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DIMENSION OF k-LEADERS

By

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Introduction.

Let X be a T_2-space. The *k-leader* kX of the space X is the set X with the topology generated by the family of all subsets of X that have closed intersections with all compact subspaces of X (see [1]).

A. Koyama [2] introduced the notion of a *c-refinable* map and showed that if $f: X\rightarrow Y$ is a c-refinable map between normal spaces, then $\dim X=\dim Y$. He asked: Is there a normal space X satisfying $\dim kX\neq\dim X$?

The purpose of this note is to give a positive answer to the question by constructing a Lindelöf non-zerodimensional space X with the property that every compact subspace of X is finite. Note that the k-leader kX of the space X is discrete.

The letter N denotes the set of positive integers.

The example.

EXAMPLE. *There exists a Lindelöf space X such that* $\dim X>0$ *and every compact subspace of X is finite.*

The real line with the natural topology is denoted by the letter R. Let S and T be subsets of R^2 satisfying:

(a) $R^2=S\cup T$, $S\cap T=\emptyset$; and

(b) $|F\cap S|=|F\cap T|=\mathfrak{c}$ for every closed uncountable subsets F of R^2. For the existence of such subsets S and T, see [3], Ch. III, 40, I, Theorem 1.

Let $\{F_\alpha: \alpha<\mathfrak{c}\}$ be a enumeration of all closed uncountable subsets of R^2.

LEMMA. *There exist $\{s_\alpha: \alpha<\mathfrak{c}\}$ and $\{t_{\alpha n}: n\in N\}$, $\alpha<\mathfrak{c}$, such that*

(a) *$s_\alpha\in F_\alpha\cap S$ and $s_\alpha\neq s_\beta$ for each α, $\beta<\mathfrak{c}$ with $\alpha\neq\beta$; and*

(b) *$\{t_{\alpha n}: n\in N\}\subset F_\alpha\cap T$ and for each $\alpha<\mathfrak{c}$, $\{t_{\alpha n}: n\in N\}$ converges to s_α.*

PROOF. For each $\alpha<\mathfrak{c}$, by the property of the sets S and T, $F_\alpha\cap T$ is uncountable and hence $(\mathrm{Cl}(F_\alpha\cap T))\cap S$ is uncountable. Thus by a transfinite

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induction on $\alpha<\mathfrak{c}$, we can obtain the desired sequences.

CONSTRUCTION. We shall construct the example $\langle X, \tau\rangle$. Define $X=R^2$ as the set. Each point of T is defined to be isolated in $\langle X, \tau\rangle$. Denote by ρ and d the natural topology and the usual distance function of the space R^2. Let $\{s_\alpha: \alpha<\mathfrak{c}\}$ and $\{t_{\alpha n}: n\in N\}$, $\alpha<\mathfrak{c}$ be the sequences obtained by the above Lemma. For each $s\in S$, we define a sequence $\{t_n^s\}\subset T$ which ρ-converges to the point s. If $s=s_\alpha$ for some $\alpha<\mathfrak{c}$, then define $t_n^s=t_{\alpha n}$ for each $n\in N$. If $s\notin \{s_\alpha: \alpha<\mathfrak{c}\}$, then t_n^s be an arbitrary sequence converging to s. We can choose the sequence because T is ρ-dense in R^2.

For each point $x\in R^2$ and $\varepsilon>0$, define $B_\varepsilon(x)=\{y\in R^2: d(x, y)<\varepsilon\}$. For each $s\in S$, $n\in N$ and a function $f: N\to N$, define

$$U(s, f, n)=\{s\}\cup\bigcup\{B_{1/f(k)}(t_k^s)-\{t_k^s\}: k\geqq n\}.$$

Now the basic neighborhood system of the point $s\in S$ in $\langle X, \tau\rangle$ is defined to be the collection

$$\{U(s, f, n): n\in N, f: N\to N\}.$$

It is easy to see that the space $\langle X, \tau\rangle$ is a regular T_1-space.

