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# HYPOELLIPTIC OPERATORS OF PRINCIPAL TYPE WITH INFINITE DEGENERACY

Dedicated to Professor Hiroki Tanabe on his sixtieth birthday in 1992

By

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## Introduction.

Let $P$ be a classical pseudodifferential operator of order $m$. We assume $P$ is of principal type, that is, the Hamilton vector field $H_p$ of the principal symbol $p$ of $P$ is not parallel to the radial direction where the principal symbol $p$ vanishes. In this paper we study the microhypoellipticity for $P$, under the following $(\overline{\Psi})$ condition given by Nirenberg-Treves [15];

$$(\overline{\Psi})\qquad \left\{\begin{array}{l}\text{the imaginary part } p_2 \text{ of the principal symbol } p\\ \text{does not change sign from } + \text{ to } - \text{ along any oriented}\\ \text{(null-) bicharacteristic of the real part } p_1 \text{ of } p.\end{array}\right.$$

Let us recall that $(\overline{\Psi})$ is necessary for adjoint operator $P^*$ of $P$ to be locally solvable (see Hörmander [6; Theorem 26.4.7], cf. Moyer [14]). Since it follows from the hypoellipticity of $P$ that $P^*$ is locally solvable, it is reasonable to assume the condition $(\overline{\Psi})$.

By supplying the missing arguments of Egorov [2], Hörmander [5] (see also [6; Chapter 27]) showed that a pseudodifferential operator $P$ of principal type is subelliptic (and hence hypoelliptic) if and only if the principal symbol $p$ of $P$ satisfies $(\overline{\Psi})$ and a finite type assumption ((27.1.8) in [6]). Without the finite type assumption, the problem of hypoellipticity seems to be difficult. For example, consider a first-order pseudodifferential operator of Egorov type as follows:

$$P_0=D_t+i(t^s D_{x_1}+t^k x_1^m|D|) \qquad \text{in } \boldsymbol{R}_t\times\boldsymbol{R}_x^n,\ |D|^2=D_t^2+|D_x|^2,$$

where $s$, $k$, $m$ are nonnegative integers. For $P_0$, condition $(\overline{\Psi})$ and the finite type assumption are expressed as

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$$s, m \text{ even}, \quad k \text{ odd}, \quad s<k .$$

Then $P_0$ is subelliptic with loss of $r/(r+1)$ derivatives $(r=k+m(s+1))$ and hence hypoelliptic. If $t^s$, $t^k$, $x_1^m$ of $P_0$ are replaced by functions infinitely vanishing then the hypoellipticity of $P_0$ is unknown. The aim of the present paper is to solve this particular problem, but we shall reply only for special cases, unfortunately, because we do not know even whether $L^2$ *a priori-estimate* holds for this modified $P_0$, in general. Actually, a remarkable counter-example given by Lerner [10] shows that we can not always expect $L^2$ *a priori-estimate* for operators satisfying $(\overline{\Psi})$.

To end the introduction, we state a few historical remarks: As a perfection of the preceding results of Nirenberg-Treves [15] in the analytic case or the finite type case, Beals-Fefferman [1] proved $L^2$ *a priori-estimate* (and hence local solvability) for pseudodifferential operators of principal type, under condition $(P)$ (i.e. the imaginary part $p_2$ of the principal symbol of $P$ does not change sign along the bicharacterisitic of the real part $p_1$, which is equivalent to $(\overline{\Psi})$ for differential operators.) Furthermore, Hörmander [6; Chapter 26] extented the local existence result of [1] to the semi-global one and fully studied the regularities of solutions for operators, of principal type, satisfying condition $(P)$. Under condition $(\overline{\Psi})$, $L^2$ *a priori-estimate* for operators in 2-dimension space was proved by Lerner [8], whoes method also plays an important role in the present paper.

