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ON CERTAIN MULTIVALENTLY STARLIKE FUNCTIONS

By

Mamoru NUNOKAWA

Let $A(p)$ denote the class of functions $f(z)=z^{p}+\sum_{n=p+1}^{\infty}a_{n}z^{n}$ which are analytic in the open unit disk $E=\{z:|z|<1\}$.

A function $f(z)\in A(p)$ is called p-valently starlike with respect to the origin iff

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0 \quad \text{in} \quad E,$$

Ozaki [2, Theorem 1] proved that if $f(z)\in A(1)$ and

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}<\frac{3}{2} \quad \text{in} \quad E, \tag{1}$$

then $f(z)$ is univalent in E.

Moreover, Umezawa [6] proved that if $f(z)\in A(1)$ satisfies the condition (1), then $f(z)$ is univalent and convex in one direction in E.

Recently, R. Singh and S. Singh [4, Theorem 6] proved that if $f(z)\in A(1)$ satisfies the condition (1), then $f(z)$ is starlike in E.

Ozaki [2, Theorem 3] proved that if $f(z)\in A(p)$ and

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}<p+\frac{1}{2} \quad \text{in} \quad E, \tag{2}$$

then $f(z)$ is p-valent in E.

It is the purpose of the present paper to prove that if $f(z)\in A(p)$ satisfies the condition (2), then $f(z)$ is p-valently starlike in E.

This is an extended result of R. Singh and S. Singh [4, Theorem 6].

In this paper, we need the following lemma.

LEMMA 1. *Let $f(z)\in A(1)$ be starlike with respect to the origin in E.*

Let $C(r,\theta)=\{f(te^{i\theta}):0\leqq t\leqq r<1\}$ and $T(r,\theta)$ be the total variation of $\arg f(te^{i\theta})$ on $C(r,\theta)$, so that

$$T(r,\theta)=\int_{0}^{r}\left|\frac{\partial}{\partial t}\arg f(te^{i\theta})\right|dt.$$

Then we have

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$$T(r,\ \theta)<\pi.$$

We owe this lemma to Sheil-Small [5, Theorem 1].

MAIN THEOREM. *Let* $f(z)\in A(p)$ *and*

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}<p+\frac{\alpha}{2} \quad in \quad E, \tag{3}$$

where $0<\alpha\leqq 1$.

Then we have

$$\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\pi}{2}\alpha \quad in \quad E,$$

or $f(z)$ *is* p*-valently starlike in* E.

PROOF. Let us put

$$\frac{2}{\alpha}\left(p+\frac{\alpha}{2}-1-\frac{zf''(z)}{f'(z)}\right)=\frac{zg'(z)}{g(z)} \tag{4}$$

where $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$.

From the assumption (3), we have

$$\mathrm{Re}\frac{zg'(z)}{g(z)}>0 \quad \text{in} \quad E.$$

This shows that $g(z)$ is starlike and univalent in E.

From (4) and by an easy calculation (see e.g. [1]), we have

$$f'(z)=pz^{p-1}\left(\frac{g(z)}{z}\right)^{-\alpha/2}.$$

Since

$$f'(z)\neq 0 \quad \text{in} \quad 0<|z|<1,$$

we easily have

$$\begin{aligned}\frac{f(z)}{zf'(z)}&=\int_0^1\frac{f'(tz)}{f'(z)}dt \\ &=\int_0^1 t^{p-1+\alpha/2}\left(\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right)^{-\alpha/2}dt\end{aligned} \tag{5}$$

where $z=re^{i\theta}$ and $0<r<1$.

Since $g(z)$ is starlike in E, from Lemma 1, we have

$$-\pi<\arg g(tre^{i\theta})-\arg g(re^{i\theta})<\pi \tag{6}$$

for $0<t\leqq 1$.

Putting

$$s=t^{p-1+\alpha/2}\left(\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right)^{-\alpha/2},$$

then we have

$$\arg s=-\frac{\alpha}{2}\arg\left(\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right). \tag{7}$$

From (6) and (7), s lies in convex sector

$$\left\{s: |\arg s|\leqq\frac{\pi}{2}\alpha\right\}$$

and the same is true of its integral mean of (5), (see e.g. [3, Lemma 1]).

Therefore we have

$$\left|\arg\frac{f(z)}{zf'(z)}\right|<\frac{\pi}{2}\alpha \quad \text{in } E,$$

or

$$\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\pi}{2}\alpha \quad \text{in } E.$$

This shows that

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0 \quad \text{in } E.$$

This completes our proof and this is an extended result of [4, Theorem 6].

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Department of Mathematics
Gunma University
Aramaki, Maebashi, 371
Japan