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A RELATION BETWEEN k-th UV^{k+1} GROUPS AND k-th STRONG SHAPE GROUPS

By

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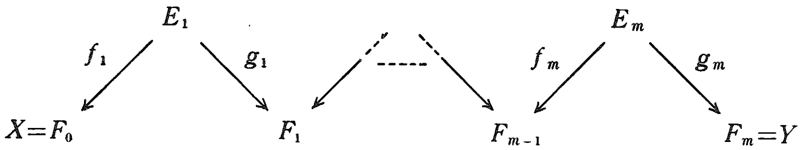
1. Introduction

Compacta X and Y are *UV^n-equivalent* provided that there exist sequences $\{E_i\}_{1\leq i\leq m}$ and $\{F_i\}_{0\leq i\leq m}$ of compacta and sequences $\{f_i\}_{1\leq i\leq m}$ and $\{g_i\}_{1\leq i\leq m}$ of UV^n-maps $f_i: E_i\rightarrow F_{i-1}$ and $g_i: E_i\rightarrow F_i$, where $F_0=X$ and $F_m=Y$. Replacing UV^n-maps with CE-maps, we have the definition of CE-equivalence.



It is well known that finite-dimensional CE-equivalent compacta are shape equivalent (see [D-S]). The first example that shows the gap between shape equivalence and CE-equivalence was found by Ferry [Fe1]. In [Fe3], it was shown that UV^m-equivalent n-dimensional compacta are shape equivalent. Next Daverman and Venema [D-V] constructed an n-dimensional LC^{n-2}-continuum which is shape equivalent but not UV^{n-1}-equivalent to S^1. Mrozik [Mr1] obtained a method to have continua which are shape equivalent but not UV^1-equivalent to each other. Moreover Mrozik [Mr2] improved the method and had a strategy to construct a LC^n-continuum Y from any LC^{n+1}-continuum X with $\pi_1(X)$ infinite such that they are shape equivalent but not UV^{n+1}-equivalent. As a criterion of UV^n-equivalence he introduced the notions of UV^n-component $\pi_0{}^{(n)}(X)$ [Mr1], k-th UV^n-homotopy group $\pi_k{}^{(n)}(X)$ and k-th CE-homotopy group $\pi_k{}^{CE}(X)$ [Mr2]. Venema [Ve] investigated the groups and showed that $\pi_k{}^{(k+1)}(X)=\pi_k{}^{(k+2)}(X)=\cdots=\pi_k{}^{CE}(X)$ for every continuum X and that $\pi_n{}^{(n)}(Y)=0$ for every UV^n-continuum Y.

In this paper we consider a relation between $\pi_k{}^{(k+1)}(X)$ and the k-th strong shape group $\underline{\underline{\pi}}_k(X)$ [Q]. We define a natural homomorphism $s_k: \pi_k{}^{(k+1)}(X)\rightarrow \underline{\underline{\pi}}_k(X)$ and show that, if pro-$\pi_1(X)$ is profinite, s_k is an isomorphism. As its

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consequence we have that if pro-$\pi_1(X)$ is profinite, and $\pi_0^{(1)}(X)=\{X\}$ and $\pi_k^{(k+1)}(X)=0$ for $k=1, \cdots, n$, then a continuum X is UV^n.

2. Definitions and lemmas.

By the Hilbert cube Q, we mean the countable product of closed unit intervals $I=[0, 1]$. By S^k and D^k, we denote the k-sphere and the k-ball, respectively. For each $k\in N$, a compactum X is a *UV^k-compactum* provided that for every embedding $i: X\to M$ of X into an ANR M and every neighborhood U of $i(X)$ in M, there is a neighborhood V of $i(X)$ in M such that $U\supset V$ and the homomorphism $\pi_j(V)\to\pi_j(U)$ induced by the inclusion is trivial for $j\leqq k$. For each compacta X and Y, a surjective map $f: X\to Y$ is UV^k provided that each point preimage $f^{-1}(y)$ is a UV^k-compactum. For a subspace Z of X and $x\in X$, by $d(x, Z)$ we denote the number $\inf\{d(x, z)\mid z\in Z\}$, and set $N_\varepsilon(Z)=\{x\in X\mid d(x, Z)<\varepsilon\}$.

