

ON COMPLETE HYPERSURFACES WITH HARMONIC CURVATURE IN A RIEMANNIAN MANIFOLD OF CONSTANT CURVATURE

Dedicated to Professor Morio Obata on his 60th birthday

By

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0. Introduction.

This paper is concerned with hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature. The classification of curvature-like tensor fields on a Riemannian manifold has been studied by K. Nomizu [10], in which the Codazzi equation for the curvature-like tensor played an important role. The subject is also treated by S. Y. Cheng and S. T. Yau [3] from the different point of view. A Riemannian curvature tensor is said to be *harmonic* if the Ricci tensor S satisfies the Codazzi equation $\delta S = 0$, namely, in local coordinates

$$(0.1) \quad R_{ij;k} = R_{ik;j},$$

where $R_{ij;k}$ denotes the covariant derivative of the Ricci tensor R_{ij} . Although the concept is closely related to a parallel Ricci tensor, it was shown by A. Derdziński [5] and A. Gray [6] that it is essentially weaker than the latter one. In the Yang-Mills theory the harmonic curvature is also weighty, and some studies for these topics are made. In particular, J. P. Bourguignon conjectured that on a 4-dimensional compact Riemannian manifold with harmonic curvature the Ricci tensor must be parallel. This is negatively answered by A. Derdziński [4], who gave an example of a 4-dimensional compact Riemannian manifold with harmonic curvature and non-parallel Ricci tensor. Certain kinds of Riemannian manifolds with harmonic curvature are investigated by J. P. Bourguignon [1], A. Derdziński [5], T. Kashiwada [7], S. Tachibana [13] and so on. In particular, A. Derdziński [5] gave also other examples of higher dimensional Riemannian manifolds.

On the other hand, hypersurfaces with parallel Ricci tensor in a Riemannian manifold of constant curvature are studied by H. B. Lawson Jr. [8] and I. Mogi

*² This research was partially supported by JSPS and KOSEF.

Received February 5, 1986

and one of the present authors [9], and hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature are recently investigated by E. Ômachi [11] and one of the present authors [15], who determined the situation of the principal curvatures, provided that the mean curvature is constant. Especially, one of the present authors [15] treated also them without the assumption that the mean curvature is constant.

In this paper a class of hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature will be considered. The purpose is to classify completely hypersurfaces with harmonic curvature in the case where a multiplicity of each principal curvature is greater than one, and to show that there exist infinitely many hypersurfaces with harmonic curvature and non-parallel Ricci tensor.

1. Preliminaries.

In order to fix the notation, the theory of hypersurfaces in a Riemannian manifold of constant curvature is prepared for. Let $\bar{M}=M^{n+1}(c)$ be an $(n+1)$ -dimensional Riemannian manifold of constant curvature c and let M be an n -dimensional connected Riemannian manifold. By ϕ the isometric immersion of M into \bar{M} is denoted. When the argument is local, M need not be distinguished from $\phi(M)$ and therefore, to simplify the discussion a point x in M may be identified with the point $\phi(x)$ and a tangent vector X at x may be also identified with the tangent vector $d\phi(X)$ at $\phi(x)$ via the differential $d\phi$ of ϕ .

To begin with, we choose an orthonormal local frame field $\{e_1, \dots, e_n, e_{n+1}\}$ in \bar{M} in such a way that, restricted to M , the vectors e_1, \dots, e_n are tangent to M and hence the other e_{n+1} is normal to M . With respect to this field of frames on \bar{M} , let $\{\bar{\omega}_1, \dots, \bar{\omega}_n, \bar{\omega}_{n+1}\}$ be the dual field. Here and in the sequel, the following convention on the range of indices are adopted, unless otherwise stated:

$$\begin{aligned} A, B, \dots &= 1, \dots, n, n+1, \\ i, j, \dots &= 1, \dots, n. \end{aligned}$$

Then, associated with the frame field $\{e_1, \dots, e_n, e_{n+1}\}$ there exist differential 1-forms $\bar{\omega}_{AB}$ on \bar{M} , which are called *connection forms* on \bar{M} , so that they satisfy the following structure equations on \bar{M} :

$$(1.1) \quad d\bar{\omega}_A + \sum_B \bar{\omega}_{AB} \wedge \bar{\omega}_B = 0, \quad \bar{\omega}_{AB} + \bar{\omega}_{BA} = 0,$$

$$(1.2) \quad d\bar{\omega}_{AB} + \sum_C \bar{\omega}_{AC} \wedge \bar{\omega}_{CB} = c \bar{\omega}_A \wedge \bar{\omega}_B.$$

By restricting these forms $\bar{\omega}_A$ and $\bar{\omega}_{AB}$ to M , they are denoted by ω_A and ω_{AB} without bar, respectively. Then

$$(1.3) \quad \omega_{n+1} = 0.$$

The metric on M induced from the Riemannian metric \bar{g} on the ambient space \bar{M} under the immersion ϕ is given by $g = 2 \sum_i \omega_i \omega_i$. Then $\{e_1, \dots, e_n\}$ is an orthonormal local field with respect to the induced metric and $\{\omega_1, \dots, \omega_n\}$ is the dual field, which consists of real valued, linearly independent 1-forms on M . They are called *canonical* forms on the hypersurface M . It follows from (1.3) and the Cartan lemma that

$$(1.4) \quad \omega_{n+1, i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

