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TILTING LATTICES OVER ORDERS ASSOCIATED WITH SIMPLE MODULES

By

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Let R be a complete discrete valuation ring with quotient field K and Λ a basic R-order in a separable K-algebra, and let e be a primitive idempotent of Λ. In this paper we shall study tilting Λ-lattices in the form of $T=(1-e)\Lambda \oplus \mathrm{Tr}_L(Je)$ where J is the Jacobson radical of Λ and Tr_L is the transpose functor for Λ-lattices.

Tilting theory was initiated by Brenner and Butler [5] and its general theory over artin algebras was given in Happel and Ringel [6] and Bongartz [4] and has been used and developed by many authors not only in the study of representations of artin algebras but also in more general situations. Among them tilting modules arising from suitable simple modules are concrete and typical ones (see [2] and [5]). While almost all general results in [4] are reconstructed in the case of orders by Roggenkamp [8], it seems also to be desirable to provide an order version of such tilting modules and study its fundamental properties, which is the aim of this paper.

In Section 1, we shall recall some definitions and notation which will be used throughout the paper. In Section 2, we shall show that $T=(1-e)\Lambda \oplus \mathrm{Tr}_L(Je)$ is a tilting Λ-lattice if and only if Je is not Λ-reflexive and Λe is not isomorphic to a direct summand of the projective cover of Je (Theorem 2.1). We call such a tilting Λ-lattice Brenner-Butler type (BB-*type* for short). We shall also show that T is a tilting Λ-lattice of BB-type if and only if T is a tilting left Γ-lattice of BB-type and $\Lambda \cong \mathrm{End}_\Gamma(T)$ where $\Gamma=\mathrm{End}_\Lambda(T)$ (Theorem 2.4). As an application of Theorem 2.1 we shall show that a non-hereditary, basic tiled R-order of finite global dimension always has tilting lattices of BB-type (Proposition 2.5). As a special class of BB-type, in Section 3, we shall introduce the notion of tilting lattices of Auslander-Platzeck-Reiten type (APR-*type* for short), which arise from almost split sequences starting from certain projective modules. It should be noted that in the case of orders we cannot consider simple projective modules. We shall replace simplicity by injectivity of its radical. (See Theorem 3.1.) In Section 4, we shall precisely describe the

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categories $\mathcal{T}$, $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{Q}$ associated with a tilting Λ-lattice T (see §1 for definition) when T is of BB-type. Let T be a tilting Λ-lattice of BB-type and put $\Gamma=\mathrm{End}_\Lambda(T)$. In general, even if Λ is of finite representation type and if T is of APR-type, Γ is not of finite representation type (see [8] or Example 7.2). However, we shall show that whenever T is a tilting left Γ-lattice of APR-type, if Λ is of finite representation type then so is Γ (Corollary 4.3). So, in Section 5, we shall consider when ${}_\Gamma T$ is of APR-type and show that ${}_\Gamma T$ is of APR-type if and only if the middle term of the almost split sequence starting from $\mathrm{Hom}_R(Je, R)$ is an injective Λ-lattice which does not contain $\mathrm{Hom}_R(\Lambda e, R)$ as a direct summand (Proposition 5.2). In Section 6, global dimension of Γ is determined when Λ is of global dimension two and T is of APR-type. Examples are gathered in Section 7.

1. Preliminaries

In this section we shall recall some definitions and notation which will be used throughout this paper.

Rings are associative with identity. Modules are finitely generated and unital over a ring, which are usually right modules unless otherwise stated. For modules M and N, we denote by $M|N$ if M is isomorphic to a direct summand of N. For a module M over a ring S, $\mathrm{pd}_S(M)$ (resp. $\mathrm{id}_S(M)$) denotes the projective (resp. injective) dimension of M.

Let Λ be an R-order in a separable K-algebra A where R is a complete discrete valuation ring with a unique maximal ideal πR and the quotient field K. J denotes the Jacobson radical of Λ and $\bar{\Lambda}=\Lambda/J$. A Λ-module is called a *Λ-lattice* if it is finitely generated free as an R-module. An R-order Λ is said to be of *finite representation type* if the number of isomorphism classes of indecomposable Λ-lattices is finite. A Λ-lattice T is said to be a *tilting Λ-lattice* provided;

(i) $\mathrm{pd}_\Lambda(T)\leqq 1$.

(ii) $\mathrm{Ext}^1_\Lambda(T, T)=0$.

(iii) There exists a short exact sequence of Λ-lattices

$$0 \longrightarrow \Lambda \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \mathrm{add}(T)=\{X: X \text{ is a direct summand of a finite direct sum of copies of } T\}$.

We denote the Λ-dual $\mathrm{Hom}_\Lambda(\ ,\Lambda)$ by $(\ )^\#$ and R-dual $\mathrm{Hom}_R(\ , R)$ by $(\ )^*$. The Morita duality functor for mod-R is denoted by $D=\mathrm{Hom}_R(\ , I_1)$ where I_1

is the minimal injective cogenerator for mod-R. A Λ-module X is called *Λ-reflexive* if the canonical map $X \rightarrow X^{\#\#}$ is an isomorphism. Tr_L denotes the transpose functor between left and right Λ-lattices. Namely, let Y be a left or right Λ-lattice and $f: P \rightarrow Y$ a projective cover of Y. Then $\mathrm{Tr}_L(Y) = \mathrm{Coker}(f^{\#})$. Besides, usual transpose functor is denoted by Tr. Namely, let X be a left or right Λ-module and $P_1 \xrightarrow{g} P_0 \rightarrow X \rightarrow 0$ a minimal projective presentation of X. Then $\mathrm{Tr}(X) = \mathrm{Coker}(g^{\#})$. The Auslander-Reiten translate for Λ-lattices is denoted by τ. Namely, for a Λ-lattice X, $\tau X = (\mathrm{Tr}_L(X))^*$ and $\tau^{-1} X = \mathrm{Tr}_L(X^*)$.

We now recall the basic results of tilting theory for orders from [8]. Let T be a tilting Λ-lattice and $\Gamma = \mathrm{End}_\Lambda(T)$. Then put

$$\mathcal{T} = \{X : X \text{ is a } \Lambda\text{-lattice and } \mathrm{Ext}^1_\Lambda(T, X) = 0\},$$

$$\mathcal{F} = \{Y : Y \text{ is a } \Lambda\text{-module and } \mathrm{Hom}_\Lambda(T, Y) = 0\},$$

$$\mathcal{S} = \{Z : Z \text{ is a } \Gamma\text{-lattice and } Z \otimes_\Gamma T \text{ is } R\text{-torsionfree}\},$$

$$\mathcal{G} = \{W : W \text{ is a } \Gamma\text{-module and } W \otimes_\Gamma T = 0\}.$$

THEOREM 1.1 ([8, Theorem 2.8]). i) *The functor* $F = \mathrm{Hom}_\Lambda(T, -)$ *induces an equivalence between* $\mathcal{T}$ *and* $\mathcal{S}$ *with its inverse* $- \otimes_\Gamma T$.

ii) *The functor* $\mathrm{Ext}^1_\Lambda(T, -)$ *induces an equivalence between* $\mathcal{F}$ *and* $\mathcal{Y}$ *with its inverse* $\mathrm{Tor}^\Gamma_1(-, T)$.

iii) *For every Λ-lattice X, there exists an exact sequence*

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0, \qquad X' \in \mathcal{T} \text{ and } X'' \in \mathcal{F}.$$

iv) *For every Γ-lattice Y, there exists an exact sequence*

$$0 \longrightarrow Y \longrightarrow Y' \longrightarrow Y'' \longrightarrow 0, \qquad Y' \in \mathcal{S} \text{ and } Y'' \in \mathcal{G}.$$

v) *T is also a tilting left Γ-lattice.*

vi) $\mathrm{End}_\Gamma({}_\Gamma T) \cong \Lambda$.