## 1. Main results

Let $P$ be a classical pseudodifferential operator on $\boldsymbol{R}^{n+1}$, of order $m$, of principal type, which satisfies the condition $(\overline{\Psi})$. We are interested in the micro-hypoellipticity of $P$; that is, for $\rho_0 \in T^*(\boldsymbol{R}^{n+1})\backslash 0$, we shall see whether

$$\rho_0 \notin \mathrm{WF}(Pu) \text{ implies } \rho_0 \notin \mathrm{WF}(u) \quad \text{for } \forall u \in \mathscr{D}'(\boldsymbol{R}^{n+1}). \tag{1.1}$$

We assume $\rho_0 \in \mathrm{Char}\, P$ because (1.1) is trivial, otherwise, where Char $P$ denotes the set of characteristic points. Let $p=p_1+ip_2$ ($p_1$, $p_2$ real-valued) be the principal symbol of $P$ and let $\Gamma$ be a subset of Char $P$ where the Poisson bracket $\{p_1, p_2\}$ vanishes. It is known by Hömander's classical theorem [4] (and also Egorov-Hörmander Theorem [6; Theorem 27.1.11]) that (1.1) is true if $\rho_0 \notin \Gamma$, because we have a subelliptic estimate with loss of 1/2 derivatives. In what follows we consider the case where $\rho_0 \in \Gamma$. We assume that in a conic neighborhood of $\rho_0$

$$(1.2)\qquad \begin{cases} \Gamma \text{ is contained in a } C^\infty\text{-hypersurface in } T^*(\boldsymbol{R}^{n+1})\backslash 0 \\ \text{to which the Hamilton vector field } H_1 \text{ of } p_1 \text{ is transversal.} \end{cases}$$

After the multiplication by an elliptic factor, we may assume $P$ is of first order. Furthermore, by homogeneous canonical transformation and Malgrange preparation theorem we may assume that $\rho_0=(0, (0, \xi_0))\in T^*(\boldsymbol{R}_t\times\boldsymbol{R}_x^n)\backslash 0$, $(|\xi_0|=1)$, and the principal symbol $p$ of $P$ is expressed as, in a small conic neighborhood $V$ of $\rho_0$,

$$(1.3)\qquad p=p(t, x, \tau, \xi)=\tau+iq(t, x, \xi),$$

where $q(t, x, \xi)\in C^\infty(\boldsymbol{R}_t\times\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n)$ is real valued, positively homogeneous of degree one for $|\xi|\geqq 1/2$; in particular $q$ satisfies:

$$(1.4)\qquad q(t, x, \xi)=\lambda q(t, x, \xi/\lambda), \qquad \text{if } |\xi|\geqq 1/2 \text{ and } 0<\lambda\leqq 1,$$

and

$$(1.5)\qquad |(D_t^k D_x^\alpha D_\xi^\beta q)(t, x, \xi)|\leqq C_{\alpha,\beta,k}(1+|\xi|)^{1-|\beta|}.$$

We may also assume that lower order terms $p_0$, $p_{-1}$, $\cdots$ in the symbol of $P$ are independent of $\tau$ in a conic neighborhood $V$ of $\rho_0$ (see the paragragh after [6; Theorem 26.4.7′]). Hence we can write

$$(1.3)'\qquad P=D_t+iQ(t, x, D_x) \qquad \text{in } V,$$

where the principal symbol of $Q$ is $q(t, x, \xi)$. In that frame work, condition $(\overline{\Psi})$ is expressed as

$$(1.6)\qquad q(t, x, \xi)>0 \text{ and } s>t \text{ imply } q(s, x, \xi)\geqq 0.$$

Moreover, the set $\Gamma$ is defined by

$$\{(t, x, 0, \xi)\in T^*(\boldsymbol{R}^{n+1})\backslash 0\,;\, \partial_t q(t, x, \xi)=q(t, x, \xi)=0\}$$

and it follows from assumption (1.2) that

$$(1.7)\qquad \begin{cases} \text{for any } \mu>0 \text{ there exists a } \delta_\mu>0 \text{ such that} \\ \left\{(t, x, 0, \xi)\,;\, \mu\leqq|t|\leqq 2\mu,\quad |x|+\left|\dfrac{\xi}{|\xi|}-\xi_0\right|<\delta_\mu\right\}\cap\Gamma=0. \end{cases}$$

because $\rho_0=(0, (0, \xi_0))\in\Gamma$.