The quadratic form $\sum_{i,j} h_{ij} \omega_i \otimes \omega_j$ is called a *second fundamental form* of M . We call also a form σ defined by

$$\sigma(X, Y) = \sum_{i,j} h_{ij} \omega_i(X) \omega_j(Y) e_{n+1}$$

for any vector fields X and Y a second fundamental form on M . A linear transformation A on the tangent bundle TM is defined by $g(AX, Y) = g(\sigma(X, Y), e_{n+1})$. Then A is called a *shape operator* of M . By the structure equations (1.1), (1.2) and (1.3), the following structure equations on the hypersurface M are given:

$$(1.5) \quad d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(1.6) \quad d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \quad \Omega_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

where ω_{ij} (resp. Ω_{ij} and R_{ijkl}) denotes a connection form (resp. a curvature form and a curvature tensor) on M . From (1.2) and (1.6) the Gauss equation

$$(1.7) \quad R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + h_{il}h_{jk} - h_{ik}h_{jl}$$

is obtained, and the Ricci tensor R_{ij} and the scalar curvature R can be expressed as follows:

$$(1.8) \quad R_{ij} = (n-1)c\delta_{ij} + h h_{ij} - \sum_k h_{ik} h_{kj},$$

$$(1.9) \quad R = n(n-1)c + h^2 - \sum_{i,j} h_{ij} h_{ij},$$

where h is a function defined by $h = \sum_i h_{ii}$, namely, for the mean curvature H it satisfies $h = nH$.

Now, the covariant derivative h_{ijk} and R_{ijk} of h_{ij} and R_{ij} are respectively defined by

$$(1.10) \quad \begin{aligned} \sum_k h_{ijk} \omega_k &= dh_{ij} - \sum_k h_{kj} \omega_{ki} - \sum_k h_{ik} \omega_{kj}, \\ \sum_k R_{ijk} \omega_k &= dR_{ij} - \sum_k R_{kj} \omega_{ki} - \sum_k R_{ik} \omega_{kj}. \end{aligned}$$

Differentiating (1.4) exteriorly, we have the Codazzi equation on the hypersurface M

$$(1.11) \quad h_{ijk} - h_{ikj} = 0,$$

since the ambient space \bar{M} is of constant curvature, and by differentiating (1.8) exteriorly the covariant derivative R_{ijk} satisfies

$$\sum_k R_{ijk} \omega_k = \sum_k (h_k h_{ij} + h h_{ijk} - \sum_l h_{ilk} h_{lj} - \sum_l h_{il} h_{ljk}) \omega_k,$$

where $dh = \sum_k h_k \omega_k$, and hence

$$(1.12) \quad \sum_{j,k} R_{ijk} \omega_k \wedge \omega_j = \sum_{j,k} (h_k h_{ij} - \sum_l h_{ilk} h_{lj}) \omega_k \wedge \omega_j.$$

A Riemannian curvature tensor is said to be *harmonic* if the Ricci tensor satisfies the Codazzi equation (0.1), namely, R_{ijk} is symmetric with respect to all indices i, j and k . It follows from (1.12) that it is necessary and sufficient for M to be of harmonic curvature that it satisfies

$$(1.13) \quad h_k h_{ij} - h_j h_{ki} - \sum_l h_{ilk} h_{lj} + \sum_l h_{ilj} h_{lk} = 0$$

for any indices.

2. The gradient of the mean curvature.

Let M be an n -dimensional hypersurface with harmonic curvature in $M^{n+1}(c)$ and let H be the mean curvature on M . In this section, assume that the gradient of H is an eigenvector associated with an eigenvalue 0 of the shape operator A . In other words, we shall assume that it satisfies

$$(2.1) \quad A \text{ grad } H = 0, \quad \text{namely } \sum_j h_{ij} h_j = 0$$

holds true. In this assumption the case where $\text{grad } H = 0$ is included, that is, the situation that the mean curvature H has critical points is admitted. For simplification, a tensor h_{ij}^m and a function h_m on M for any integer m are introduced as follows;

$$(2.2) \quad \begin{aligned} h_{ij}^m &= \sum_{i_1, \dots, i_{m-1}} h_{ii_1} h_{i_1 i_2} \cdots h_{i_{m-1} j} \\ h_m &= \sum_i h_{ii}^m, \end{aligned}$$

where $h_1 = h = nH$. By taking account of the second Bianchi identity, it is easily seen that the scalar curvature is constant, and therefore the function $h^2 - h_2$ is constant. This implies

$$(2.3) \quad dh_2 = 2h dh.$$

First of all, the generalization of (2.3) is requested. Namely, the relation

$$(2.4) \quad dh_m = m h_{m-1} dh$$

is true for any integer $m (\geq 2)$. In fact, the relation (2.4) is proved by induction on m . At first, (2.3) shows that the case where $m=2$ in (2.4) holds. By the property of derivations for the exterior differential, it is easily seen that the following equation

$$dh_m = \sum_{i,j} m h_{ij}^{m-1} dh_{ij}.$$

The definition (1.10) of the covariant derivative h_{ijk} and the above equation imply

$$\begin{aligned} dh_m &= m \sum_{i,j,k} (h_{ijk} \omega_k + h_{kj} \omega_{ki} + h_{ik} \omega_{kj}) h_{ij}^{m-1} \\ &= m \sum_{i,j,k} h_{ijk} h_{ij}^{m-1} \omega_k + 2m \sum_{i,j} h_{ij}^m \omega_{ij}. \end{aligned}$$