If X and Y are compact metric spaces and $j: Y\to W$ is an embedding into a compact AR W, then an *approaching map* $\underline{f}: X\to Y$ is a pair (f, j), where f is a map $f: X\times[0, \infty)\to W$ such that for each neighborhood U of $j(Y)$, there is an $m\in N$ such that $f(X\times[m, \infty))\subset U$. Two approaching maps $\underline{f}, \underline{g}: X\to Y$ ($\underline{f}=(f, j)$, $\underline{g}=(g, j)$) are *homotopic through approaching maps,* if there is an approaching map $\underline{H}: X\times I\to Y$ ($\underline{H}=(H, j)$) such that $\underline{H}|X\times\{0\}=\underline{f}$ and $\underline{H}|X\times\{1\}=\underline{g}$ [Fe2].

Let $h: X\to Y$ be a map and let $i: X\to Q$ and $j: Y\to Q$ be embeddings. Define an embedding $l: X\to Q\times Q$ by $l(x)=(j\circ h(x), i(x))$ and the projection proj: $Q\times Q\to Q$ by $\mathrm{proj}(a, b)=a$. We assume that $X\subset Q\times Q$ by the above embedding l, and $\mathrm{proj}|X=h$. We take the metric on $Q\times Q$ to be the supremum of the metrics on two factors,

LEMMA 1. *Let $h: X\to Y$ be a UV^k-map as above. If P is a finite k-dimensional polyhedron, S is a subpolyhedron of P, $\underline{f}=(f, j): P\to Y$ is an approaching map, and $\underline{g}=(g, l): S\to X$ is an approaching map with* $\mathrm{proj}\circ g=f|S\times[0, \infty)$, *then there is an extension $g^*: P\times[0, \infty)\to Q\times Q$ of g such that (g^*, l) is an approaching map and that $\underline{f}$ and* $(\mathrm{proj}\circ g^*, j)$ *are homotopic through approaching maps.*

PROOF. By Corollary 1.2 of [Fe3], we get a sequence $\{\delta_n\}_{n\geqq 0}$ of positive numbers satisfying:

(1) $\delta_n<\min\{\delta_{n-1}, 1/2^n\}$ for $n\geqq 1$, $\delta_0<1$ and

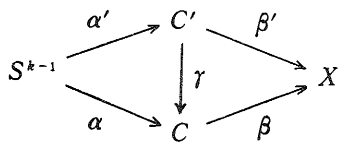
(2) for any finite $(k+1)$-dimensional polyhedron K, subpolyhedron L of K, map $\alpha: K\to N_{\delta n}(Y)$ and map $\alpha_0: L\to N_{\delta n}(X)$ with $\mathrm{proj}\circ\alpha_0=\alpha|L$, there exists an extension $\alpha^*: K\to N_{\delta n-1}(X)$ of α_0 such that $\mathrm{proj}\circ\alpha^*=\alpha$.

Since f is an approaching map, there is a monotone increasing sequence $\{i_n\}_{n\geq 1}$ with $f(P\times[i_n, \infty))\subset N_{\delta n+1}(X)$ for $n\geqq 1$. For each $n\in N$ set $f_n=f|P\times[i_n, i_{n+1}]$. By (2) we get an extension $g_n: P\times[i_n, i_{n+1}]\to N_{\delta n}(X)$ of $g|S\times[i_n, i_{n+1}]$ with $\mathrm{proj}\circ g_n=f_n$. For each $n\in N$, define $H_n: P\times I\to N_{\delta n}(Y)$ by $H_n(x, t)=f(x, i_{n+1})$ for each $x\in P$ and $t\in I$, and $H_{n,0}: P\times\{0, 1\}\to N_{\delta n}(X)$ by $H_{n,0}(x, 0)=g_n(x, i_{n+1})$, $H_{n,0}(x, 1)=g_{n+1}(x, i_{n+1})$ for each $x\in P$. And by (2) there exists an extension $H^*_n: P\times I\to N_{\delta n-1}(X)$ of $H_{n,0}$ with $\mathrm{proj}\circ H^*_n=H_n$. Define $g^*_n: P\times[i_n, i_{n+1}]\to Q\times Q$ as

$$g^*_n(x, (1-t)i_n+ti_{n+1})=\begin{cases} g_n(x, (1-2t)i_n+2ti_{n+1}) & \text{if } t\in[0, 1/2] \\ H^*_n(x, 2t-1) & \text{if } t\in[1/2, 1] \end{cases}$$

Then $g^*=\bigcup_{n\in N} g^*_n: P\times[i_1, \infty)\to Q\times Q$ is a desired extension of g and the proof is finished.