In order to state a sufficient condition for (1.1), we define a microlocalized operator of $P$ at $\rho_0$ as follows: Let $h(x)$ be a $C_0^\infty(\boldsymbol{R}^n)$ function such that $0\leqq h\leqq 1$, $h(x)=1$ for $|x|\leqq 1/5$ and $h(x)=0$ for $|x|\geqq 7/24$. For a $\delta>0$ we set $h_\delta(x)=h(x/\delta)$ and $H_\delta(x, \xi\,;\,\lambda)=h_\delta(x)h_\delta(\lambda\xi-\xi_0)$, where $0<\lambda\leqq 1$ is a parameter. Let $\delta_1$ be a small positive such that the projection of $V$ into $\boldsymbol{R}_t\times\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n$ contains $\{|t|\leqq 2\delta_1\}\times\operatorname{supp} h_{2\delta_1}(x)h_{2\delta_1}(\lambda\xi-\xi_0)$. For a parameter $0<\lambda\leqq 1$, we set

$$P_\lambda = D_t + i h_{\delta_1}(x) Q(t, x, D_x) h_{\delta_1}(\lambda D_x - \xi_0) \equiv D_t + i Q_\lambda(t, x, D_x).$$

THEOREM 1. *Let $\Gamma$ be the above set in* Char $P$ *and assume* (1.2). *Let* $\rho_0 = (0, (0, \xi_0)) \in \Gamma$ *and let $P$ be a pseudodifferential operator of the form* (1.3)′ *in a conic neighborhood $V$ of $\rho_0$. Let $\delta$ be a small positive such that* $100\delta < \delta_1$ *for the above $\delta_1$. Assume that for each $\delta$ there exist non-negative symbols* $\varphi(x, \xi; \lambda) \in S^0_{1,0}$ *and* $\alpha(t, x, \xi) \in C^\infty(\boldsymbol{R}_t; S^0_{1,0})$ *such that* $\{\varphi(x, \xi; \lambda); 0 < \lambda \leqq 1\}$ *is a bounded set of* $S^0_{1,0}$ *and we have*

$$(1.8) \qquad \begin{cases} \varphi \geqq 1 & \textit{outside of } \operatorname{supp} H_{5\delta}(x, \xi; \lambda) \\ \varphi = 0 & \textit{on } \operatorname{supp} H_\delta(x, \xi; \lambda), \end{cases}$$

$$(1.9) \qquad |(H_q\varphi)(t, x, \xi; \lambda)| \leqq \alpha(t, x, \xi) \quad \textit{on} \quad \{|t| \leqq \delta_1\} \times \operatorname{supp} H_{100\delta}(x, \xi; \lambda)$$

*and the following estimate: For any* $\varepsilon > 0$ *there exists a* $C_\varepsilon > 0$ *independent of* $0 < \lambda \leqq 1$ *such that*

$$(1.10) \qquad \begin{aligned} &\|u\|^2 + (\log\lambda)^2 \|\alpha(t, x, D_x)u\|^2 \\ &\leqq \varepsilon \|P_\lambda u\|^2 + C_\varepsilon(\lambda\|u\|^2 + \lambda^{-2}\|(1 - H_{20\delta}(x, D_x; \lambda))u\|^2) \end{aligned}$$

*if* $u \in C^\infty_0([-\delta_1, \delta_1]; \mathcal{S}(\boldsymbol{R}^n_x))$. *Then we have* (1.1).

COROLLARY. *The same conclusion of Theorem* 1 *follows if we replace* (1.9) *and* (1.10), *respectively, by*

$$(1.9)' \qquad |(H_q\varphi)(t, x, \xi; \lambda)|^2 \leqq \alpha(t, x, \xi) \quad \textit{on} \quad \{|t| \leqq \delta_1\} \times \operatorname{supp} H_{100\delta}(x, \xi; \lambda)$$

*and*

$$(1.10)' \qquad \begin{aligned} &\|u\|^2 + (\log\lambda)^2 \operatorname{Re}(\alpha(t, x, D_x)u, u) \\ &\leqq \varepsilon \|P_\lambda u\|^2 + C_\varepsilon(\lambda\|u\|^2 + \lambda^{-2}\|(1 - H_{20\delta}(x, D_x; \lambda))u\|^2). \end{aligned}$$

REMARK 1. The function $h(x)$ in Theorem 1 and Corollary is not necessary to be homogeneous spatially. For example, we can replace it by $h(x_1/\nu)h(x')$ for any $\nu > 0$ and for $h(x_1)$, $h(x')$ similar as $h(x)$.