2. Tilting lattices of BB-type

In this section we shall consider an analogue of a result of Brenner and Butler [5, Theorem IX] for orders. A short exposition of [5, Theorem IX] can be found in [9, § 2].

THEOREM 2.1. *Let Λ be a basic R-order in A, $J = \mathrm{rad}(\Lambda)$, $\bar{\Lambda} = \Lambda/J$, e a primitive idempotent of Λ and $f: P \rightarrow Je$ a projective cover of Je. Assume that Je is not projective and put $T = (1-e)\Lambda \oplus \mathrm{Tr}_L(Je)$. Then T is a tilting Λ-lattice* *if and only if Je is not Λ-reflexive and Λe is not isomorphic to a direct summand of P.*

In this case, Je and $\mathrm{Tr}_L(Je)$ are indecomposable.

DEFINITION. A tilting Λ-lattice T satisfying the conditions of Theorem 2.1 is called BB-*type*.

The proof of Theorem 2.1 follows from the next two lemmas.

LEMMA 2.2. *The following statements are equivalent.*

(a) $\mathrm{pd}_\Lambda(T)=1$.

(b) *Je is not Λ-reflexive.*

(c) $\mathrm{Hom}_\Lambda(Je, \Lambda)\cong\mathrm{Hom}_\Lambda(\Lambda e, \Lambda)\cong e\Lambda$, *canonically.*

(d) $\mathrm{Ext}^1_\Lambda(\bar{\Lambda}\bar{e}, \Lambda)=0$.

In this case, Je and $\mathrm{Tr}_L(Je)$ are indecomposable.

PROOF. (a) is equivalent to $\mathrm{pd}_\Lambda(\mathrm{Tr}_L(Je))=1$. By the exact sequence $0\to(Je)^{\#}\xrightarrow{f^{\#}}P^{\#}\to\mathrm{Tr}_L(Je)\to 0$, this is equivalent to $(Je)^{\#}$ being projective. Since $Je\subset(Je)^{\#\#}\subset\Lambda e$, if Je is not Λ-reflexive then $(Je)^{\#\#}=\Lambda e$, hence $(Je)^{\#}=e\Lambda$, which is projective. If $(Je)^{\#}$ is projective and if Je is Λ-reflexive then $Je\cong(Je)^{\#\#}$ is projective, which contradicts to the assumption. Thus (a), (b) and (c) are equivalent. Apply $(\ )^{\#}$ to $0\to Je\to\Lambda e\to\bar{\Lambda}\bar{e}\to 0$. Then we obtain an exact sequence

$$0=(\bar{\Lambda}\bar{e})^{\#}\longrightarrow(\Lambda e)^{\#}\longrightarrow(Je)^{\#}\longrightarrow\mathrm{Ext}^1_\Lambda(\bar{\Lambda}\bar{e}, \Lambda)\longrightarrow 0.$$

Hence (c) is equivalent to (d). It follows from (d) and [1, Chapter I, Lemma 9.1] that Je and hence $\mathrm{Tr}_L(Je)$ are indecomposable.

LEMMA 2.3. *If the conditions of Lemma 2.2 hold then the following statements are equivalent.*

(a) $\mathrm{Ext}^1_\Lambda(T, T)=0$.

(b) $\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_L(Je), \mathrm{Tr}_L(Je))=0$.

(c) *Λe is not isomorphic to a direct summand of P.*

(d) $\mathrm{Ext}^1_\Lambda(\bar{\Lambda}\bar{e}, \bar{\Lambda}\bar{e})=0$.

PROOF. First we show that $\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_L(Je), (1-e)\Lambda)=0$. By Lemma 2.2(c), we obtain the following exact sequence:

$$\text{(e)}\qquad 0\longrightarrow(\Lambda e)^{\#}\longrightarrow P^{\#}\longrightarrow\mathrm{Tr}_L(Je)\longrightarrow 0$$

Since P and Λe are finitely generated projective, applying $\mathrm{Hom}_\Lambda(-, (1-e)\Lambda)$ to (e), we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_\Lambda(P^\#, (1-e)\Lambda) & \longrightarrow & \mathrm{Hom}_\Lambda((\Lambda e)^\#, (1-e)\Lambda) & \longrightarrow & \mathrm{Ext}^1_\Lambda(\mathrm{Tr}_L(Je), (1-e)\Lambda) & \longrightarrow & 0 \\
\downarrow\wr & & \downarrow\wr & & & & \\
(1-e)\Lambda\otimes_\Lambda P & \longrightarrow & (1-e)\Lambda\otimes_\Lambda \Lambda e & \longrightarrow & (1-e)\Lambda\otimes_\Lambda\bar{\Lambda}\bar{e} & \longrightarrow & 0
\end{array}$$

Hence $\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_L(Je), (1-e)\Lambda)\cong(1-e)\Lambda\otimes_\Lambda\bar{\Lambda}\bar{e}=0$, because Λ is basic. Thus (a) and (b) are equivalent. Next apply $\mathrm{Hom}_\Lambda(-, \mathrm{Tr}_L(Je))$ to (e). Then similarly we obtain $\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_L(Je), \mathrm{Tr}_L(Je))\cong\mathrm{Tr}_L(Je)\otimes_\Lambda\bar{\Lambda}\bar{e}$. Hence (b) is equivalent to that $e\Lambda$ is not isomorphic to a direct summand of $P^\#$. This is equivalent to (c). The equivalence of (c) and (d) is well known. This completes the proof.

THEOREM 2.4. *Suppose the same assumptions of Theorem* 2.1 *and put* $\Gamma=\mathrm{End}_\Lambda(T)$. *Then T is a tilting Λ-lattice of BB-type if and only if T is a tilting left Γ-lattice of BB-type and* $\Lambda\cong\mathrm{End}_\Gamma(T)$.

PROOF. Put $J'=\mathrm{rad}(\Gamma)$ and $\bar{\Gamma}=\Gamma/J'$, and let $e'\in\Gamma$ be the map of the composition of the projection $T\to\mathrm{Tr}_L(Je)$ and the injection $\mathrm{Tr}_L(Je)\to T$. Assume that T is a tilting Λ-lattice of BB-type. We first claim that $Te\cong\mathrm{Tr}_L(e'J')$. By Lemma 2.2, we have an exact sequence:

(e) $$0\longrightarrow e\Lambda\longrightarrow P^\#\longrightarrow\mathrm{Tr}_L(Je)\longrightarrow 0$$

Applying the functor F to (e) we obtain an exact sequence:

(*) $$0\longrightarrow F(e\Lambda)\longrightarrow F(P^\#)\longrightarrow F(\mathrm{Tr}_L(Je))\longrightarrow \mathrm{Ext}^1_\Lambda(T, e\Lambda)\longrightarrow 0$$