For each pointed compactum (X, x_0) and each $k\geqq 1$, let $UV^m{}_k(X, x_0)$ be the class of all triples $\Delta=(C, \alpha, \beta)$, where C is a UV^m compactum and $\alpha: S^{k-1}\to C$, $\beta: C\to X$ are maps with $\beta\circ\alpha(S^{k-1})=\{x_0\}$. Given two such triples $\Delta=(C, \alpha, \beta)$ and $\Delta'=(C', \alpha', \beta')$, we write $\Delta'\geqq\Delta$ if there exists a map $\gamma: C'\to C$ such that commutativity holds in each triangle of the following diagram.



Let $\equiv$ denote the equivalence relation generated by $\geqq$ (i.e. $\Delta'\equiv\Delta$ iff there exists a sequence of triples $\Delta_1=\Delta, \Delta_2, \cdots \Delta_{2r+1}=\Delta'$ in $UV^m{}_k(X, x_0)$ such that $\Delta_{2i}\geqq\Delta_{2i\pm 1}$, $i=1, \cdots, r$) and let $\pi_k{}^{(m)}(X, x_0)=UV^m{}_k(X, x_0)/\equiv$. The equivalence class of $\Delta=(C, \alpha, \beta)$ in $\pi_k{}^{(m)}(X, x_0)$ will be denoted by $[\Delta]=[C, \alpha, \beta]$.

Let $\kappa: S^{k-1}\to(S^{k-1}, *)\vee(S^{k-1}, *)$ denote the usual comultiplication map on the H-cogroup S^{k-1} and $\mu: (X, x_0)\vee(X, x_0)\to X$ the folding map. For $[\Delta_i]=[C_i, \alpha_i, \beta_i]\in\pi_k{}^{(m)}(X, x_0)$, $i=1, 2$, define a multiplication by

($) $\qquad [\Delta_1][\Delta_2]=[(C_1, \alpha_1(*))\vee(C_2, \alpha_2(*)), (\alpha_1\vee\alpha_2)\circ\kappa, \mu\circ(\beta_1\vee\beta_2)]$.

Obviously this is a group multiplication on $\pi_k{}^{(m)}(X, x_0)$: The neutral element is $\Delta x_0=[\{*\}, \text{const}, \text{const}]$, where const is the constant map. An inverse for $[\Delta]=[C, \alpha, \beta]$ is given by $[\Delta^{-1}]$, where $\Delta^{-1}=(C, \alpha\circ\nu, \beta)$ and $\nu: S^{k-1}\to S^{k-1}$ is the usual homotopy inverse on the H-cogroup S^{k-1} (see [Mr2]).

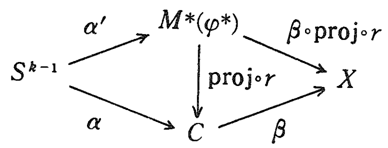
LEMMA 2. *Let (X, x_0) be a pointed compactum and $k\geqq 1$. Then for each $[C, \alpha, \beta]\in\pi_k{}^{(k+1)}(X, x_0)$, there exists a $[C', \alpha', \beta']\in\pi_k{}^{(k+1)}(X, x_0)$ such that*

(1) *$\alpha' : S^{k-1} \to C'$ is an embedding,*
(2) $\dim C' \leqq k+2$ *and*
(3) $[C, \alpha, \beta]=[C', \alpha', \beta']$.

PROOF. By Theorem 2.1.9 of [Be], there exists a compactum C^* with $\dim C^* \leqq k+2$, and a UV^{k+1}-map $f: C^* \to C$. Since C is UV^{k+1}, C^* is UV^{k+1}. Let $i: C^* \to Q$ and $j: C \to Q$ be embeddings. Define $l: C^* \to Q \times Q$ by $l(x)=(j \circ f(x), i(x))$. For convenience we may assume that $\mathrm{proj} | C^*=f$ as before. Moreover define $\varphi: S^{k-1} \times [0, \infty) \to Q$ by $\varphi(x, t)=\alpha(x)$ for each $x \in S^{k-1}$ and $t \in [0, \infty)$. By the proof of Lemma 1 there exists a map $\varphi^*: S^{k-1} \times [0, \infty) \to Q \times Q$ such that (φ^*, l) is an approaching map and $\mathrm{proj} \circ \varphi^*=\varphi$. The mapping cylinder $M(\varphi^*)$ of φ^* is the space obtained from $(S^{k-1} \times [0, \infty) \times I) \oplus (\varphi^*(S^{k-1} \times [0, \infty)) \cup C^*)$ by identifying for each $y \in \varphi^*(S^{k-1} \times [0, \infty))$ the set $(\varphi^{*-1}(y) \times \{1\}) \cup \{y\}$ to a single point. Identifying of C^* and $S^{k-1} \times [0, \infty) \times [0, 1)$ as subspaces of $M(\varphi^*)$, we set

$$M^*(\varphi^*)=C^* \cup \{[x, s, s/(1+s)] \in M(\varphi^*) \mid x \in S^{k-1}, s \in [0, \infty)\}.$$