REMARK 2. As criteria of hypoellipticity, logarithmic regularity up estimates were used in Morimoto [11-13]. The simple proof of Theorem 1 in the present paper is inspired by Kajitani-Wakabayashi [7; Theorem 1.2] (see also [16]) and Hörmander [6; Lemma 26.9.3].

As an application of Theorem 1, we consider a pseudodifferential operator of principal type which has the following form:

$$(1.11) \qquad P = D_t + i a(t, x, D_x)(D_{x_1} + f(t)|D_x|),$$

in a conic neighborhood $V$ of $\rho_0=(0,(0,\xi_0))$, where $a(t,x,\xi)\in C^\infty(\boldsymbol{R}_t; S^0_{1,0})$, $f(t)\in C^\infty$ satisfy

$$a(t,x,\xi)\geqq 0,\quad f'(t)>0\ (t\neq 0),\quad f'(0)=0. \tag{1.12}$$

It follows from (1.12) that $P$ satisfies $(\overline{\Psi})$.

THEOREM 2. *Let $\rho_0$ be $(0,(0,\xi_0))\in T^*(\boldsymbol{R}^{n+1})$. Let $P$ be of the form* (1.11) *in a conic neigborhood of $V$ of $\rho_0$ and satisfy* (1.12). *Then we have* (1.1) *if the following conditions are fulfilled;*

$$\exists\alpha(t),\ \beta(t)\in C^\infty;\ \beta(t)>0\ (t\neq 0),\ t\alpha'(t)\geqq 0, \tag{1.13}$$

$$\beta(t)\leqq a(t,x,\xi)\leqq\alpha(t)\quad in\ V, \tag{1.14}$$

$$|\nabla_x a(t,x,\xi)|+|\nabla_\xi a(t,x,\xi)|\xi||\leqq a(t)\quad in\ V, \tag{1.15}$$

$$\lim_{t\to 0} t\alpha(t)\log f'(t)=0\quad and \tag{1.16}$$

$$\lim_{t\to 0} t\alpha(t)\log\beta(t)=0\,. \tag{1.17}$$

It follows from (1.13) and (1.14) that $a(t,x,\xi)>0$ $(t\neq 0)$ and hence we see, together with (1.12),

$$\Gamma\subset\{\tau=t=0\}$$

Consequently we have (1.2) (and hence (1.7)). The operator of the form (1.11) is infinitely degenerate model corresponding to the case of $m=0$ in the operator $P_0$ of Egorov type stated in the introduction. We do not know the microhypoellipticity for a simple operator with $f(t)$ in (1.11) replaced by $f(t)x_1^2$, because of the difficulty in deriving $L^2$ *a priori estimate*.†

However, if $a(t,x,\xi)$ in (1.12) does not vanish we can treat infinitely degenerate model of $P_0$ a little more generally. This case is geometrically stated as

(1.18) $H_1$, $H_2$ and the radial direction are linearly independent in $V\cap\mathrm{Char}\,P$,

which is invariant condition under the multiplication of elliptic factors. If (1.18) is valid then it follows from condition $(\overline{\Psi})$ that we have the maximal hypoelliptic estimate, in a sense of Helffer-Nourrigat [3], as follows;

$$\|D_t u\|^2+\|Q_\lambda(t,x,D_x)u\|^2\leqq C(\|P_\lambda u\|^2+\|u\|^2)\quad (\text{cf., } (4.9)).$$

By means of this estimate, the problem of hypoellipticity for $D_t+iQ$ can be reduced to the similar one for second operator $D_t^2+Q^2$ as in [13]. From now

† Some special cases will be studied in the forthcoming paper [17].