Applying $-\otimes_\Gamma T$ to (*) we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
F(P^\#)\otimes_\Gamma T & \longrightarrow & F(\mathrm{Tr}_L(Je))\otimes_\Gamma T & \longrightarrow & \mathrm{Ext}^1_\Lambda(T, e\Lambda)\otimes_\Gamma T & \longrightarrow & 0 \\
\downarrow\wr & & \downarrow\wr & & & & \\
P^\# & \longrightarrow & \mathrm{Tr}_L(Je)\otimes_\Gamma T & \longrightarrow & 0 & &
\end{array}$$

so that $\mathrm{Ext}^1_\Lambda(T, e\Lambda)\otimes_\Gamma T=0$. Since $\mathrm{Tr}_L(Je)=\mathrm{Tr}(\bar{\Lambda}\bar{e})$, $\mathrm{Ext}^1_\Lambda(T, e\Lambda)\cong\underline{\mathrm{Hom}}_\Lambda(T, \mathrm{Tr}_L(Je))\cong\underline{\mathrm{Hom}}_\Lambda(\bar{\Lambda}\bar{e}, \bar{\Lambda}\bar{e})$ is a division ring, so that (*) is a minimal projective resolution of a simple Γ-module $\mathrm{Ext}^1_\Lambda(T, e\Lambda)\cong\bar{e}'\bar{\Gamma}$. Hence we have an exact sequence:

(e_1) $$0\longrightarrow e'J'\longrightarrow e'\Gamma\longrightarrow\mathrm{Ext}^1_\Lambda(T, e\Lambda)\longrightarrow 0$$

Apply $-\otimes_\Gamma T$ to (e_1). Since $\mathrm{Ext}^1_\Lambda(T, e\Lambda)\otimes_\Gamma T=0$, we obtain an exact sequence:

(e_2) $$0\longrightarrow\mathrm{Tor}^\Gamma_1(\mathrm{Ext}^1_\Lambda(T, e\Lambda), T)\longrightarrow e'J'\otimes_\Gamma T\longrightarrow e'\Gamma\otimes_\Gamma T\longrightarrow 0$$

Since T is R-torsionfree, by (e_2) we get an isomorphism $\mathrm{Hom}_\Lambda(e'J'\otimes_\Gamma T, T)\cong\mathrm{Hom}_\Lambda(e'\Gamma\otimes_\Gamma T, T)$. Thus we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow \mathrm{Hom}_\Gamma(e'J', \Gamma) & \longrightarrow & \mathrm{Hom}_\Gamma(\mathrm{F}(P^*), \Gamma) & \longrightarrow & \mathrm{Tr}_L(e'J') \longrightarrow 0 \\
\downarrow\wr & & \downarrow\wr & & \\
0 \longrightarrow \mathrm{Hom}_\Lambda(e'J'\otimes_\Gamma T, T) & \longrightarrow & \mathrm{Hom}_\Lambda(\mathrm{F}(P^*)\otimes_\Gamma T, T) & & \\
\downarrow\wr & & \downarrow\wr & & \\
0 \longrightarrow \mathrm{Hom}_\Lambda(\mathrm{Tr}_L(Je), T) & \longrightarrow & \mathrm{Hom}_\Lambda(P^*, T) & \longrightarrow & \mathrm{Hom}_\Lambda(e\Lambda, T) \longrightarrow 0
\end{array}$$

Hence $\mathrm{Tr}_L(e'J')\cong Te$. Since $T(1-e)\cong \mathrm{Hom}_\Lambda((1-e)\Lambda, T)=\mathrm{Hom}_\Lambda((1-e')T, T)\cong \Gamma(1-e')$, $T\cong\Gamma(1-e')\oplus\mathrm{Tr}_L(e'J')$ is a tilting left Γ-lattice of BB-type, because $e'J'$ is not projective. This completes the proof.

We end this section with an application of Theorem 2.1 to tiled R-orders of finite global dimension. For the definitions and elementary properties of tiled R-orders and their quivers, we refer the reader to [7] and [10].

PROPOSITION 2.5. *Let Λ be a basic tiled R-order in $(K)_n$ of finite global dimension and not hereditary. Then there exists a primitive idempotent e of Λ such that $(1-e)\Lambda\oplus\mathrm{Tr}_L(Je)$ is a tilting Λ-lattice.*

PROOF. Let $e_1, \cdots, e_n$ be the primitive idempotents of Λ. Since Λ is of finite global dimension, the quiver of Λ has no loop, i.e., $\mathrm{Ext}^1_\Lambda(\bar{\Lambda}\bar{e}_i, \bar{\Lambda}\bar{e}_i)=0$ for any e_i $(1\leqq i\leqq n)$. Suppose that every Je_i is Λ-reflexive $(1\leqq i\leqq n)$. Then $\mathrm{Ext}^1_\Lambda(\bar{\Lambda}\bar{e}_i, \Lambda)\neq 0$. Hence, there exists a left Λ-module E_i such that $\Lambda\subset E_i\subset \pi^{-1}\Lambda$ and $E_i/\Lambda\cong\bar{\Lambda}\bar{e}_i$, so that $\bar{\Lambda}\bar{e}_i$ is embedded in $\Lambda/\pi\Lambda$. Since Λ is of finite global dimension, $\Lambda/\pi\Lambda$ has finite $\Lambda/\pi\Lambda$-injective dimension. Let

$$0\longrightarrow \Lambda/\pi\Lambda\longrightarrow E_0\longrightarrow\cdots\longrightarrow E_t\longrightarrow 0$$

be a minimal injective resolution of $\Lambda/\pi\Lambda$, and let S be a simple $\Lambda/\pi\Lambda$-submodule of E_t. Then $\mathrm{Ext}^t_{\Lambda/\pi\Lambda}(S, \Lambda/\pi\Lambda)\neq 0$. Since S is embedded in $\Lambda/\pi\Lambda$, we obtain an exact sequence

$$\mathrm{Ext}^t_{\Lambda/\pi\Lambda}(\Lambda/\pi\Lambda, \Lambda/\pi\Lambda)\longrightarrow \mathrm{Ext}^t_{\Lambda/\pi\Lambda}(S, \Lambda/\pi\Lambda)\longrightarrow 0.$$

Hence $\mathrm{Ext}^t_{\Lambda/\pi\Lambda}(\Lambda/\pi\Lambda, \Lambda/\pi\Lambda)\neq 0$, so that $t=0$. This contradicts to Λ being not hereditary. Thus Theorem 2.1 completes the proof.

3. Tilting lattices of APR-type

In this section we shall introduce the notion of a tilting Λ-lattice of APR-*type*, which arises from a property of an almost split sequence. This is an order version of the tilting modules first studied in Auslander, Platzeck and Reiten [2].

THEOREM 3.1. *Let e be a primitive idempotent of a basic R-order Λ and* *put* $J=\mathrm{rad}(\Lambda)$. *Then the following statements are equivalent.*

(a) $(Je)^* \cong e\Lambda$ *and* Je *is not projective as a left* Λ*-module.*

(b) *There exists an almost split sequence*

$$0 \longrightarrow e\Lambda \xrightarrow{\psi} E \xrightarrow{\varphi} \tau^{-1}(e\Lambda) \longrightarrow 0$$

such that E *is projective.*

In this case $T=(1-e)\Lambda \oplus \tau^{-1}(e\Lambda)$ *is a tilting* Λ*-lattice of BB-type.*

DEFINITION. A tilting Λ-lattice T satisfying the conditions of Theorem 3.1 is called APR-*type*.

The next lemma explains the conditin (a) of Theorem 3.1, which will play a key role in the proof of the theorem.

LEMMA 3.2. *Let* e *be a primitive idempotent of an* R*-order* Λ. *Then the following statements are equivalent.*

(a) $(eJ)^* \cong \Lambda e$.

(b) eJ *is an indecomposable injective* Λ*-lattice.*

(a′) $(Je)^* \cong e\Lambda$.