Then $M^*(\varphi^*)$ is UV^{k+1}. Let $r: M(\varphi) \to \varphi^*(S^{k-1} \times [0, \infty)) \cup C^*$ be the natural retraction of the mapping cylinder and define an embedding $\alpha': S^{k-1} \to M^*(\varphi^*)$ by $\alpha'(x)=[x, 0, 0]$. Since we can obtain a commutative diagram:



we infer $[M^*(\varphi^*), \alpha', \beta \circ \mathrm{proj} \circ r | M^*(\varphi^*)]=[C, \alpha, \beta] \in \pi_k^{(k+1)}(X, x_0)$.

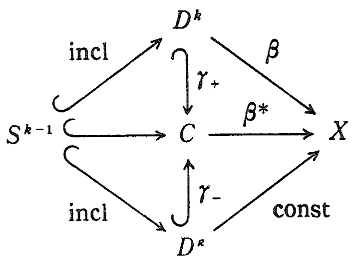
If X and Y are compact metric ANR's, a map $p: X \to Y$ is said to have *the approximate homotopy lifting property* (AHLP) with respect to a compact space Z if for every homotopy $f: Z \times I \to Y$, map $F_0: Z \to X$ with $p \circ F_0=f | Z \times \{0\}$, and $\varepsilon>0$ there is a map $F: Z \times I \to X$ such that $F_0=F | Z \times \{0\}$ and $d(p \circ F(z, t), f(z, t))<\varepsilon$ for each $(z, t) \in Z \times I$. We will call p an *AF^n-map* if p has the AHLP for all n-dimensional compacta.

For a finite or infinite inverse sequence $\{(X_i, f_i)\}$ of compacta, $\mathrm{CMap}^+((X, f_i))$ is defined by S. Ferry, (Definition 5.2, [Fe2]). We remark that the inverse limit $\varprojlim (X_i, f_i)$ is regarded a subspace of $\mathrm{CMap}^+((X_i, f_i))$ and that if the spaces X_i's are ANR's, then $\mathrm{CMap}^+((X_i, f_i))$ is an AR.

Next we shall define a homomorphism $t_k: \pi_k(X, x_0) \to \pi_k^{(k+1)}(X, x_0)$. For each $[\beta]\in\pi_k(X, x_0)$, where $\beta: D^k\to X$ is a map with $\beta(S^{k-1})=\{x_0\}$, define $t_k([\beta])=[D^k, \mathrm{incl}, \beta]$. Here, incl: $S^{k-1}\to D^k$ is the inclusion map [Mr2].

LEMMA 3. *If* $X=\varprojlim(K_i, f_i)$, *where each* K_i *is a finite polyhedron and each* f_i *is an* AF^i*-map, then the homomorphism* $t_k: \pi_k(X, x_0)\to\pi_k{}^{(k+1)}(X, x_0)$ *is isomorphic for each* $k\geq 1$.

PROOF. a) Injectivity. Let β be a map $\beta: D^k\to X$ such that $t_k([\beta])=[D^k, \mathrm{incl}, \beta]=0\in\pi_k{}^{(k+1)}(X, x_0)$. By the proof of Theorem 2.7 in [Mr2], there exist UV^{k+1}-compactum C and maps satisfying the following commutative diagram:



Define $\gamma: S^k\to C$ by $\gamma|$the upper hemisphere $=\gamma_+$, $\gamma|$the lower hemisphere $=\gamma_-$. Let $i: C\to Q$ be an embedding. Since C is UV^{k+1}, we get a map $\gamma^*: D^{k+1}\times[0, \infty)\to Q$ such that (γ^*, i) is an approaching map and $\gamma^*(x, t)=\gamma(x)$ for each $x\in S^k$, $t\in[0, \infty)$. There is an extension $\beta^{**}: Q\to\mathrm{CMap}^+((K_i, f_i))$ of β^*. By Corollary 5.5 of [Fe2], there exists a map $g^*: D^{k+1}\times[0, \infty]\to\mathrm{CMap}^+((K_i, f_i))$ such that $g^*(x, \infty)=\beta^*\circ\gamma(x)$ for each $x\in S^k$, and that $g^*(S^k\times\{\infty\})\subset X$. Since $[g^*|S^k\times\{\infty\}]=[\beta^*\circ\gamma]=[\beta]\in\pi_k(X, x_0)$, $[\beta]=0$.