Hence

$$(2.5) \quad dh_m = m \sum_{i,j,k} h_{ijk} h_{ij}^{m-1} \omega_k,$$

because h_{ij}^m is symmetric with respect to i and j and the connection form ω_{ij} is skew-symmetric with respect to i and j . This yields

$$\begin{aligned} dh_m &= m \sum_{i,j,k,l} h_{ijk} h_{jl} h_{li}^{m-2} \omega_k \\ &= m \sum_{i,k,l} (\sum_j h_{ijl} h_{jk} + h_k h_{il} - h_l h_{ik}) h_{li}^{m-2} \omega_k \\ &= m (\sum_{i,j,k,l} h_{ijl} h_{li}^{m-2} h_{jk} \omega_k + h_{m-1} dh - \sum_{k,l} h_k l^{m-1} h_l \omega_k), \end{aligned}$$

where we have used (1.10) and (1.13). By the assumption (2.1) the last term in the right hand side vanishes identically. It follows from the case where $m-1$ in (2.5) and the supposition of the induction that we get

$$\sum_{i,j,l} h_{ijl} h_{li}^{m-2} \omega_j = \frac{1}{m-1} dh_{m-1} = h_{m-2} dh,$$

which yields

$$\sum_{i,l} h_{ijl} h_{li}^{m-2} = h_{m-2} dh(e_j).$$

This means that the first term in the right hand side of the above equation vanishes also identically, which completes the proof.

A function H_m for any integer $m (\geq 2)$ is next defined by

$$(2.6) \quad H_m = \sum_{k=0}^m (-1)^k \binom{m}{k} h^{m-k} h_k, \quad h_0 = 1.$$

By making use of (2.4) it follows from the straightforward calculation that

$$\begin{aligned} dH_m &= \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} (m-k) h^{m-k-1} h_k dh + \sum_{k=1}^m (-1)^k k \binom{m}{k} h^{m-k} h_{k-1} dh \\ &= \sum_{k=0}^{m-1} (-1)^k \left\{ (m-k) \binom{m}{k} - (k+1) \binom{m}{k+1} \right\} h^{m-k-1} h_k dh, \end{aligned}$$

which shows that H_m is constant on M . Thus we have

LEMMA 2.1. *Let M be a hypersurface with harmonic curvature in $M^{n+1}(c)$. If the shape operator A of M satisfies $A \operatorname{grad} H = 0$, then H_m is constant on M for any integer $m (\geq 2)$.*

By rewriting (2.6), the relation

$$(2.7) \quad h_m = h^m + \sum_{k=2}^m (-1)^k \binom{m}{k} H_k h^{m-k}$$

is true for any integer $m \geq 2$. In fact, the equation is also verified by induction on m . At first, the case where $m=2$ in (2.6) is considered. Then it shows that (2.7) holds for $m=2$. Next, suppose that (2.7) holds for integers less than m . Since the constant H_m is expressed as

$$H_m = \binom{m}{0} h^m - \binom{m}{1} h^{m-1} h + \sum_{k=2}^{m-1} (-1)^k \binom{m}{k} h^{m-k} h_k + (-1)^m h_m,$$

the supposition of the induction is applied to the third term in the right hand side, so it is reduced to

$$\begin{aligned} H_m &= (-1)^m h_m + \left\{ \binom{m}{0} - \binom{m}{1} \right\} h^m \\ &\quad + \sum_{k=2}^{m-1} (-1)^k \binom{m}{k} h^{m-k} \left\{ h^k + \sum_{l=2}^k (-1)^l \binom{k}{l} H_l h^{k-l} \right\} \\ &= (-1)^m h_m + \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} h^m \\ &\quad + \sum_{l=2}^{m-1} (-1)^l \left\{ \sum_{k=l}^{m-1} (-1)^k \binom{m}{k} \binom{k}{l} \right\} H_l h^{m-l}. \end{aligned}$$

On the other hand, the binomial theorem $(1-x)^m = \sum_{k=0}^m (-1)^k \binom{m}{k} x^k$ and the derivative of l -order for variable x yield the following relation for the binomial coefficients:

$$\sum_{k=l}^m (-1)^k \binom{m}{k} \binom{k}{l} = 0.$$

Accordingly we have

$$\begin{aligned} H_m &= (-1)^m h_m + \left\{ \sum_{k=0}^m (-1)^k \binom{m}{k} - (-1)^m \right\} h^m \\ &\quad + \sum_{l=2}^{m-1} (-1)^l \left\{ \sum_{k=l}^m (-1)^k \binom{m}{k} \binom{k}{l} - (-1)^m \binom{m}{l} \right\} H_l h^{m-l} \\ &= (-1)^m h_m - (-1)^m h^m - (-1)^m \sum_{l=2}^{m-1} (-1)^l \binom{m}{l} H_l h^{m-l}, \end{aligned}$$

which implies that (2.7) holds for any integer $m \geq 2$.