(b′) Je *is an indecomposable injective left* Λ*-lattice.*

PROOF. By the exact sequence $0 \to (e\Lambda)^* \to (eJ)^* \to \bar{\Lambda}\bar{e} \to 0$, we can show that (a) and (b) are equivalent. Next, we show that (a) implies (a′), which will complete the proof. By (a), we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (e\Lambda)^* \arrow[r] & (eJ)^* \arrow[r] \arrow[d, "\wr"] & \bar{\Lambda}\bar{e} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Je \arrow[r] & \Lambda e \arrow[r] & \bar{\Lambda}\bar{e} \arrow[r] & 0
\end{tikzcd}$$

Hence (a′) holds.

PROOF OF THEOREM 3.1. Put $P=e\Lambda$ and $S=e\Lambda/eJ$. Assume that (a) $(Je)^* \cong e\Lambda$ and that ${}_{\Lambda}Je$ is not projective. Then there exists an almost split sequence

$$0 \longrightarrow P \xrightarrow{\psi} E \xrightarrow{\varphi} \tau^{-1}P \longrightarrow 0. \tag{e}$$

Note that $\tau^{-1}P=\mathrm{Tr}_L(P^*) \cong \mathrm{Tr}_L(Je)^{**} \cong \mathrm{Tr}_L(Je)$.

We first show that $T=(1-e)\Lambda \oplus \mathrm{Tr}_L(Je)$ is a tilting Λ-lattice. By [1, Chapter I, Lemmas 7.7 and 7.8], we have an exact sequence

$$0 \longrightarrow Je \longrightarrow (Je)^{\#\#} \longrightarrow \mathrm{Ext}^2_{\Lambda}(\mathrm{Tr}(Je), \Lambda) \longrightarrow 0.$$

Let $P_1 \to P_0 \to Je \to 0$ be a minimal projective presentation of ${}_{\Lambda}Je$. Then we have an exact sequence $0 \to \mathrm{Tr}_L(Je) \to P_1^* \to \mathrm{Tr}(Je) \to 0$. Thus $\mathrm{Ext}^2_{\Lambda}(\mathrm{Tr}(Je), \Lambda) \cong \mathrm{Ext}^1_{\Lambda}(\mathrm{Tr}_L(Je), \Lambda) \cong \mathrm{Ext}^1_{\Lambda}(\tau^{-1}P, \Lambda) \neq 0$, so that Je is not Λ-reflexive. Consider an exact sequence of Λ-lattices

$$0 \longrightarrow \tau^{-1}P \longrightarrow X \longrightarrow \tau^{-1}P \longrightarrow 0.$$

If it does not split then we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \tau^{-1}P \arrow[r] \arrow[d, "f"'] & X \arrow[r] \arrow[d] & \tau^{-1}P \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] & E \arrow[r] & \tau^{-1}P \arrow[r] & 0
\end{tikzcd}$$

Since $\tau^{-1}P$ is not projective, we have $\mathrm{Im}\, f \subset eJ$. By Lemma 3.2, eJ is an injective Λ-lattice. Thus f can be extended to X, which implies that the second row splits, a contradiction. Therefore $\mathrm{Ext}^1_{\Lambda}(\tau^{-1}P, \tau^{-1}P)=0$, so that it follows from Theorem 2.1 and its proof that T is a tilting Λ-lattice of BB-type.

Next, we show that E is projective. Let $f: P' \to \tau^{-1}P$ be a projective cover of $\tau^{-1}P$ and put $Y = \mathrm{Ker}\, f$. Then we have a commutative diagram with exact rows:

$$(*) \qquad \begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "g"] \arrow[d, "h"'] & P' \arrow[r, "f"] \arrow[d, "k"'] & \tau^{-1}P \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r, "\psi"] & E \arrow[r, "\varphi"] & \tau^{-1}P \arrow[r] & 0
\end{tikzcd}$$

Then by Lemma 3.2, h is surjective, so that it splits, i.e., there is $h': P \to Y$ such that $hh' = 1_P$. Since T is a tilting Λ-lattice, P is not a direct summand of P'. Hence gh' is not a splitting monomorphism. Hence there exists k': $E \to P'$ with $k'\psi = gh'$, so that there is $l: \tau^{-1}P \to \tau^{-1}P$ with $l\varphi = fk'$. Attach the almost split sequence (e) just above to the diagram (*) by the maps l, k', h'. Then, if l is not surjective then we have a map $m: \tau^{-1}P \to E$ with $l = \varphi m$ and hence we have a map $\alpha: E \to P$ with $\alpha\psi = hh' = 1_P$, a contradiction. Thus l is surjective, so that it is an isomorphism. Hence $\varphi = l^{-1}fk'$ factors through P'. Hence there is $k'': E \to P'$ such that $\varphi = fk''$. Then we have $\varphi kk'' = \varphi$. Since φ is right minimal, kk'' is an isomorphism, so that E is projective.

Conversely, assume the condition (b). Then E is projective. If $P \mid E$, then there exists an irreducible map $P \to \tau^{-1}P$. Thus $\tau^{-1}P \mid E$, a contradiction, so that $P \nmid E$. By the exact sequence

$$0 \longrightarrow (\Lambda e)^* \longrightarrow (Je)^* \xrightarrow{p} S \longrightarrow 0$$

we have $f: P \to (Je)^*$ such that $pf: P \to S$ is a projective cover of S. If f is not a splitting monomorphism, then there is $g: E \to (Je)^*$ with $f = g\phi$. Thus we have $pg\phi = pf \neq 0$, so that $0 \neq pg: E \to S$ and $P \mid E$, a contrapiction. Therefore f is a splitting monomorphism, so that $P \cong (Je)^*$, because $\mathrm{rank}_R(\Lambda e) = \mathrm{rank}_R(e\Lambda)$.

4. The categories $\mathcal{T}$, $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{G}$

In this section we shall give a precise description of the categories $\mathcal{T}$, $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{G}$ for a tilting lattice of BB-type. Let e be a primitive idempotent of Λ and $T = (1-e)\Lambda \oplus \mathrm{Tr}_L(Je)$ a tilting Λ-lattice of BB-type. Put $S = e\Lambda/eJ$ a simple Λ-module, $\Gamma = \mathrm{End}_\Lambda(T)$, and $\tilde{S} = \mathrm{Ext}^1_\Lambda(T, e\Lambda)$. It follows from the proof of Theorem 2.4 that $\tilde{S}$ is a simple Γ-module such that $\tilde{S} \cong e'\Gamma/e'J'$ where $J' = \mathrm{rad}(\Gamma)$ and e' is the map $T \to \mathrm{Tr}_L(Je) \to T$.

PROPOSITION 4.1. *Let $T = (1-e)\Lambda \oplus \mathrm{Tr}_L(Je)$ be a tilting Λ-lattice of BB-type. Then*

(a) $\mathcal{T} = \{X : X$ *is a Λ-lattice with* $\mathrm{Hom}_\Lambda(X, S) = 0\}$,

(b) $\mathcal{F} = \{Y : Y \cong S^{(m)}$ *for some integer* $m \geqq 0\}$,

(c) $\mathcal{S} = \{Z : Z$ *is a Γ-lattice with* $\mathrm{Hom}_\Gamma(Z^*, D(\tilde{S})) = 0\}$
$= \{Z : Z$ *is a Γ-lattice with* $\mathrm{Ext}^1_\Gamma(\tilde{S}, Z) = 0\}$,

(d) $\mathcal{G} = \{W : W \cong \tilde{S}^{(n)}$ *for some integer* $n \geqq 0\}$.