CLAIM 1. *X is Lindelöf.*

PROOF. Let $\mathcal{U}$ be an open cover of $\langle X, \tau\rangle$. Define $\mathcal{V}=\{\mathrm{Int}_\rho U: U\in\mathcal{U}\}$. Since $\langle R^2, \rho\rangle$ is hereditarily Lindelöf, there exists a countable subcollection $\mathcal{U}'$ of $\mathcal{U}$ such that $\bigcup\mathcal{V}=\bigcup\{\mathrm{Int}_\rho U: U\in\mathcal{U}'\}$. We need only show that the set $R^2-\bigcup\mathcal{V}$ is countable. Suppose the contrary, there is a closed uncountable subset F of $R^2-\bigcup\mathcal{V}$ which is dense in itself with respect to the ρ-topology. Then $F=F_\alpha$ for some $\alpha<\mathfrak{c}$. By the construction of Lemma, there are $s=s_\alpha\in F_\alpha\cap S$ and $\{t_n^s\}=\{t_{\alpha n}\}\subset F_\alpha\cap T$. Since $\mathcal{U}$ is a cover of $\langle X, \tau\rangle$, there is an open set $U\in\mathcal{U}$ and a basic neighborhood $U(s, f, n)$ of s such that $s\in U(s, f, n)\subset U$. Then $(\mathrm{Int}_\rho U(s, f, n))\cap F_\alpha\neq\emptyset$, because $B_{1/f(n)}(t_n^s)-\{t_n^s\}\subset\mathrm{Int}_\rho U(s, f, n)$ and t_n^s is a non-isolated point of F_α with respect to the ρ-topology. Thus $(\mathrm{Int}_\rho U)\cap F_\alpha\neq\emptyset$. On the other hand, $(\mathrm{Int}_\rho U)\cap F_\alpha=\emptyset$, because $F_\alpha\subset R^2-\bigcup\mathcal{V}$ and $\mathrm{Int}_\rho U\in\mathcal{V}$. Contradiction.

CLAIM 2. $\dim X>0$.

PROOF. For every Lindelöf space, the condition $\mathrm{ind}\, X=0$, $\mathrm{Ind}\, X=0$ and $\dim X=0$ are equivalent (see [1]). We need only show that $\mathrm{ind}\, X>0$. To see this, we claim that if U is τ-open, bounded with respect to the usual metric d

of R^2 and $\mathrm{Int}_\rho U \neq \emptyset$, then $\mathrm{Bd}_\tau U \neq \emptyset$. Let U be a τ-open set with the above properties. Put $V = \mathrm{Int}_\rho U$. Then V is bounded with respect to the usual metric of R^2 and hence $\mathrm{Bd}_\rho V$ is a closed uncountable subset of $\langle R^2, \rho\rangle$. Therefore there is a ρ-closed uncountable subset F of $\mathrm{Bd}_\rho V$ which is dense in itself with respect to the ρ-topology. Then $F = F_\alpha$ for some $\alpha < \mathfrak{c}$. By the construction of Lemma, there are $s = s_\alpha \in F_\alpha \cap S$ and $\{t_n^s\} = \{t_{\alpha n}\} \subset F_\alpha \cap T$ satisfying the condition of Lemma. Since $t_n^s \in \mathrm{Bd}_\rho V$, $B_{1/f(n)}(t_n^s) - \{t_n^s\} \cap V \neq \emptyset$ for each $f: N \to N$ and each $n \in N$. Thus $s \in \mathrm{Cl}_\tau V \subset \mathrm{Cl}_\tau U$. It remains to show that $s \notin U$. Suppose $s \in U$, then there is a basic neighborhood $U(s, f, n)$ of s such that $s \in U(s, f, n) \subset U$. But then $B_{1/f(n)}(t_n^s) - \{t_n^s\} \subset \mathrm{Int}_\rho U \subset V$. On the other hand, $B_{1/f(n)}(t_n^s) \cap F_\alpha$ is infinite because $t_n^s \in F_\alpha$ and F_α is dense in itself with respect to the ρ-topology. This contradicts the fact that $F_\alpha \subset X - V$.

Claim 3. *Every compact subset of X is finite.*

Proof. Suppose that there exists a compact subset C of X of infinite cardinality. Then there is a non-isolated point s of C. Since every point of T is isolated, $s \in S$. By the definition of the neighborhood system of s, it is easy to find an increasing sequence $\{n_k : n \in N\}$ of positive integers and a sequence $\{c_k : k \in N\}$ of points of C such that $c_k \in B_{1/2^k}(t_{n_k}^s) \cap C - \{t_n^s : n \in N\}$ for each $k \in N$. Then $\{c_k : k \in N\}$ converges to s with respect to the ρ-topology. But a simple observation of the basic neighborhood system of the point $s \in S$ verifies that $s \notin \mathrm{Cl}_\tau\{c_k : k \in N\}$. Hence $\{c_k : k \in N\}$ is a closed infinite subset of C, which contradicts the compactness of C. The proof is completed.

Remark. In a recent letter to A. Koyama, E. van Douwen announced that for each $n = 1, 2, \cdots, \infty$, there is a normal space X_n with the property that $\dim X_n = n$ and every compact subset of X_n is finite. However his examples are not Lindelöf and $\mathrm{ind}\, X_n = 0$ for each n.

Question. Is there a Lindelöf space X with $\dim X = n$ and $\dim kX = 0$ for each $n = 2, 3, \cdots, \infty$?

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References

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