b) Surjectivity. Let $[C, \alpha, \beta]\in\pi_k{}^{(k+1)}(X, x_0)$. By Lemma 2 we may assume that $\dim C\leq k+2$ and α is an embedding. Since C is UV^k, we get a map $\varphi: D^k\times[0, \infty)\to Q$ such that $\varphi(x, t)=\alpha(x)$ for each $x\in S^{k-1}$ and $t\in[0, \infty)$, and that (φ, i) is an approaching map, where $i: C\to Q$ is an embedding. The mapping cylinder $M(\varphi)$ of φ is the space obtained from $(S^{k-1}\times[0, \infty)\times I)\oplus(\varphi(D^k\times[0, \infty))\cup C)$ by identifying for each $y\in\varphi(D^k\times[0, \infty))$ the set $(\varphi^{*-1}(y)\times\{1\})\cup\{y\}$ to a single point. Identifying of C and $D^k\times[0, \infty)\times[0, 1)$ as subspaces of $M(\varphi)$, we set

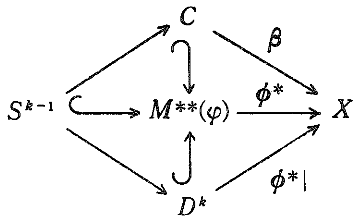
$$M^{**}(\varphi)=C\cup\{[x, s, s/(1+s)]\in M(\varphi)\mid x\in D^k, s\in[0, \infty)\}\supset M^*(\varphi)$$

(see Lemma 2).

Let $j: M^{**}(\varphi)\to Q$ be an embedding. We will construct a map $\phi: M^{**}(\varphi)\times[0, \infty)\to Q$ with (ϕ, j) an approaching map satisfying the following condition:

(♯) $\phi(x, t)=x$ for each $x\in M^*(\varphi)$ and $t\in[0, \infty)$.

For a while, we assume that there exists a map ϕ as above. Let $\beta^*: Q\to \mathrm{CMap}^+((K_i, f_i))$ be an extension of β satisfying $\beta^*([x, s, s/(1+s)])=\beta(x)$ for each $x\in S^{k-1}$ and $s\in[0, \infty)$ and apply Corollary 5.5 of [Fe2] to $\beta^*\circ\phi$, then there exists a map $\phi^*: M^{**}(\varphi)\to X$ with $\phi^*|M^*(\varphi)=\beta^*|M^*(\varphi)$. Identifying D^k with $\{[x, 0, 0]\in M(\varphi)|x\in D^k\}$, and from the following commutative diagram:



and the fact that $M^{**}(\varphi)$ and C are shape equivalent, we have $t_k([\phi^*|D^k])=[C, \alpha, \beta]$. Therefore it is sufficient to construct a map ϕ with the condition (♯).

Since C and $M^*(\varphi)$ are shape equivalent, $M^*(\varphi)$ is UV^{k+1}. There exists a sequence $\{U_n\}_{n\geq -2}$ of neighborhoods of $M^*(\varphi)$ in Q such that

(1) $U_n\supset U_{n+1}$ for each $n\geqq -2$, and

(2) for each $n\geqq -2$, $l\leqq k+1$ and map $\alpha: S^l\to U_{n+1}$, there exists an extension $\alpha^*: D^{l+1}\to U_n$ of α.

Since $M(\varphi)\supset M^*(\varphi)$, there exists a monotone sequence $\{s_m\}_{m\geq 0}$ of positive numbers such that $D^k\times\{s_m\}=\{[x, s_m, s_m/(1+s_m)]\in M(\varphi)|x\in D^k\}\subset U_{3m+1}$ for each $m\geqq 0$. By (2), there exists a map $\alpha_m: D^k\to U_{3m+1}$ with $\alpha_m(x)=[x, 0, 0]\in M^*(\varphi)$ for each $x\in S^{k-1}$. Identifying $D^k\times[0, s_m]$ with $\{[x, s, s/(1+s)]\in M(\varphi)|x\in D^k, s\in[0, s_m]\}$, by (2) we have a map $\phi'_m: D^k\times[0, s_m]\to U_{3m}$ such that $\phi'_m(x, 0)=\alpha_m(x)$ for each $x\in D^k$, and $\phi'_m(x, t)=[x, t, t/(1+t)]$ for each $(x, t)\in S^{k-1}\times[0, s_m]\cup D^k\times\{s_m\}$. Since $\phi'_m(D^k\times\{0\})\cup\phi'_{m+1}(D^k\times\{0\})\subset U_{3m}$, by (2) there exists a map $\phi''_m: D^{k+1}\to U_{3m-1}$ with $\phi''_m|$the upper hemisphere $=\phi'_m|D^k\times\{0\}$ and $\phi''_m|$the lower hemisphere $=\phi'_{m+1}|D^k\times\{0\}$. Applying (2) to $D^k\times[s_m, s_{m+1}]\subset U_{3m+1}$, and three maps ϕ'_m, ϕ'_{m+1} and ϕ''_m, then we get a map $\phi^*_{m,m+1}: D^k\times[0, s_m]\times[m, m+1]\to U_{3m-2}$ satisfying that