Furthermore, we have $\tilde{S} \cong \mathrm{Ext}^1_\Lambda(T, S) \cong \mathrm{Ext}^1_\Lambda(T, (Je)^*)$.

PROOF. (a) Since S does not appear in the top of $\mathrm{Tr}_L(Je)$, we have $\mathrm{Hom}_\Lambda(T, S) = 0$. Thus if $X \in \mathcal{T}$ then $\mathrm{Hom}_\Lambda(X, S) = 0$. Conversely, suppose $\mathrm{Hom}_\Lambda(X, S) = 0$. Then $e\Lambda \nmid \boldsymbol{P}(X)$ where $\boldsymbol{P}(X)$ is a projective cover of X. Hence $\boldsymbol{P}(X) \in \mathcal{T}$, so that $X \in \mathcal{T}$.

(b) Let $Y \in \mathcal{F}$, i.e., $\mathrm{Hom}_\Lambda(T, Y) = 0$ and $Y \neq 0$. Then the top of Y is a finite direct sum of copies of S. Note that YJ is also in $\mathcal{F}$. By Lemma 2.3, we have $\mathrm{Ext}^1_\Lambda(S, S) \cong \mathrm{Ext}^1_\Lambda(D(S), D(S)) = 0$. Hence the short exact sequence $0 \to YJ/YJ^2 \to Y/YJ^2 \to Y/YJ \to 0$ splits. Hence $YJ = 0$ and $Y \cong S^{(m)}$ for some positive integer m.

(c) Let Z be a Γ-lattice. We first claim that $\mathrm{Ext}^1_\Gamma(\tilde{S}, Z) \cong \mathrm{Hom}_\Gamma(Z^*, D(\tilde{S}))$. Apply $\mathrm{Hom}_\Gamma(\ , Z)$ and $\mathrm{Hom}_\Gamma(Z^*,\ )$ to the exact sequences

$$0 \to e'J' \to e'\Gamma \to \tilde{S} \to 0 \quad \text{and} \quad 0 \to (e'\Gamma)^* \to (e'J')^* \to D(\tilde{S}) \to 0,$$

respectively. Then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_\Gamma(e'\Gamma, Z) & \longrightarrow & \mathrm{Hom}_\Gamma(e'J', Z) & \longrightarrow & \mathrm{Ext}^1_\Gamma(\tilde{S}, Z) & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow\wr & & & & \\ 0 & \longrightarrow & \mathrm{Hom}_\Gamma(Z^*, (e'\Gamma)^*) & \longrightarrow & \mathrm{Hom}_\Gamma(Z^*, (e'J')^*) & \longrightarrow & \mathrm{Hom}_\Gamma(Z^*, \mathrm{D}(\tilde{S})) & \longrightarrow & 0 \end{array}$$

Hence $\mathrm{Ext}^1_\Gamma(\tilde{S}, Z) \cong \mathrm{Hom}_\Gamma(Z^*, D(\tilde{S}))$. It follows from [8, (2.10)] that $Z \in \mathcal{S}$ if and only if $\mathrm{Ext}^1_\Gamma(T, Z^*)=0$. So, since T is a tilting left Γ-lattice of BB-type by Theorem 2.4, (a) completes the proof of (c).

(d) Let $W \in \mathcal{G}$. By the category equivalence $\mathcal{T} \approx \mathcal{G}$ and (a), $W \cong \mathrm{Ext}^1_\Lambda(T, S^{(m)})$ for some $m \geqq 0$. Since $\mathrm{Ext}^1_\Lambda(S, S)=0$, $\mathrm{Hom}_\Lambda(eJ, S)=0$, so that $eJ \in \mathcal{T}$ by (a). Hence from the short exact sequence $0 \to eJ \to e\Lambda \to S \to 0$, we have $\tilde{S} \cong \mathrm{Ext}^1_\Lambda(T, e\Lambda) \cong \mathrm{Ext}^1_\Lambda(T, S)$.

Finally, note that $\mathrm{Ext}^1_\Lambda(T, (Je)^*) \cong \mathrm{Ext}^1_\Lambda(T, S)$ from the exact sequence $0 \to (\Lambda e)^* \to (Je)^* \to S \to 0$. This completes the proof.

COROLLARY 4.2. *Let $T=(1-e)\Lambda \oplus \mathrm{Tr}_L(Je)$ be a tilting Λ-lattice of APR-type. Then $\mathcal{T}=\{X$: X is a Λ-lattice with $e\Lambda \nmid X\}$.*

PROOF. There exists an almost split sequence

$$0 \longrightarrow e\Lambda \longrightarrow E \longrightarrow \tau^{-1}(e\Lambda) \longrightarrow 0$$

such that E is projective. This induces an exact sequence of functors:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_\Lambda(\tau^{-1}(e\Lambda), -) \longrightarrow \mathrm{Hom}_\Lambda(E, -) \longrightarrow \mathrm{Hom}_\Lambda(e\Lambda, -) \\ \longrightarrow \mathrm{Ext}^1_\Lambda(\tau^{-1}(e\Lambda), -) \longrightarrow 0 \end{aligned}$$

Then by [1, Chapter II, Proposition 4.4], $\mathrm{Ext}^1_\Lambda(\tau^{-1}(e\Lambda), -)$ is a simple functor over the category of Λ-lattices such that for an indecomposable Λ-lattice Y, $\mathrm{Ext}^1_\Lambda(\tau^{-1}(e\Lambda), Y) \neq 0$ if and only if $Y \cong e\Lambda$. Hence $X \in \mathcal{T}$ if and only if $e\Lambda \nmid X$.

COROLLARY 4.3. *Let $T=(1-e)\Lambda \oplus \mathrm{Tr}_L(Je)$ be a tilting Λ-lattice of BB-type and a tilting left Γ-lattice of APR-type. Then if Λ is of finite representation type then so is Γ.*

PROOF. Put ${}_\Gamma\mathcal{T}=\{L$: L is a left Γ-lattice with $\mathrm{Ext}^1_\Gamma(T, L)=0\}$. Since $\mathcal{T} \approx \mathcal{S}=\{L^*: L \in {}_\Gamma\mathcal{T}\}$ by [8, (2.10)], ${}_\Gamma\mathcal{T}$ has only finitely many isomorphism classes of indecomposable objects. By Corollary 4.2, $\Gamma e'$ is the only indecomposable left Γ-lattice outside ${}_\Gamma\mathcal{T}$. This completes the proof.

REMARK. When T is a tilting Λ-lattice of APR-type and Λ is of finite representation type, Γ is not necessarily of finite representation type. (See Example 7.2.)

5. When is ${}_{\Gamma}T$ of APR-type?

Let $T=(1-e)\Lambda\oplus\mathrm{Tr}_L(Je)$ be a tilting Λ-lattice of BB-type. In this section we shall study when T is a tilting left Γ-lattice of APR-type. To this end, we need a *connecting sequence* for $F((\Lambda e)^*)=\mathrm{Hom}_\Lambda(T, (\Lambda e)^*)$. Namely, we shall prove the following.

