (iii). Thus it vanishes from the Bianchi identity (3.3).

(iii)⇔(iv) The invariance of $F^{2,0}$ by # gives that

$$(3.4) \qquad \frac{1}{(n-2)!}(\partial_A^* F^{2,0}) \wedge \Phi^{n-2} = -*\bar{\partial}_A F^{2,0} \cdot$$

Since $\#(F^0 \otimes \Phi) = (n-1)(F^0 \otimes \Phi)$, we have that

$$(3.5) \qquad *\partial_A(F^0 \otimes \Phi) = \frac{1}{(n-1)!} *\partial_A *(F^0 \otimes \Phi^{n-1}) = -\frac{1}{(n-1)!}(\bar{\partial}_A^* F^0 \otimes \Phi) \wedge \Phi^{n-2},$$

where we have used the definition of # and $\bar{\partial}_A^* = -*\partial_A *$.

Now we suppose the assumption (iii). Then (iii) implies $-*\bar{\partial}_A F^{2,0} = (n/2)*\partial_A(F^0 \otimes \Phi)$, from which, and using the invariance of the nondegeneracy of Φ^{n-2} to (3.4) and (3.5), it follows that

$$\partial_A^* F^{2,0} = -\frac{n}{2(n-1)} \bar{\partial}_A^*(F^0 \otimes \Phi) = -\frac{n}{2(n-1)} i \partial_A F^0 .$$

Conversely, the condition (iv) gives $-*\bar{\partial}_A F^{2,0} = (n/2)*\partial_A(F^0 \otimes \Phi)$ by virtue of (3.4) and (3.5). Thus the condition (iii) holds immediately.

Note. *M. Itoh obtained the above results for the case* $n=2$ *in the paper* [3].

DEFINITION. A connection A is said to be irreducible when it admits no nontrivial covariantly constant Lie algebra valued 0-form.

By using the above proposition we get the following.

COROLLARY 3.2. *Let A be an irreducible Yang-Mills connection and its curvature is* (1.1) *type, then it is anti-self-dual.*

PROOF. Anti-self-dual Yang-Mills connection is characterized by the self-dual part $F^+ = F^{2,0} + F^{0,2} + F^0 \otimes \Phi$ vanishes. Since F is of type (1, 1), $F^{2,0}$ and $F^{0,2}$ vanishes. By Proposition 3.1 (iii) $\partial_A(F^0 \otimes \Phi) = 0$ (or $\bar{\partial}_A(F^0 \otimes \Phi) = 0$), which implies $F^0 \otimes \Phi = 0$ by the irreducibility of A. Thus the self-dual part F^+ vanishes.

Using Proposition 3.1, we also obtain the following Lemma·

LEMMA 3.3. *Let A be a Yang-Mills connection. Then* $\square_A F^{2,0} = \frac{n}{2(n-1)} i[F^0 \wedge F^{2,0}]$, *where* $\square_A$ *means* $\partial_A \partial_A^* + \partial_A^* \partial_A$.

PROOF. By Proposition 3.1 (iv) we have $\Box_A F^{2,0}=-\frac{n}{2(n-1)}i\partial_A\partial_A F^0$. From this and the formula $d_Ad_AF^0=[F_A\wedge F^0]$ we obtain the above fact.

Applying Ricci formula for the $\mathfrak{g}_P^{\mathbf{C}}$-valued (2, 0) form Ψ, then we obtain ([12])

$$(3.6)\qquad (\Box_A\Psi)_{\mu\nu}=-\sum g^{\bar{\sigma}\tau}\nabla_{\bar{\sigma}}\nabla_\tau\Psi_{\mu\nu}-\sum g^{\bar{\sigma}\tau}[F_{\mu\bar{\sigma}},\Psi_{\tau\nu}]+\sum g^{\bar{\sigma}\tau}[F_{\nu\bar{\sigma}},\Psi_{\tau\mu}] +\sum(R_\mu{}^\varepsilon\Psi_{\varepsilon\nu}-R_\nu{}^\varepsilon\Psi_{\varepsilon\mu}).$$

With this formula and Lemma 3.3 we will show here Theorem A in the introduction.

PROOF OF THEOREM A. For the component $F^{2,0}$ of type (2, 0) the above formula (3.6) becomes

$$(3.7)\qquad (\Box_AF^{2,0})_{\mu\nu}=(\nabla_A^*\nabla_AF^{2,0})_{\mu\nu}-\sum g^{\bar{\sigma}\tau}[F_{\mu\bar{\sigma}},F_{\tau\nu}]+\sum g^{\bar{\sigma}\tau}[F_{\nu\bar{\sigma}},F_{\tau\mu}]+(\lambda_\mu+\lambda_\nu)F_{\mu\nu},$$

where λ_μ means the eigenvalues of the Ricci operator R.

Computing the inner product of $\Box_AF^{2,0}$ and $F^{2,0}$, then under the assumption $[F^{2,0}\wedge F^{0,2}]=0$ we obtain the following integral formula

$$\int_M(|\nabla_AF^{2,0}|^2+\sum_{\mu<\nu}(\lambda_\mu+\lambda_\nu)|F^{2,0}_{\mu\nu}|^2)dv=0.$$

Here we used Lemma 3.3 and the fact that $\langle\langle i[F^0\wedge F^{2,0}], F^{2,0}\rangle\rangle dv=\langle i[F^0\wedge F^{2,0}]\wedge *F^{0,2}\rangle=\langle F^0, i[F^{2,0}\wedge F^{0,2}]\rangle=0$. Thus $F^{2,0}$ vanishes because of $\sum_{\mu<\nu}(\lambda_\mu+\lambda_\nu)>0$. So is $F^{0,2}$. Hence Proposition 3.1 (iii) and the irreducibility of Yang-Mills connection implies $F^0\otimes\Phi=0$. This means $F=F_0^{1,1}$. That is, A is anti-self-dual.