3. No simple roots.

This section is devoted to the study the case where the hypersurfaces with harmonic curvature in $M^{n+1}(c)$ has principal curvatures all of whose multiplicities are greater than one. The second fundamental form may be diagonalized so that $\sum_i h_{ij} \omega_i \otimes \omega_j = \sum_i \lambda_i \omega_i \otimes \omega_i$. A principal curvature λ_i is called a *simple root* at x if the multiplicity at x is equal to 1.

First of all, we prove

LEMMA 3.1. *Let M be a hypersurface of $M^{n+1}(c)$ with harmonic curvature. If the shape operator has no simple roots on M , then $A \operatorname{grad} H = 0$.*

PROOF. Since we have $h_{ij} = \lambda_i \delta_{ij}$ at a point x on M , then equation (1.13) says that

$$(3.1) \quad \lambda_j h_k \delta_{ij} - \lambda_k h_j \delta_{ki} + (\lambda_k - \lambda_j) h_{ijk} = 0$$

at x , where we have used

$$(3.2) \quad \sum_i h_i h_{ij} = \lambda_j h_j.$$

Because of the assumption that the second fundamental form h_{ij} has no simple roots, for any fixed index j there is an index k different from j such that $\lambda_j = \lambda_k$, and therefore (3.1) reduces to

$$\lambda_j (h_k \delta_{ij} - h_j \delta_{ki}) = 0$$

at the point x , which implies that if x is not a zero point of the principal curvature λ_j , then we have $h_j = 0$ at x . From these data, we conclude, using (3.2), that $\sum_i h_i h_{ij} = 0$. This completes the proof of the lemma.

In the next place, using Lemma 3.1 we are going to prove that the mean curvature H of M is constant.

By taking account of (2.7), it is easily seen that

$$(3.3) \quad \begin{aligned} h_{n+1} - h h_n &= \sum_{k=2}^n (-1)^k \left\{ \binom{n+1}{k} - \binom{n}{k} \right\} H_k h^{n+1-k} + (-1)^{n+1} H_{n+1} \\ &= \sum_{k=2}^{n+1} (-1)^k \binom{n}{k-1} H_k h^{n+1-k}, \end{aligned}$$

which is a polynomial of degree $n-1$ with respect to h with constant coefficient, because of Lemma 2.1. Since $\lambda_1, \dots, \lambda_n$ are the principal curvatures of the second fundamental form h_{ij} , h_m can be written as

$$(3.4) \quad h_0=1, \quad h_1=h=\sum_{i=1}^n \lambda_i, \quad h_m=\sum_{i=1}^n \lambda_i^m, \quad m=2, 3, \dots.$$

Now, let $f_1(\lambda), \dots, f_n(\lambda)$ be elementary symmetric functions of $\lambda=(\lambda_1, \dots, \lambda_n)$, namely,

$$(3.5) \quad \begin{cases} f_1=f_1(\lambda)=(-1) \sum_i \lambda_i, \\ f_2=f_2(\lambda)=(-1)^2 \sum_{i<j} \lambda_i \lambda_j, \\ \vdots \\ f_n=f_n(\lambda)=(-1)^n \lambda_1 \lambda_2 \cdots \lambda_n. \end{cases}$$

Then it is well known that f_1, \dots, f_n and h_1, \dots, h_n, h_{n+1} are related by the Newton formulas (cf. [14]) as follows:

$$(3.6) \quad \begin{cases} h_1+f_1=0, \\ h_2+f_1h_1+2f_2=0, \\ \vdots \\ h_n+f_1h_{n-1}+\cdots+f_{n-1}h_1+nf_n=0, \\ h_{n+1}+f_1h_n+\cdots+f_{n-1}h_2+h_1f_n=0. \end{cases}$$

When these formulas are regarded as the linear homogeneous simultaneous equations with respect to $(1, f_1, \dots, f_n)$, we see, using the principle of elimination, that the determinant of coefficients vanishes identically. If we take account of (3.3) and the Laplace expansion to this determinant, we can get

$$(3.7) \quad ((n+1)!/2)H_2h^{n-1}-((n-1)(n+1)!/3)H_3h^{n-2}+\cdots=0.$$

Therefore, it follows from (3.7) that h_1 is the root of the algebraic equation with constant coefficients unless all H_m vanishes. According to Lemma 2.1, we have

LEMMA 3.2. *Let M be a hypersurface with harmonic curvature in $M^{n+1}(c)$. If the shape operator A of M satisfies $A \operatorname{grad} H=0$, then the mean curvature of M is constant, provided that there exists a nonzero H_m defined by (2.6).*

On the other hand, if all H_m 's are zero and the shape operator of M has no simple roots, then it is easily derived, by using (2.7), (3.5) and (3.6), that h is also constant.