PROPOSITION 5.1. *Let*

$$(*) \qquad 0 \longrightarrow (Je)^* \xrightarrow{i} E \longrightarrow \mathrm{Tr}_L(Je) \longrightarrow 0$$

be an almost split sequence starting from $(Je)^*$. Then there is an almost split sequence of Γ-lattices

$$(**) \qquad 0 \longrightarrow F((\Lambda e)^*) \longrightarrow F(E) \longrightarrow e'J' \longrightarrow 0.$$

PROOF. By the exact sequence $0\to(\Lambda e)^*\to(Je)^*\to S\to 0$, we have $F((\Lambda e)^*)\cong F((Je)^*)$. Applying F to $(*)$, we have an exact sequence

$$0 \to F((Je)^*) \to F(E) \to F(\mathrm{Tr}_L(Je)) \xrightarrow{\delta} \mathrm{Ext}^1_\Lambda(T, (Je)^*) \to \mathrm{Ext}^1_\Lambda(T, E) \to 0.$$

It follows from Proposition 4.1 that $\mathrm{Ext}^1_\Lambda(T, (Je)^*)\cong\tilde{S}$. Hence $\mathrm{Ker}\,\delta=e'J'$ and we obtain the exact sequence $(**)$. Since δ is surjective, $\mathrm{Ext}^1_\Lambda(T, E)=0$, i.e., $E\in\mathcal{T}$. It follows from Proposition 4.1(c) that $e'J'\notin\mathcal{S}$. Hence $(**)$ does not split. It follows from the proof of Theorem 2.4 that $e'J'\cong\mathrm{Tr}_L(Te)$. Hence $\tau^{-1}F((\Lambda e)^*)=\mathrm{Tr}_L(F((\Lambda e)^*)^*)\cong\mathrm{Tr}_L(Te)\cong e'J'$. Finally we claim that the exact sequence $(**)$ belongs to the socle of the left $\mathrm{End}_\Gamma(F((\Lambda e)^*))$-module $\mathrm{Ext}^1_\Gamma(e'J', F((\Lambda e)^*))$. Take $\alpha\in\mathrm{rad}(\mathrm{End}_\Gamma(F((\Lambda e)^*)))$ and consider the following pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & F((\Lambda e)^*) \arrow[r, "\beta"] \arrow[d, "\alpha"] & F(E) \arrow[r] \arrow[d] & e'J' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F((\Lambda e)^*) \arrow[r] & X \arrow[r] & e'J' \arrow[r] & 0
\end{tikzcd}$$

where the first row is $(**)$. Then there exists $f\in\mathrm{End}_\Lambda((\Lambda e)^*)$ such that $\alpha=F(f)$ and that $f=f_1f_2$ for some $f_1\colon (Je)^*\to(\Lambda e)^*$ where $f_2\colon (\Lambda e)^*\to(Je)^*$ is the map induced from the inclusion map $Je\to\Lambda e$. Since $(\Lambda e)^*$ is an injective Λ-lattice, there exists $g\colon E\to(\Lambda e)^*$ such that $f_1=gi$ where $i\colon (Je)^*\to E$. Thus $\alpha=F(f_1f_2)=F(gif_2)=F(g)F(if_2)=F(g)\beta$, so that the second row splits. This completes the proof.

PROPOSITION 5.2. *Let* $T=(1-e)\Lambda\oplus \mathrm{Tr}_L(Je)$ *be a tilting* Λ*-lattice of BB-type, and let*

$$0\longrightarrow (Je)^* \longrightarrow E \longrightarrow \mathrm{Tr}_L(Je)\longrightarrow 0$$

be an almost split sequence. Then the following statements are equivalent.

(a) E *is an injective* Λ*-lattice and* $(\Lambda e)^* \nmid E$.

(b) $F(E)$ *is an injective* Γ*-lattice.*

(c) $e'J'$ *is an injective* Γ*-lattice.*

(d) T *is a tilting left* Γ*-lattice of APR-type.*

PROOF. Let $f\in\Lambda$ be a primitive idempotent with $f\neq e$ and let $f'\in\Gamma$ be the map $T\to f\Lambda\to T$. Then $F((\Lambda f)^*)\cong(Tf)^*\cong(\Gamma f')^*$. Hence (a) implies (b). Conversely assume (b). It is shown in the proof of Proposition 5.1 that $E\in\mathcal{T}$. Hence $F(E)\in\mathcal{S}$. Since $(\Gamma e')^*\notin\mathcal{S}$ by Proposition 4.1 (c), $(\Gamma e')^*\nmid F(E)$. Hence by Theorem 1.1 (i), (a) holds.

It follows from Theorem 2.4 and Lemma 2.2 that $e'J'$ is an indecoposable Γ-lattice. Hence by Theorem 3.1, (c) and (d) are equivalent.

Let X be a Γ-lattice. By Theorem 1.1 and Proposition 4.1 (d) we have an exact sequence

$$(\mathrm{e}_1)\qquad 0\longrightarrow X\longrightarrow F(Y)\longrightarrow \tilde{S}^{(n)}\longrightarrow 0$$

where $Y\in\mathcal{T}$. From Proposition 5.1, we have an almost split sequence

$$(\mathrm{e}_2)\qquad 0\longrightarrow F((\Lambda e)^*)\longrightarrow F(E)\longrightarrow e'J'\longrightarrow 0.$$

Note that by [8, (2.3) (iv)], for any $Z\in\mathcal{T}$, $\mathrm{Ext}^i_\Gamma(F(Z), F((\Lambda e)^*))\cong\mathrm{Ext}^i_\Lambda(Z, (\Lambda e)^*)=0$ $(i=1, 2, \cdots)$. Now assume that $F(E)$ is an injective Γ-lattice. Then by (e_2), $\mathrm{Ext}^1_\Gamma(X, e'J')\cong\mathrm{Ext}^2_\Gamma(X, F((\Lambda e)^*))$. By (e_1), we have an exact sequence

$$0=\mathrm{Ext}^2_\Gamma(F(Y), F((\Lambda e)^*))\to\mathrm{Ext}^2_\Gamma(X, F((\Lambda e)^*))\to\mathrm{Ext}^3_\Gamma(\tilde{S}, F((\Lambda e)^*))^{(n)}.$$

On the other hand, by (e_2) we have an exact sequence

$$\mathrm{Ext}^1_\Gamma(F((\Lambda e)^*), F((\Lambda e)^*))\to\mathrm{Ext}^2_\Gamma(e'J', F((\Lambda e)^*))\to\mathrm{Ext}^2_\Gamma(F(E), F((\Lambda e)^*))$$

Since both ends are zero, $\mathrm{Ext}^3_\Gamma(\tilde{S}, F((\Lambda e)^*))\cong\mathrm{Ext}^2_\Gamma(e'J', F((\Lambda e)^*))=0$. Hence $\mathrm{Ext}^1_\Gamma(X, e'J')=0$. Thus $e'J'$ is an injective Γ-lattice.

Conversely assume that $e'J'$ is an injective Γ-lattice. Then using (e_2), we can show that for any $Z\in\mathcal{T}$, $\mathrm{Ext}^1_\Gamma(F(Z), F(E))=0$. By (e_1), we have an exact sequence

$$0=\mathrm{Ext}^1_\Gamma(F(Y), F(E))\longrightarrow\mathrm{Ext}^1_\Gamma(X, F(E))\longrightarrow\mathrm{Ext}^2_\Gamma(\tilde{S}, F(E))^{(n)}.$$

Since (a_2) is an almost split sequence, applying $\mathrm{Hom}_\Gamma(\ , F(E))$ to (e_2), we obtain that $\mathrm{Ext}^1_\Gamma(e'J', F(E))=0$. Hence $\mathrm{Ext}^2_\Gamma(\tilde{S}, F(E))=0$ and hence $\mathrm{Ext}^1_\Gamma(X, F(E))=0$. This completes the proof.