Since $F^{1,1}=F^0\otimes\Phi+F_0^{1,1}$, where $F^0=(1/n)\langle F_\Phi^{1,1},\Phi\rangle$, Lemma 3.3 and the formula (3.7) give that

$$(\nabla_A^*\nabla_AF^{2,0})_{\mu\nu}-\frac{5n-4}{2(n-1)}i[F^0,F_{\mu\nu}]+(\lambda_\mu+\lambda_\nu)F_{\mu\nu}+\sum_\sigma([(F_0)_{\mu\bar{\sigma}},F_{\sigma\mu}] -[(F_0)_{\mu\bar{\sigma}},F_{\sigma\nu}])=0.$$

Applying this formula, by the similar way as in Theorem A we have Theorem B.

DEFINITION. A connection on a complex n-dimensional Kaehler manifold is said to be with harmonic curvature if $F^{2,0}$ is harmonic, i.e., $\partial_A^*F^{2,0}=0$.

Then a Yang-Mills connection with harmonic curvature by Proposition 3.1 (iv) satisfies that $F^0=0$ and $F^{1,1}=F_0^{1,1}$. From this fact and Theorem B we can also assert that

COROLLARY 3.4. *Let M be a compact Kaehler manifold with the same assumption as in Theorem A. Let A be an irreducible Yang-Mills connection with harmonic curvature. If $[F_0^{1,1}\wedge F^{2,0}]=0$, then A is anti-self-dual.*

4. Another characterization of anti-self-dual connection.

Let $\boldsymbol{P}$ be a principal fibre bundle over compact Kaehler manifold M with structure group $G=SU(r)$. And let A be a connection in $\boldsymbol{P}$. Then it is well known that Yang-Mills functional $\mathfrak{YM}(A)$ is given by

$$\mathfrak{YM}(A)=\frac{1}{2}\int_M(-Tr)(F\wedge *F)=\frac{1}{2}\int_M|F|^2\frac{\Phi^n}{n!}.$$

where $\Phi^n/n!$ is the volume form of compact Kaehler manifold M.

Now we assert the following formula.

LEMMA 4.1.

$$-Tr\,F\wedge *F=Tr\,F\wedge F\wedge\frac{\Phi^{n-2}}{(n-2)!}+2|F^{2,0}+F^{0,2}|^2 vol_\Phi+n|F^0\otimes\Phi|^2 vol_\Phi,$$

where $vol_\Phi=\Phi^n/n!$.

PROOF. Since the curvature is decomposed as $F=F^{2,0}+F^{0,2}+F^0\otimes\Phi+F_0^{1,1}$, Lemma 1.5 yields $\#F=*\left(F\wedge\frac{\Phi^{n-2}}{(n-2)!}\right)=F^{2,0}+F^{0,2}+(n-1)F^0\otimes\Phi-F_0^{1,1}$. Then we get

$$*(F^{2,0}+F^{0,2})=(F^{2,0}+F^{0,2})\wedge\frac{\Phi^{n-2}}{(n-2)!},\quad (n-1)*(F^0\otimes\Phi)=(F^0\otimes\Phi)\wedge\frac{\Phi^{n-2}}{(n-2)!},$$

$$*F_0^{1,1}=-F_0^{1,1}\wedge\frac{\Phi^{n-2}}{(n-2)!}.$$

Thus it follows that

$$*F=(F^{2,0}+F^{0,2})\wedge\frac{\Phi^{n-2}}{(n-2)!}+F^0\otimes\frac{\Phi^{n-1}}{(n-1)!}-F_0^{1,1}\wedge\frac{\Phi^{n-2}}{(n-2)!}.$$

Then by a direct calculation we have

$$\begin{aligned}(4.1)\qquad F\wedge *F=&(F^{2,0}+F^{0,2})\wedge(F^{2,0}+F^{0,2})\wedge\frac{\Phi^{n-2}}{(n-2)!}+\frac{1}{(n-1)!}F^0\otimes F^0\Phi^n\\&-F_0^{1,1}\wedge F_0^{1,1}\wedge\frac{\Phi^{n-2}}{(n-2)!},\end{aligned}$$

$$\begin{aligned}(4.2)\qquad Tr\,F\wedge F\wedge\frac{\Phi^{n-2}}{(n-2)!}=&Tr\,(F^{2,0}+F^{0,2})\wedge(F^{2,0}+F^{0,2})\wedge\frac{\Phi^{n-2}}{(n-2)!}\\&+Tr\,F^0\otimes F^0\cdot\frac{\Phi^n}{(n-2)!}+Tr\,F_0^{1,1}\wedge F_0^{1,1}\wedge\frac{\Phi^{n-2}}{(n-2)!}.\end{aligned}$$

Thus, combining (4.1) and (4.2), we obtain Lemma 4.1.

From the above Lemma 4.1 we obtain

THOREM 4.2. *Let M be a compact Kaehler manifold. Let A be a connection in the principal fibre bundle $\boldsymbol{P}$ over M with structure group $G=SU(r)$. Then* $\mathfrak{YM}(A)\geqq\frac{1}{2}\int_M C(\boldsymbol{P})\wedge\frac{\Phi^{n-2}}{(n-2)!}$, *where* $C(\boldsymbol{P})=Tr\, F\wedge F=8\pi^2 c_2(E)$, $E=\boldsymbol{P}\times_{SU(r)}C^r$. *The equality holds if and only if A is anti-self-dual.*

REMARK. H. J. Kim showed that the Yang-Mills functional is bounded below by a topological constant and this minimum is achieved if and only if the curvature is Einstein ([4]).

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