$$\phi^*_{m,m+1}(x, t, m)=\phi'_m(x, t) \quad \text{if } (x, t)\in D^k\times[0, s_m],$$

$$\phi^*_{m,m+1}(x, t, m)=[x, t, t/(1+t)] \quad \text{if } (x, t)\in D^k\times[s_m, s_{m+1}] \text{ and}$$

$$\phi^*_{m,m+1}(x, t, m+1)=\phi'_{m+1}(x, t) \quad \text{if } (x, t)\in D^k\times[0, s_m].$$

For each $m\geqq 0$ define $p_m: \{D^k\times[s_m, \infty)\cup C\}\times[m, m+1]\to M^{**}(\varphi)$ by $p_m(x, t, s)=[x, t, t/(1+t)]$ for each $(x, t, s)\in D^k\times[s_m, \infty)\times[m, m+1]$, and

$p_m(y, s)=y$ for each $(y, s)\in C\times[m, m+1]$. We set

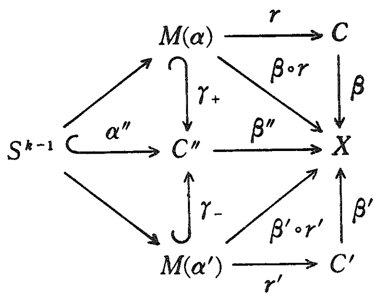
$$\phi_{m,m+1}=\phi^*_{m,m+1}\cup p_m: M^{**}(\varphi)\times[m, m+1] \longrightarrow U_{3m-2}, \quad \text{and}$$

$$\phi=\bigcup_{m\in N}\phi_{m,m+1}: M^{**}(\varphi)\times[0, \infty) \longrightarrow U_{-2}.$$

Clearly by the construction as above, the map ϕ satisfies the condition (♯).

3. Main results

The k-th homotopy pro-group, the k-th shape group and the strong shape group of a space X are denoted pro-$\pi_k(X)$, $\underline{\pi}_k(X)$ and $\underline{\underline{\pi}}_k(X)$, respectively. We will construct a homomorphism $s_k: \pi_k^{(k+1)}(X, x_0)\to\underline{\underline{\pi}}_k(X, x_0)$. Let $[C, \alpha, \beta]\in\pi_k^{(k+1)}(X, x_0)$ and let $i: C\to Q$ be an embedding. Since C is UV^{k+1}, there exists a map $\phi_C: D^k\times[0, \infty)\to Q$ such that $\phi_C(x, t)=\alpha(x)$ for each $x\in S^{k-1}$ and $t\in[0, \infty)$, and that (ϕ_C, i) is an approaching map. Suppose that $X=\varprojlim(K_i, f_i)$, where K_i's are finite polyhedra, then there exisits a map $\beta^*: Q\to \mathrm{CMap}^+((K_i, f_i))$ which is an extension of β. Define $s_k: \pi_k^{(k+1)}(X, x_0)\to\underline{\underline{\pi}}_k(X, x_0)$ by $s_k([C, \alpha, \beta])=[\beta^*\circ\phi_C]$. Since C is UV^{k+1}, the definition as above is independent of a choice of ϕ_C. By the proof of Theorem 2.7 in [Mr2], if $[C, \alpha, \beta]=[C', \alpha', \beta']$, there exists the following commutative diagram:



Here γ_+ and γ_- are embeddings and $[C'', \alpha'', \beta'']\in\pi_k^{(k+1)}(X, x_0)$. By the commutative diagram as above,

$$[\beta\circ\phi_C]=[\beta\circ r\circ\phi_{M(\alpha)}]=[\beta''\circ\phi_{C''}]=[\beta'\circ r'\circ\phi_{M(\alpha')}]=[\beta'\circ\phi_{C'}]\in\underline{\underline{\pi}}_k(X, x_0).$$

s_k turns out to be well-defined. Clearly s_k is a homomorphism.