Combining this fact, Lemma 3.1 and Lemma 3.2, we have

PROPOSITION 3.3. *Let M be a hypersurface with harmonic curvature in $M^{n+1}(c)$. If the shape operator of M has no simple roots, then the mean curvature of M is constant.*

Under the property of Proposition 3.3, (2.4) means that each principal curvature of M is constant and hence, by means of Umehara's theorem [15] the number of distinct principal curvatures is at most two, say λ and μ , such that $c + \lambda\mu = 0$, which is applied to the situation where the ambient space is a sphere, a Euclidean one or a hyperbolic one. So, in the case, the above result for the number of distinct principal curvatures is simply proved from a different point of view. In fact, M is an isoparametric hypersurface in the sense of E. Cartan and the basic identity for principal curvatures shows that the above is true, provided that $c \leq 0$ [2]. If $c > 0$, then it is evident in [11]. Moreover, the second fundamental form of M is parallel.

By the way, we shall here give a model of hypersurfaces with parallel Ricci tensor in a hyperbolic space $H^{n+1}(c)$ (cf. Lawson [8]). $H^{n+1}(c)$ is covered by a coordinate system $\{x_1, \dots, x_{n+1}\}$ such that the Riemannian metric ds^2 of $H^{n+1}(c)$ is given by

$$ds^2 = \sum_{\alpha=1}^{n+1} dx_{\alpha}^2 - (\sum_{\alpha=1}^{n+1} x_{\alpha} dx_{\alpha})^2 / (r^2 + \sum_{\alpha=1}^{n+1} x_{\alpha}^2),$$

where $r^2 = -1/c$. The space $H^{n+1}(c)$ is a complete and simply connected Riemannian manifold of constant negative curvature c . A family of hypersurfaces $M(s)$ in $H^{n+1}(c)$ is defined by

$$M(s) = \{x \in H^{n+1}(c) : \sum_{\alpha=1}^{n+1} x_{\alpha}^2 = s^2 - r^2\}$$

for $s > r$. Then a hypersurface $M(s)$ for a fixed s is a space of constant curvature $c_1 = 1/(s^2 - r^2)$ in $H^{n+1}(c)$, which is totally umbilic. As another family of hypersurfaces $M(t)$, the following subject is defined:

$$M(t) = \{x \in H^{n+1}(c) : x_1 = t \geq 0\}.$$

The hypersurface $M(t)$ for an arbitrary fixed t is totally umbilic and hence it is a hyperbolic space of constant curvature $c_1 = -1/(r^2 + t^2)$. A flat hypersurface F^n is constructed as follows:

$$F^n = \{x \in H^{n+1}(c) : \sum_{i=1}^n x_i^2 = 2rx_{n+1}\}.$$

Then F^n is covered by one coordinate system $\{x_1, \dots, x_n\}$ such that the Riemannian metric induced from the Riemannian metric in $H^{n+1}(c)$ is given by $ds^2 = \sum_{i=1}^n dx_i^2$. Accordingly, F^n is flat. Lastly, a family of product hypersurfaces $S^k(c_1) \times H^{n-k}(c_2)$ in $H^{n+1}(c)$ is considered. They are defined by

$$S^k(c_1) \times H^{n-k}(c_2) = \{x \in H^{n+1}(c) : \sum_{i=1}^{k+1} x_i^2 = 1/c_1\},$$

where c_1 is positive constant and $1/c_1 + 1/c_2 = 1/c$, and $1 \leq k \leq n-1$. Any hypersurface of the family is the product manifold of a sphere of constant curvature

c_1 and a hyperbolic space of constant curvature c_2 and consequently it has exactly two distinct principal curvatures $(c_1 - c)^{1/2}$ and $(c_2 - c)^{1/2}$ of multiplicity k and $n - k$, respectively.

Combining Proposition 3.3 together with Umehara's theorem, we can see the following

THEOREM 3.4. *Let M be an n (≥ 3)-dimensional complete and simply connected Riemannian manifold with harmonic curvature and let ϕ be an isometric immersion of M into an $(n+1)$ -dimensional complete and simply connected Riemannian manifold of constant curvature c . If the multiplicity of each principal curvature is greater than one, then $\phi(M)$ is isometric to one of the following spaces:*

- (1) *The case where $c > 0$. The great sphere, the small sphere and $S^k(c_1) \times S^{n-k}(c_2)$, where $2 \leq k \leq n-2$ and $1/c_1 + 1/c_2 = 1/c$. In particular, ϕ is an imbedding.*
- (2) *The case where $c = 0$. The sphere, the Euclidean space and $S^k \times R^{n-k}$.*
- (3) *The case where $c < 0$. The sphere, the hyperbolic space, the flat space F^n and $S^k(c_1) \times H^{n-k}(c_2)$, where $2 \leq k \leq n-2$ and $1/c_1 + 1/c_2 = 1/c$. In particular, ϕ is an imbedding.*

4. Hypersurfaces with harmonic curvature and non-parallel Ricci tensor.

This section is devoted to the investigation of examples of hypersurfaces with harmonic curvature and non-parallel Ricci tensor in $M^{n+1}(c)$. By taking account of Theorem 3.4, it is seen that at least one principal curvatures ought to be of multiplicity 1.