on we shall consider the case corresponding to [13; Theorem 4]. Let $\Gamma$ be a $C^\infty$ submanifold of codimension 2 in Char $P$ and symplectic, that is,

$$T\Gamma \cap T\Gamma^{\perp}=0 \text{ at every point of } \Gamma. \tag{1.19}$$

It follows from (1.19) that both $H_1$ and $H_2$ are transversal to $\Gamma$ because $H_1, H_2 \in T(\text{Char } P)^{\perp} \subset T\Gamma^{\perp}$. Hence (1.2) holds and so (1.7). In order to state the additional condition we have to fix a special coordinate. By a symplectic linear transformation, it follows from (1.18) that $q(t, x, \xi)$ of (1.3) satisfies $\partial_{\xi_1} q(t, x, \xi) \neq 0$. It follows from the implicit function theorem that there exist $a(t, x, \xi) \in C^\infty(\boldsymbol{R}_t; S^0_{1,0})$ and $b(t, x, \xi') \in C(\boldsymbol{R}_t; S^1_{1,0})$ such that

$$q(t, x, \xi)=a(t, x, \xi)(\xi_1+b(t, x, \xi')), \quad a(t, x, \xi) \neq 0, \quad \text{in } V, \tag{1.20}$$

where $\xi'=(\xi_2, \cdots, \xi_n)$. Let $\gamma_\rho(t)$ be the bicharacteristic of $p_1=\tau$ through $\rho=(0, x, 0, \xi_1, \xi') \in V \cap \text{Char } P$, that is, $\gamma_\rho(t)=\{(t, x, 0, \xi_1, \xi')\}$. By setting $\xi_1=-b(t, x, \xi')$ we define a projection $\pi\gamma_\rho(t)$ into Char $P$ of the bicharacteristic. We assume that

$$\left\{\begin{array}{l}\text{there exist a } \delta_0>0 \text{ and a } 0 \neq e(t, x, \xi) \in C^\infty(\boldsymbol{R}_t; S^0_{1,0}) \\ \text{such that for any } \rho=(0, x, 0, \xi) \in V \cap \text{Char } P,\ F_\rho(t) \equiv \\ (e\partial_t q)|_{\text{Char } P}(\pi\gamma_\rho(t)) \text{ has a unique extremum at } t=s(\rho) \\ \text{in } (-\delta_0, \delta_0), \text{ and } s(\rho) \text{ belongs to } C^\infty \text{ with respect to } \rho.\end{array}\right. \tag{1.21}$$

THEOREM 3. *Let $\rho_0$ be $(0, (0, \xi_0)) \in \Gamma$ and let $P$ be the form* (1.3)′ *and satisfy* (1.18) *in a conic neighborhood $V$ of $\rho_0$. Assume that $\Gamma$ is a $C^\infty$-symplectic submanifold and of codimension* 2 *in* Char $P$. *Then we have* (1.1) *if the condition* (1.21) *holds with $q(t, x, \xi)$ expressed as* (1.20).

As a typical example of Theorem 3 we have the following:

$$p(t, x, \tau, \xi)=\tau+i\left\{\xi_1+\int_0^t \exp-(s^2+x_1^2)^{-\delta/2} ds|\xi|\right\}, \quad \delta>0.$$

## 2. Proof of Theorem 1

Let $\chi(t)$ be a $C_0^\infty(\boldsymbol{R}_t)$ function such that $0 \leqq \chi(t) \leqq 1$, $\chi(t)=1$ for $|t| \leqq 1$, $\chi(t)=0$ for $|t| \geqq 2$. Set $\Phi(\tau, \xi; \mu)=\chi(|\tau|/\mu|\xi|)(1-\chi)(|\xi|)$ for a small $\mu>0$. For cutting $\boldsymbol{R}^n_\xi$ we define the following:

DEFINITION 1. For $\delta>0$ and $\xi_0 \in \boldsymbol{R}^n$ ($|\xi_0|=1$) we say that a function $\psi(\xi) \in C^\infty(\boldsymbol{R}^n)$ belongs to $\Psi_{\delta, \xi_0}$ if $0 \leqq \psi \leqq 1$ satisfies

$$\begin{cases} \psi(\xi)=1 & \text{for } |\xi/|\xi|-\xi_0|\leqq\delta/12 \text{ and } |\xi|\geqq 2/3, \\ \psi(\xi)=0 & \text{for } |\xi/|\xi|-\xi_0|\geqq\delta/10 \text{ or } |\xi|\leqq 1/2, \\ \psi(\xi)=\psi(\xi/\lambda) & \text{for } 0<\lambda\leqq 1 \text{ and } |\xi|\geqq 1. \end{cases}$$