An inverse sequence $\{G_i, h_i\}$ of groups and homomorphisms is *profinite* if for each i there is a $j>i$ such that $\mathrm{im}\, h_{i+1}\circ\cdots\circ h_j(G_j)\subset G_i$ is finite. A continuum X has pro-$\pi_1(X)$ *profinite* if whenever X is written as an inverse limit $X=\varprojlim(K_i, \alpha_i)$ of finite CW complexes, the system $\{\pi_1(K_i), \alpha_{i*}\}$ is profinite.

MAIN THEOREM. *If (X, x_0) is a pointed continuum with* pro-$\pi_1(X)$ *profinite,* *then* $\pi_k^{(k+1)}(X, x_0)$ *and* $\underline{\underline{\pi}}_k(X, x_0)$ *are isomorphic for each* $k \geqq 1$.

PROOF. We will show that s_k is an isomorphism.

First we may consider a special case that f_i is an AF^i-map for each $i \geqq 1$. Then we will construct a homomorphism $u_k: \underline{\underline{\pi}}_k(X, x_0) \to \pi_k^{(k+1)}(X, x_0)$. Let $\varphi: S^k \times [0, \infty) \to \mathrm{CMap}^+((K_i, f_i))$ such that $\varphi(\{s_0\} \times [0, \infty)) = \{x_0\}$, where s_0 is the basepoint of S^k, and such that (φ, j) is an approaching map, where $j: X \to \mathrm{CMap}^+((K_i, f_i))$ is the inclusion. By Corollary 5.5 of [Fe2], there exists a map $\varphi': S^k \to X$ such that defining $\varphi'': S^k \times [0, \infty) \to X$ by $\varphi''(x, t) = \varphi'(x)$ for each $x \in S^k$ and $t \in [0, \infty)$, $[\varphi''] = [\varphi] \in \underline{\underline{\pi}}_k(X, x_0)$. Define $u_k: \underline{\underline{\pi}}_k(X, x_0) \to \pi_k^{(k+1)}(X, x_0)$ by $u_k([\varphi]) = [D^k, \mathrm{incl}, \varphi' \circ p]$, where $\mathrm{incl}: S^{k-1} \to D^k$ is the inclusion and $p: D^k \to D^k/S^{k-1} = S^k$ is the projection. Because of Corollary 5.5 of [Fe2] and [Mr2], u_k is well-defined. It is clear that $s_k \circ u_k = id$. Since $t_k: \pi_k(X, x_0) \to \pi_k^{(k+1)}(X, x_0)$ is an isomorphism by Lemma 3, for each $[C, \alpha, \beta] \in \pi_k^{(k+1)}(X, x_0)$ there exists a map $\gamma: D^k \to X$ such that $\gamma(S^{k-1}) = \{x_0\}$ and $[C, \alpha, \beta] = [D^k, \mathrm{incl}, \gamma]$, where $\mathrm{incl}: S^{k-1} \to D^k$ is the inclusion. Because $u_k \circ s_k([D^k, \mathrm{incl}, \gamma]) = [D^k, \mathrm{incl}, \gamma]$, $u_k \circ s_k = id$. That is, s_k is an isomorphism.

Next we consider the general case. Since pro-$\pi_1(X)$ is profinite, by Theorem 3′ and Lemma 3.2 of [Fe2], there exists a continuum X' such that X' and X are shape equivalent and $X' = \varprojlim (K'_i, f'_i)$, where K'_i's are finite polyhedra and f'_i's are AF^i-maps. Moreover, by Theorem 2 of [Fe3], X' and X are UV^n-equivalent for each $n \geqq 0$. There exist a compactum X'' and UV^{k+1}-maps $\xi_1: X'' \to X$, $\xi_2: X'' \to X'$. Let $x''_0 \in X''$ with $\xi_1(x''_0) = x_0$.

$$\begin{array}{ccccc}
\pi_k^{(k+1)}(X, x_0) & \xleftarrow{\ \xi_{1*}\ } & \pi_k^{(k+1)}(X'', x''_0) & \xrightarrow{\ \xi_{2*}\ } & \pi_k^{(k+1)}(X', \xi''_2(x''_0)) \\
\downarrow s_k & & \downarrow s_k'' & & \downarrow s_k' \\
\underline{\underline{\pi}}_k(X, x_0) & \xleftarrow{\ \underline{\underline{\xi}}_{1*}\ } & \underline{\underline{\pi}}_k(X'', x''_0) & \xrightarrow{\ \underline{\underline{\xi}}_{2*}\ } & \underline{\underline{\pi}}_k(X', \xi''_2(x''_0)).
\end{array}$$

By Theorem 1.6 of [Mr2], ξ_{1*} and ξ_{2*} are isomorphisms, and by Lemma 1, $\underline{\underline{\xi}}_{1*}$ and $\underline{\underline{\xi}}_{2*}$ are isomorphisms. Since this diagram is commutative and s_k' is an isomorphism, so is s_k.