Let M be a hypersurface immersed in $M^{n+1}(c)$, and assume that the principal curvatures λ_i on M satisfy

$$(4.1) \quad \begin{cases} \lambda_1 = \cdots = \lambda_{n-1} = \lambda \neq 0, \\ \lambda_n = \mu, \end{cases}$$

such that $\lambda \neq \mu$. Without loss of generality, we may suppose that $\lambda > 0$. As is well known, the distribution of the space of eigenvectors corresponding to the eigenvalue λ is completely integrable, because the multiplicity of each principal curvature is constant. Now, since λ and μ are smooth functions on M , we have, using the covariant derivative h_{ijk} ,

$$(4.2) \quad d\lambda = d\lambda_a = h_{aaa}\omega_a + \sum_{b \neq a} h_{aab}\omega_b + h_{aan}\omega_n,$$

where indices a, b, \dots run over the range $\{1, \dots, n-1\}$. Because of $\omega_{n+1}{}_a = \lambda_a\omega_a$, we have

$$\begin{aligned} d\omega_{n+1\ a} &= d\lambda_a \wedge \omega_a + \lambda_a d\omega_a \\ &= d\lambda \wedge \omega_a + \lambda(-\sum_b \omega_{ab} \wedge \omega_b - \omega_{an} \wedge \omega_n). \end{aligned}$$

On the other hand, the structure equation (1.2) yields

$$\begin{aligned} d\omega_{n+1\ a} &= -\sum_k \omega_{n+1\ k} \wedge \omega_{ka} \\ &= -\lambda \sum_b \omega_b \wedge \omega_{ba} - \mu \omega_n \wedge \omega_{na}. \end{aligned}$$

Combining with above two equations, we have

$$(4.3) \quad \sum_b \lambda_{,b} \omega_b \wedge \omega_a + \{(\mu - \lambda) \omega_{an} - \lambda_{,n} \omega_a\} \wedge \omega_n = 0$$

for a fixed index a , where $d\lambda = \sum_b \lambda_{,b} \omega_b + \lambda_{,n} \omega_n$. This implies

$$(4.4) \quad \begin{cases} \lambda_{,a} = 0, \\ (\mu - \lambda) \omega_{an} - \lambda_{,n} \omega_a = \sigma_a \omega_n \end{cases}$$

for any index a , where σ_a is a function on M . From (4.2) and the first equation of (4.4) it follows that we have

$$h_{aaa} \omega_a + \sum_{b \neq a} h_{aab} \omega_b + h_{aan} \omega_n = \lambda_{,n} \omega_n,$$

and hence

$$h_{aaa} = 0, \quad h_{aab} = 0 \ (b \neq a), \quad h_{aan} = \lambda_{,n}.$$

Similarly, for the other μ we have

$$d\mu = \sum_b h_{nbn} \omega_b + h_{nnn} \omega_n.$$

Because of $\omega_{n+1\ n} = \mu \omega_n$, by the same argument as that of λ we have

$$d\omega_{n+1\ n} = -\lambda \sum_b \omega_{nb} \wedge \omega_b = d\mu \wedge \omega_n - \mu \sum_b \omega_{nb} \wedge \omega_b,$$

and hence

$$d\mu \wedge \omega_n + (\lambda - \mu) \sum_b \omega_{nb} \wedge \omega_b = 0.$$

This together with (4.4) implies

$$(4.5) \quad \mu_{,a} = \sigma_a \quad \text{for any index } a.$$

On the other hand, for distinct indices a and b , we have

$$(4.6) \quad h_{abk} = 0.$$

In the case where M is with harmonic curvature, principal curvatures λ_j satisfy (3.1) and because of $h = (n-1)\lambda + \mu$ and $dh = \sum_k h_k \omega_k$, we see

$$h_k = (n-1)\lambda_{,k} + \mu_{,k}$$

for any index k . Considering the case where $j=a$ and $k=n$ in (3.1), one gets

$$\lambda h_n \delta_{ai} - \mu h_a \delta_{ni} - (\lambda - \mu) h_{ani} = 0$$

for any indices a and i . This means that it follows from the above equation and (4.5) that

$$(4.7) \quad \begin{cases} \{(n-2)\lambda + \mu\}\lambda_{,n} + \lambda\mu_{,n} = 0, \\ \mu h_a + (\lambda - \mu)h_{ann} = 0. \end{cases}$$

Consequently, making use of the above relations, we have $h_a = \mu_{,a} = h_{ann}$ and $\lambda h_{ann} = 0$, namely

$$h_a = 0, \quad \sigma_a = 0.$$

Thus, by (4.4) we have

$$(4.8) \quad \begin{aligned} h_{nnn} &= \mu_{,n}, \\ \omega_{na} &= \frac{\lambda_{,n}}{\lambda - \mu} \omega_a. \end{aligned}$$

Accordingly, in order for M to be with harmonic curvature, principal curvatures λ and μ must satisfy (4.7) and (4.8). Moreover we have $d\omega_n = 0$, which shows that we may put

$$(4.9) \quad \omega_n = dv.$$

Thus we have

$$(4.10) \quad \omega_{na} = \frac{\lambda'}{\lambda - \mu} \omega_a,$$

where the prime denotes the derivative with respect to v . This means that the integral submanifold $M^{n-1}(v)$ corresponding to λ and v is umbilic in M and hence in $M^{n+1}(c)$.