REMARK. If T is a tilting Λ-lattice of APR-type then in (a) of Proposition 5.2, we can delete the condition $(\Lambda e)^* \nmid E$. In fact, assume $(\Lambda e)^* | E$. Then there is an irreducible map $(\mathrm{Tr}_L(Je))^* \to \Lambda e$. Thus $(\mathrm{Tr}_L(Je))^* | Je$. Since Je is indecomposable, $\mathrm{Tr}_L(Je) \cong (Je)^* \cong e\Lambda$, a contradiction.

LEMMA 5.3. *Let L be an indecomposable Γ-lattice and let $f: e'J' \to L$ be an irreducible map. Then $L \cong e'\Gamma$ or else $L \notin \mathcal{S}$.*

PROOF. Let $L \in \mathcal{S}$. Then it follows from Proposition 4.1(c) that by the exact sequence $0 \to e'J' \xrightarrow{i} e'\Gamma \to \tilde{S} \to 0$, $\mathrm{Hom}_\Gamma(e'\Gamma, L) \cong \mathrm{Hom}_\Gamma(e'J', L)$. Hence there exists $g \in \mathrm{Hom}_\Gamma(e'\Gamma, L)$ such that $f=gi$. Since f is irreducible, g is a splitting epimorphism, so that $L \cong e'\Gamma$.

Decompose $E=\bigoplus_{i=1}^{m} E_i$ where E_i are indecomposable. It follows from [3, 6.1 Corollary] that if Λ is of finite representation type then $m \leqq 4$ and when $m=4$ one of the E_i is a projective and injective Λ-lattice.

COROLLARY 5.4. *Let T be a tilting Λ-lattice of APR-type and let Λ be of finite representation type and $m=3$ or 4. Then if E_i $(1 \leqq i \leqq 3)$ and $\mathrm{Tr}_L(Je)$ are not injective Λ-lattices then Γ is of infinite representation type.*

PROOF. It follows from Proposition 5.2 that $e'J'$ is not an injective Γ-lattice. Hence there is an almost split sequence

$$0 \longrightarrow e'J' \longrightarrow \bigoplus_{i=1}^{n} L_i \oplus e'\Gamma \longrightarrow \tau^{-1}(e'J') \longrightarrow 0.$$

Since $F(E_i)$ are not injective Γ-lattices, we obtain that $n \geqq 3$. By Lemma 5.3, $L_i \in \mathcal{S}$ $(1 \leqq i \leqq n)$. Thus none of L_i $(i=1, 2, 3)$ and $e'\Gamma$ are projective and injective Γ-lattices. Hence by [3, 6.1 Corollary], Γ is of infinite representation type.

6. A remark on global dimension of Γ

Let T be a tilting Λ-lattice and $\Gamma=\mathrm{End}_\Lambda(T)$. Then we have gl. dim $\Gamma \leqq$ gl. dim $\Lambda+1$ by [8, (2.13)]. In this section we shall study global dimension of Γ in our cases.

For a Λ-lattice X, $\mathcal{L}\mathrm{id}_\Lambda(X)$ denotes the Λ-lattice injective dimension. Namely, $\mathcal{L}\mathrm{id}_\Lambda(X)=n<\infty$ if $\mathrm{Ext}^{n+1}_\Lambda(Y, X)=0$ for all Λ-lattices Y and $\mathrm{Ext}^n_\Lambda(Z, X)$

$\neq 0$ for some Λ-lattice Z. It is known that $\mathcal{L}\mathrm{id}_\Lambda(X)=\mathrm{pd}_\Lambda(X^*)=\mathrm{id}_\Lambda(X)-1$ if one of them is finite.

PROPOSITION 6.1. *Let* $T=(1-e)\Lambda\oplus\mathrm{Tr}_L(Je)$ *be a tilting* Λ*-lattice of BB-type and put* $\Gamma=\mathrm{End}_\Lambda(T)$. *If* $F((\Lambda e)^*)$ *is a projective* Γ*-lattice, that is,* $(\Lambda e)^*\cong \mathrm{Tr}_L(Je)$ *or* $(\Lambda e)^*|(1-e)\Lambda$, *then* $\mathrm{gl.dim}\,\Gamma\leqq\mathrm{gl.dim}\,\Lambda$.

PROOF. We first show that for $X\in\mathcal{T}$ if $\mathcal{L}\mathrm{id}_\Lambda(X)=t$ then $\mathcal{L}\mathrm{id}_\Gamma(F(X))\leqq t$. By the duality $(\ )^*$, we have a minimal injective Λ-lattice resolution

$$(*)\qquad 0\longrightarrow X\xrightarrow{f_0} I_0\longrightarrow\cdots\xrightarrow{f_t} I_t\longrightarrow 0.$$

Suppose that $(\Lambda e)^*|I_t$ and let $p: I_t\to(\Lambda e)^*$ be a projection. Then the map $F(p)F(f_t): F(I_{t-1})\to F((\Lambda e)^*)$ splits by assumption. It follows from the category equivalence $\mathcal{T}\approx\mathcal{S}$ that pf_t also splits. This contradicts to the minimality of $(*)$. Thus $(\Lambda e)^*\nmid I_t$, so that $F(I_t)$ is an injective Γ-lattice by the proof of Proposition 5.2. On the other hand, for an injective Λ-lattice I, $\mathcal{L}\mathrm{id}_\Gamma(F(I))\leqq 1$, because $F(\Lambda^*)\cong T^*$ and $\mathrm{pd}_\Gamma(T)\leqq 1$. Since X, $I_i\in\mathcal{T}$, we obtain an exact sequence of Γ-lattices

$$0\longrightarrow F(X)\longrightarrow F(I_0)\longrightarrow\cdots\longrightarrow F(I_t)\longrightarrow 0.$$

Therefore, $\mathcal{L}\mathrm{id}_\Gamma(F(X))\leqq t$. Now suppose that $\mathrm{gl.dim}\,\Lambda=n<\infty$. Then $\mathcal{L}\mathrm{id}_\Lambda(T)\leqq n-1$, so that $\mathcal{L}\mathrm{id}_\Gamma(\Gamma)=\mathcal{L}\mathrm{id}_\Gamma(F(T))\leqq n-1$. Hence $\mathrm{id}_\Gamma(\Gamma)\leqq n$. Since $\mathrm{gl.dim}\,\Gamma$ is finite, we have $\mathrm{gl.dim}\,\Gamma=\mathrm{id}_\Gamma(\Gamma)\leqq n$. This completes the proof.

When Λ has global dimension two, $\mathrm{gl.dim}\,\Gamma$ is completely determined as follows, provided that T is of APR-type.

COROLLARY 6.2. *Assume that* $\mathrm{gl.dim}\,\Lambda=2$ *and that* T *is of APR-type. Then* (a) *if* $(\Lambda e)^*\cong\mathrm{Tr}_L(Je)$ *or* $(\Lambda e)^*|(1-e)\Lambda$, *then* $\mathrm{gl.dim}\,\Gamma=2$, *and* (b) *otherwise,* $\mathrm{gl.dim}\,\Gamma=3$.

PROOF. (a) follows from Proposition 6.1. (b) It follows from Proposition 5.1 that there exists an almost split sequence

$$0\longrightarrow F((\Lambda e)^*)\longrightarrow F(E)\longrightarrow e'J'\longrightarrow 0.$$

Since T is a tilting Λ-lattice of APR-type, $E\in\mathrm{add}((1-e)\Lambda)$, so that $F(E)$ is projective. Hence, since $F((\Lambda e)^*)$ is not projective, $\mathrm{pd}_\Gamma(\tilde{S})\geqq 3$. Therefore $\mathrm{gl.dim}\Gamma=3$ by [8, (2.13)].