A space X will be called *UV^n-connected* provided that for any two points $x, x' \in X$ there exist a UV^n-compactum C and a map $\gamma: C \to X$ with $x, x' \in \gamma(X)$. By a *UV^n-component* of X we mean a maximal UV^n-connected subspace of X. Denote $\pi_0^{(n)}(X)$ the set of all UV^n-components of X.

LEMMA 4. *Let X be a continuum. If $\pi_0^{(1)}(X) = \{X\}$, then $\underline{\underline{\pi}}_0(X, x_0) = 0$ for* *each* $x_0 \in X$.

PROOF. Let $x_0 \in X$ be an arbitrary point. Since $\pi_0^{(1)}(X)=0$, for each $x_1 \in X$ there exist a UV^1-compactum C and a map $\gamma: C \to X$ with $x_0, x_1 \in \gamma(C)$. Let M and M' be AR's, $i: C \to M$ and $j: X \to M'$ be embeddings and $\gamma^*: M \to M'$ be an extension of γ. Taking points $y_0, y_1 \in C$ with $\gamma^*(y_0)=x_0$, $\gamma^*(y_1)=x_1$, since C is UV^1, there exists a map $\phi: I \times [0, \infty) \to M$ such that (ϕ, i) is an approaching map and $\phi(\delta, t)=y_\delta$ for each $t \in [0, \infty)$ and $\delta \in \{0, 1\}$. Since $(\gamma^* \circ \phi, j)$ is an approaching map, $\underline{\underline{\pi}}_0(X, x_0)=0$.

COROLLARY. *Let X be a continuum with* pro-$\pi_1(X)$ *profinite. If* $\pi_0^{(1)}(X)=\{X\}$ *and* $\pi_k^{(k+1)}(X, x_0)=0$ *for each* $x_0 \in X$ *and* $k=1, 2, \cdots, n$, *then X is UV^n.*

PROOF. It follows from Main theorem and Lemma 4 that $\underline{\underline{\pi}}_k(X, x_0)=0$ for each $x_0 \in X$ and $k=0, 1, \cdots, n$. By [Wa] $\lim^1(\text{pro-}\pi_{k+1}(X, x_0))=0=\underline{\underline{\pi}}_k(X, x_0)$. Moreover by Theorem 11 and Lemma 2 of Theorem 12 in §6.2 [M-S], pro-$\pi_k(X, x_0)=0$ for each $x_0 \in X$ and $k=1, 2, \cdots, n$. Since X is connected, X is UV^n.

4. Remarks and problems.

Mrozik [Mr2] and Venema [Ve] gave fundamental properties of k-th UV^n-groups for an arbitrary continuum X: $\pi_k^{(1)}(X)=\pi_k^{(2)}(X)=\cdots=\pi_k^{(k-1)}(X)=0$ and $\pi_k^{(k+1)}(X)=\pi_k^{(k+2)}(X)=\cdots=\pi_k^{CE}(X)$. Thus the groups have some meaning only in the cases $n=k$ and $k+1$. Moreover Venema showed that, for every UV^n-compactum X, $\pi_n^{(n)}(X)=0$. Considering Corollary and Venema's result, we have a natural problem:

PROBLEM 1. *Is a continuum X with* $\pi_k^{(k)}(X)=0$ *for* $k=1, \cdots, n$, *a UV^n-compactum?*

On the other hand, we clearly have a natural homomorphism $h_{k,k+1}$: $\pi_k^{(k+1)}(X, x_0) \to \pi_k^{(k)}(X, x_0)$ as follows: for each $[C, \alpha, \beta] \in \pi_k^{(k+1)}(X, x_0)$ where C is UV^{k+1}, and $\alpha: S^{k+1} \to C$ and $\beta: C \to X$, define

$$h_{k,k+1}([C, \alpha, \beta])=[C, \alpha, \beta].$$

However, we do not have any information about $h_{k,k+1}$. It is obvious that if $h_{k,k+1}$ is a monomorphism, Problem 1 has the affirmative answer. Therefore we pose the following problem:

PROBLEM 2. *When is the homomorphism $h_{k,k+1}$ a monomorphism? In particular, consider the case that* pro-$\pi_1(X)$ *is profinite.*

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