By the simple calculation the following properties for the Ricci tensor are obtained:

$$\begin{aligned} R_{abc} &= 0, \quad R_{ann} = 0, \quad R_{abn} = [\{2(n-2)\lambda + \mu\}\lambda_{,n} + \lambda\mu_{,n}]\delta_{ab}, \\ R_{nnn} &= (n-1)(\lambda\mu_{,n} + \mu\lambda_{,n}). \end{aligned}$$

Therefore, in order for M to be with parallel Ricci tensor, it is necessary and sufficient that λ and μ are both constant.

EXAMPLE. $M = S^{n-1}(c_1) \times S^1(c_2) \subset \mathbf{R}^n \times \mathbf{R}^2$ such that $1/c_1 + 1/c_2 = 1$. The principal curvatures λ_j are given by

$$\begin{aligned} \lambda_1 = \cdots = \lambda_{n-1} &= \lambda = \pm(c_1 - 1)^{1/2}, \\ \lambda_n = \mu &= \mp(c_2 - 1)^{1/2}. \end{aligned}$$

Actually M is with harmonic curvature and parallel Ricci tensor. In particular, when $c_1 = n/(n-2)$ and $c_2 = n/2$, λ and μ are given by $\lambda = \pm(2/(n-2))^{1/2}$ and

$\mu = \mp((n-2)/2)^{1/2}$ and moreover they satisfy $(n-2)\lambda + \mu = 0$. In the latter case, the scalar curvature R is equal to $n(n-1)$.

Now, substituting (4.10) into the structure equation

$$d\omega_{na} + \sum_b \omega_{nb} \wedge \omega_{ba} = (c + \lambda\mu)\omega_n \wedge \omega_a,$$

we have

$$d\left(\frac{\lambda'}{\lambda-\mu}\omega_a\right) = -\frac{\lambda'}{\lambda-\mu}\sum_b \omega_b \wedge \omega_{ba} + (c + \lambda\mu)\omega_n \wedge \omega_a.$$

Since the left hand side is reduced to

$$\left(\frac{\lambda'}{\lambda-\mu}\right)' \omega_n \wedge \omega_a + \frac{\lambda'}{\lambda-\mu}(-\sum_b \omega_{ab} \wedge \omega_b - \omega_{an} \wedge \omega_n),$$

the following equation is obtained:

$$\left(\frac{\lambda'}{\lambda-\mu}\right)' - \left(\frac{\lambda'}{\lambda-\mu}\right)^2 - (c + \lambda\mu) = 0,$$

and hence we have

$$(4.11) \quad \lambda''(\lambda-\mu) - \lambda'(\lambda'-\mu') - \lambda'^2 - (c + \lambda\mu)(\lambda-\mu)^2 = 0.$$

Furthermore, under the condition (4.7) we have

$$(4.12) \quad \{(n-2)\lambda + \mu\}\lambda' + \lambda\mu' = 0.$$

Thus the distinct principal curvatures λ and μ satisfy a system of ordinary differential equations (4.11) and (4.12) of order 2. (4.12) is however equivalent to

$$\{(n-2)\lambda^2 + 2\lambda\mu\}' = 0,$$

which yields

$$(n-2)\lambda^2 + 2\lambda\mu = c_1,$$

where c_1 is the integral constant. Then the scalar curvature R is given by $R = n(n-1)c + (n-1)c_1$, and by taking account of (4.12), the ordinary differential equation of order 2 for λ is given by

$$(4.13) \quad \begin{aligned} &4\lambda(n\lambda^2 - c_1)\lambda'' - 4\{(n+2)\lambda^2 + c_1\}\lambda'^2 \\ & - (n\lambda^2 - c_1)^2\{2c + c_1 - (n-2)\lambda^2\} = 0, \end{aligned}$$

where $n\lambda^2 - c_1 \neq 0$. Putting $\omega = \lambda^{-2/n}$, (4.13) can be replaced by

$$\frac{d^2\omega}{dv^2} + \frac{(n+1)c_1\omega^{n-1}}{n-c_1\omega^n}\omega'^2 + \frac{\omega}{2n}(n-c_1\omega^n)\left(2c + c_1 - \frac{n-2}{\omega^n}\right) = 0.$$

Integrating the differential equation of degree 2, we obtain

$$\left(\frac{d\omega}{dv}\right)^2 = (n-c_1\omega^n)^{2(n+1)/n} \left\{ c_2 - \frac{1}{n} \int \omega(n-c_1\omega^n)^{-(n+2)/n} \left(2c + c_1 - \frac{n-2}{\omega^n}\right) d\omega \right\},$$

where c_2 is the integral constant. In the case where $c_1=0$, this is reduced to

$$\left(\frac{d\omega}{dv}\right)^2 + \frac{1}{\omega^{n-2}} + c\omega^2 = c_2,$$

which is the differential equation similar to that treated by T. Otsuki [12]. Thus there exist infinitely many hypersurfaces with harmonic curvature in $M^{n+1}(c)$ corresponding to the constants c_1 and c_2 , and the hypersurfaces have non-parallel Ricci tensor and the scalar curvatures are equal to $n(n-1)c + (n-1)c_1$.