In the proof of the theorem we may assume $u\in\mathcal{E}'$ and hence $u$ belongs to $H_{-N}$ for an integer $N>0$. Suppose that $\rho_0\notin\mathrm{WF}(Pu)$. Then for a sufficiently small $\mu>0$ we have

$$\chi(t/2\mu)\Phi(D_t, D_x; 2\mu)\psi_\mu(D_x)h_\mu(x)Pu\in H_s$$

for any real $s$, where $\psi_\mu(\xi)\in\Psi_{\mu,\xi_0}$. If we set $v=\chi(t/\mu)\Phi(D_t, D_x; \mu)u$ then it follows from (1.7) that $\psi_{\delta_\mu}(D_x)h_{\delta_\mu}(x)Pv\in H_s$ for a $\psi_{\delta_\mu}(\xi)\in\Psi_{\delta_\mu,\xi_0}$ because $P$ is microhypoelliptic on the intersection of $\mathrm{supp}\, h_{\delta_\mu}(x)\psi_{\delta_\mu}(\xi)$ and the support of derivatives of $\chi(t/\mu)\Phi(\tau, \xi; \mu)$. Fix a positive $\delta$ such that $100\delta<\min(\delta_\mu, \delta_1)$. We shall show $\psi_\delta(D_x)h_\delta(x)v\in H_s$, which will yield (1.1).

For the above $\delta$ we take $\varphi(x, \xi; \lambda)$ in the assumption of the theorem. For an integer $l>s+N+1$ we denote a pseudodifferential operator with a symbol $\lambda^{l\varphi(x,\xi;\lambda)}$ by $K(x, D_x; \lambda)$. If $\lambda$ varies $0<\lambda\leqq 1$ then $K(x, D_x; \lambda)H_{10\delta}(x, D_x; \lambda)$ belongs to a bounded set of $S^0_{1,\varepsilon_1}$ for any small $\varepsilon_1>0$. For any real $a$, $[K(x, D_x; \lambda), H_{10\delta}(x, D_x; \lambda)]\lambda^{-a}$ belongs to a bounded set of $S^{-(l-a-\varepsilon_1)}_{1,0}$ because of (1.8). Furthermore, $h_{10\delta}(x)$, $h_{10\delta}(\lambda D_x-\xi_0)$ and $K(x, D_x; \lambda)$ are commutative, each other, as a product of three factors, neglecting term in $\lambda^a\times S^{-(l-a-\varepsilon_1)}_{1,0}$.

Let $w\in\mathcal{S}$ satisfy

$$\mathrm{supp}\, w\subset\{|t|\leqq 2\mu\} \tag{2.1}$$

and substitute $K(x, D_x; \lambda)H_{10\delta}(x, D_x; \lambda)w$ into (1.10) in place of $u$. Then

$$\begin{aligned} &\|KH_{10\delta}w\|^2+(\log\lambda)^2\|\alpha(t, x, D_x)KH_{10\delta}w\|^2 \\ &\leqq 2\varepsilon\{\|h_{10\delta}(\lambda D_x-\xi_0)h_{10\delta}(x)KPw\|^2+\|H_{10\delta}[Q(t, x, D_x), K]w\|^2\} \\ &\quad+C_\varepsilon(\lambda\|KH_{10\delta}w\|^2+\lambda^{2s+1}\|w\|^2_{(0,-N)}) \end{aligned} \tag{2.2}$$

because the same commutative argument as above follows for $H_{10\delta}$ and $KQ$ by means of (1.8). Here for real $a$ we have set $\|w\|_{(0,a)}=\|(1+\Lambda)^a w\|$, $\Lambda^2=1+|D_x|^2$ and by this norm we define the space $H_{(0,a)}$. Note that the principal symbol of $[Q, K]$ is equal to

$$-il(\log\lambda)(H_q\varphi)\lambda^{l\varphi(x,\xi;\lambda)}$$

and symbols of lower orders are a sum of $\lambda^{1/2+l\varphi(x,\xi;\lambda)}$ multiplied by symbols in a bounded set of $S^0_{1,0}$ uniformly with respect to $0<\lambda\leqq 1$. It follows from (1.9) that

$$\|H_{10\delta}[Q(t,x,D_x),K]w\|^2 \leqq l^2(\log\lambda)^2\|\alpha(t,x,D_x)KH_{10\delta}w\|^2 + C_l(\lambda\|KH_{10\delta}w\|^2+\lambda^{2s+1}\|w\|^2_{(0,-N)}). \tag{2.3}$$