7. Examples

All examples here are tiled R-orders in the full matrix ring $(K)_n$ for some $n\geqq 4$. For $1\leqq i\leqq n$, let e_i be the matrix in $(K)_n$ such that the (i, i)-entry is 1 and the others are 0. For an integer $m\geqq 1$, an ideal $\pi^m R$ of R is denoted by $\boldsymbol{\pi}^m$.

It is often said that R-orders of global dimension two have a lot of similar properties as hereditary artin algebras. While such algebras always have APR-tilts, there is a tiled R-order of global dimension two which does not have tilting lattices of APR-type. Note also that we have shown in § 2 that every tiled R-order of finite global dimension has tilting lattices of BB-type.

EXAMPLE 7.1. Let Λ be the tiled R-order

$$\begin{pmatrix} R & R & R & R \\ \boldsymbol{\pi} & R & R & R \\ \boldsymbol{\pi}^2 & \boldsymbol{\pi} & R & R \\ \boldsymbol{\pi}^3 & \boldsymbol{\pi}^2 & \boldsymbol{\pi} & R \end{pmatrix}$$

in $(K)_4$. Then $\mathrm{gl.dim}\Lambda=2$. By Theorem 3.1, we can verify that Λ does not have tilting Λ-lattices of APR-type.

Next we give an R-order Λ with a tilting Λ-lattice T of APR-type such that $\mathrm{gl.dim}\Lambda=\mathrm{gl.dim}\Gamma=2$ where $\Gamma=\mathrm{End}_\Lambda(T)$ and that while Λ is of finite representation type, Γ is not.

EXAMPLE 7.2. Let Λ be the tiled R-order

$$\begin{pmatrix} R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\ \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\ \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} \\ R & R & R & R & \boldsymbol{\pi} \\ R & R & R & R & R \end{pmatrix}$$

in $(K)_5$. Then $\mathrm{gl.dim}\Lambda=2$. The Auslander-Reiten quiver of Λ is given by

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & {}^{p}R\boldsymbol{\pi\pi\pi\pi} \arrow[r] & R\boldsymbol{\pi}R\boldsymbol{\pi\pi} \arrow[r] & \boldsymbol{\pi}RR\boldsymbol{\pi\pi}_i \arrow[dr] & & & \\
{}^{p}RRRRR \arrow[ur] \arrow[r] \arrow[dr] \arrow[rrrrrr, no head, bend right=40, "\text{identify}"'] & {}^{p}\boldsymbol{\pi}R\boldsymbol{\pi\pi\pi} \arrow[r] & \| \arrow[r] & R\boldsymbol{\pi}R\boldsymbol{\pi\pi}_i \arrow[r] & RRR\boldsymbol{\pi\pi}_i \arrow[r] & {}^{p}RRRR\boldsymbol{\pi}_i \arrow[r] & {}^{p}RRRRR \\
 & {}^{p}\boldsymbol{\pi\pi}R\boldsymbol{\pi\pi} \arrow[r] & \boldsymbol{\pi}RR\boldsymbol{\pi\pi} \arrow[r] & RR\boldsymbol{\pi\pi\pi}_i \arrow[ur] & & &
\end{tikzcd}$$

where $\begin{matrix} R \\ \| \\ R \end{matrix}$ stands for the set $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : x, y\in R \text{ and } x\equiv y \mod \boldsymbol{\pi}\right\}$ and "p" ("i") means that it is a projective (injective) Λ-lattice. Put $T=(1-e_5)\Lambda\oplus \begin{matrix} R\pi R\pi\pi \\ \| \\ \pi RR\pi\pi \end{matrix}$ and $\Gamma=\mathrm{End}_{\Lambda}(T)$. Since $(Je_5)^*\cong e_5\Lambda$, T is a tilting Λ-lattice of APR-type. Hence, it follows from Corollary 6.2 that gl. dim$\Gamma=2$. Further, it follows from Corollary 5.4 that Γ is of infinite representation type.

We remark that the example in [8] is similar to ours, but in that example Γ has global dimension three by Corollary 6.2.

The last example provides an R-order Λ having no tilting Λ-lattices of APR-type, but it has a tilting Λ-lattice T of BB-type such that ${}_{\Gamma}T$ is of APR-type where $\Gamma=\mathrm{End}_{\Lambda}(T)$.

EXAMPLE 7.3. Let Λ be the tiled R-order

$$\begin{pmatrix} R & \pi & \pi^2 & \pi^2 & \pi^2 \\ R & R & \pi & \pi & \pi \\ R & \pi & R & \pi & \pi \\ R & \pi & R & R & \pi \\ R & R & R & R & R \end{pmatrix}$$

in $(K)_5$. Then gl. dim$\Lambda=3$ and Λ has no tilting Λ-lattice of APR-type by Theorem 3.1. Let $T=(1-e_2)\Lambda\oplus(R\pi\pi\pi\pi)$. Then T is a tilting Λ-lattice of BB-type. The Auslander-Reiten quiver of Λ is given by

$$\begin{tikzcd}
 & {}^{p}R\pi RR\pi_i \arrow[dr] & & {}^{p}RRRRR_i \arrow[dr] & & {}^{p}R\pi R\pi\pi \\
{}^{p}R\pi R\pi\pi \arrow[ur] \arrow[dr] & & RRRR\pi \arrow[ur] \arrow[dr] & & R\pi\pi\pi\pi \arrow[ur] \arrow[dr] & \\
 & RRR\pi\pi \arrow[ur] \arrow[dr] & & R\pi\pi\pi\pi^2{}_i \arrow[ur] & & {}^{p}RR\pi\pi\pi \\
{}^{p}RR\pi\pi\pi \arrow[ur] \arrow[dr] \arrow[urrrrr, dash, bend right, "\text{identify}"'] & & R\pi\pi\pi^2\pi^2{}_i \arrow[ur] & & & \\
 & {}^{p}R\pi\pi^2\pi^2\pi^2{}_i \arrow[ur] & & & &
\end{tikzcd}$$

Put $\Gamma=\mathrm{End}_{\Lambda}(T)$. It follows from Proposition 5.2 that T is a tilting left Γ-lattice of APR-type. Further,

$$\Gamma=\begin{pmatrix} R & \pi & \pi^2 & \pi^2 & \pi^2 \\ R & R & \pi & \pi & \pi \\ R & R & R & \pi & \pi \\ R & R & R & R & \pi \\ R & R & R & R & R \end{pmatrix}$$

and the Auslander-Reiten quiver of Γ is given by

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & {}^{p}RRRRR_i \arrow[dr] & & \\
 & & {}^{p}RRRR\pi \arrow[ur] \arrow[dr] & & R\pi\pi\pi\pi_i \arrow[r] & {}^{p}RR\pi\pi\pi \\
 & {}^{p}RRR\pi\pi \arrow[ur] \arrow[dr] & & R\pi\pi\pi\pi^2{}_i \arrow[ur] & & \\
{}^{p}RR\pi\pi\pi \arrow[ur] \arrow[dr] \arrow[urrrrr, dash, bend right=40, "\text{identify}"'] & & R\pi\pi\pi^2\pi^2{}_i \arrow[ur] & & & \\
 & {}^{p}R\pi\pi^2\pi^2\pi^2{}_i \arrow[ur] & & & &
\end{tikzcd}
$$

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