By the same method as that of Otsuki's theory, we have the following construction theorem concerning for hypersurfaces with harmonic curvature.

THEOREM 4.1. *Let M be an n (≥ 3)-dimensional hypersurface with scalar curvature $n(n-1)c$ and the harmonic curvature immersed in $M^{n+1}(c)$. If it has exactly two distinct principal curvatures, one's multiplicity of which is equal to 1, and the other has no zero points, then the following assertions are true:*

- (1) *M is a locus of moving $(n-1)$ -dimensional submanifold $M^{n-1}(v)$ along which the principal curvature λ of multiplicity $n-1$ is constant and which is umbilic in M and of constant curvature $(d/dv(\log(n\lambda^2 - c_1)^{1/n}))^2 + \lambda^2 + c$, where v is the arc length of an orthogonal trajectory of the family $M^{n-1}(v)$, and $\lambda = \lambda(v)$ satisfies the ordinary differential equation (4.13) of order 2.*
- (2) *If $\bar{M} = S^{n+1}(c) \subset \mathbf{R}^{n+2}$, then $M^{n-1}(v)$ is contained in an $(n-1)$ -dimensional sphere $S^{n-1}(v) = E^n(v) \cap S^{n+1}$ of the intersection of S^{n+1} and an n -dimensional linear subspace $E^n(v)$ in \mathbf{R}^{n+2} which is parallel to a fixed E^n . The center q moves on a plane curve in a plane \mathbf{R}^2 through the origin of \mathbf{R}^{n+2} and orthogonal to E^n .*

COROLLARY. *There exist infinitely many hypersurfaces with harmonic curvature and non-parallel Ricci tensor in $M^{n+1}(c)$, which is not congruent to each other in it.*

In the next place, the condition under which the plane curve figured with the center q is controlled will be required. Since the matter discussed in [12, section 4] can be completely applied to this case, the necessary subjects for the explanation of the statement of the theorem are only quoted from [12], and the precise argument is omitted. The sphere S^{n+1} is regarded as $S^{n+1} \subset \mathbf{R}^{n+2} = \mathbf{R}^n \times \mathbf{R}^2$, and $\{\bar{e}_1, \dots, \bar{e}_n\}$ denotes the orthonormal frame in \mathbf{R}^n at the origin. Let C be a plane curve in \mathbf{R}^2 with a given supporting function $h(\theta)$, then the generic point $q(\theta)$ of C is given by

$$(4.14) \quad q(\theta) = e^{i(\theta - \pi/2)}(h(\theta) + ih'(\theta))$$

by considering \mathbf{R}^2 as the complex plane. The Frenet formula of C at $q(\theta)$ is given by $\bar{e}_{n+1}=e^{i\theta}$ and $\bar{e}_{n+2}=e^{i(\theta+\pi/2)}$. Suppose that the curve C is contained in the unit circle. Then a positive function ρ can be defined by $\rho^2=1-\|q\|^2$, and a hypersurface M is defined in $S^{n+1}(1)$ by

$$(4.15) \quad p=q+\rho\bar{e}_n.$$

A unit vector e_n is defined by

$$e_n=(\rho'\bar{e}_n+(h+h'')\bar{e}_{n+1})/((\rho')^2+(h+h'')^2)^{1/2}.$$

If the hypersurface M in $S^{n+1}(1)$ is with harmonic curvature and $R=n(n-1)$, then the function h satisfies the following ordinary differential equation

$$(4.16) \quad nh(1-h^2)\frac{d^2h}{d\theta^2}+2\left(\frac{dh}{d\theta}\right)^2+(1-h^2)(nh^2-2)=0.$$

Conversely, if a function $h(\theta)$ satisfying (4.16) gives a plane curve by the equation (4.14) in \mathbf{R}^2 contained in the unit circle, then a hypersurface M with harmonic curvature and $R=n(n-1)$ is obtained by (4.15). The hypersurfaces depend completely on properties of $h(\theta)$.

THEOREM 4.2. *Any complete hypersurface M with harmonic curvature and $R=n(n-1)$ in $S^{n+1}(1)$ of the type of Theorem 4.1 is given by the following method.*

(1) C is a plane curve in \mathbf{R}^2 given by

$$q(\theta)=e^{i(\theta-\pi/2)}(h(\theta)+ih'(\theta)),$$

where $h(\theta)$ is a solution of the differential equation (4.16) with $0 < h(0) \leq (2/n)^{1/2}$ and $h'(0)=0$.

(2) $M \ni p=(1-h(\theta)^2-h'(\theta)^2)^{1/2}\bar{e}_n+q(\theta)$, where $\bar{e}_n \in \mathbf{R}^n$, $\|\bar{e}_n\|=1$ and $S^{n+1} \subset \mathbf{R}^n \times \mathbf{R}^2$.

There exist countable number of compact hypersurfaces of this type, and the special case $S^{n-1}(n/(n-2)) \times S^1(n/2)$ corresponds to $h(0)=(2/n)^{1/2}$ and $h'(0)=0$.

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