Choose $2\varepsilon l^2<1$, then for a constant $C_l'$ we have

$$(1-2\varepsilon\lambda C_l)\|KH_{10\delta}w\|^2$$
$$\leqq 2\varepsilon\|h_{10\delta}(\lambda D_x-\xi_0)h_{10\delta}(x)KPw\|^2+C_l'\lambda^{2s+1}\|w\|^2_{(0,-N)}.$$

It follows from (1.8) that $\|h_\delta(\lambda D_x-\xi_0)h_\delta(x)w\|^2\leqq\|KH_{10\delta}w\|^2+\tilde{C}_l\lambda^{2s+1}\|w\|^2_{(0,-N)}$. Take a $\lambda_0$ satisfying $\lambda_0(2\varepsilon C_l+C_\varepsilon)<1/4$. Then for $0<\lambda\leqq\lambda_0$ we have

$$\|h_\delta(\lambda D_x-\xi_0)h_\delta(x)w\|^2$$
$$\leqq 4\varepsilon\{\|h_{10\delta}(\lambda D_x-\xi_0)h_{10\delta}(x)Pw\|^2+C_l''\lambda^{2s+1}\|w\|^2_{(0,-N)}).$$

Multiplying $\lambda^{-2s}(1+\kappa\lambda^{-1})^{-2(l+1)}$ with a parameter $\kappa>0$ by both sides, for $0<\lambda\leqq\lambda_0$ we have

$$\|h_\delta(\lambda D_x-\xi_0)(1+\kappa\Lambda)^{-(l+1)}h_\delta(x)w\|_s^2$$
$$\leqq 4\varepsilon(\|h_{10\delta}(\lambda D_x-\xi_0)(1+\kappa\Lambda)^{-(l+1)}h_{10}(x)Pw\|_s^2+C_l''\lambda\|w\|^2_{(0,-N)})$$

because $\lambda^{-1}$ is equivalent to $|\xi|$ on supp $h(\lambda\xi-\xi_0)$. Integrate $\lambda$ from 0 to $\lambda_0$ after dividing both sides by $\lambda$. Then by means of [12; Proposition 1.7] we have for suitable $\phi_\delta(\xi)\in\Psi_{\delta,\xi_0}$ and $\tilde{\phi}_\delta(\xi)\in\Psi_{70\delta,\xi_0}$,

$$\|(1+\kappa\Lambda)^{-(l+1)}\phi_\delta(D_x)h_\delta(x)w\|^2_{(0,s)}$$
$$\leqq C(\|(1+\kappa\Lambda)^{-(l+1)}\tilde{\phi}_\delta(D_x)h_{10\delta}(x)Pw\|^2_{(0,s)}+\|w\|^2_{(0,-N)}).$$

It follows from $u\in H_{-N}$ that one can find a sequence $\{u_j\}$ in $\mathcal{S}$ satisfying $u_j\to u\in H_{-N}$. If $w_j=\chi(t/\mu)\Phi(D_t,D_x;\mu)u_j$ then $w_j\to v$ in $H_{(0,-N)}$ and $Pw_j\to Pv$ in $H_{(0,-(N+1))}$. Letting $j\to\infty$ in the above estimate with $w=w_j$, in view of $Pv\in H_s$ we get for $\kappa>0$

$$\|(1+\kappa\Lambda)^{-(l+1)}\phi_\delta(D_x)h_\delta(x)v\|^2_{(0,s)}\leqq C(\|\tilde{\phi}_\delta(D_x)h_{10\delta}(x)Pv\|_s^2+\|u\|^2_{-N}).$$

Making $\kappa\to0$ we see $\phi_\delta(D_x)h_\delta(x)v\in H_s$ because $v=\chi(t/\mu)\Phi(D_t,D_x;\mu)u$. Thus we have proved that $Pu\in H_s$ at $\rho_0$ implies $u\in H_s$ at $\rho_0$.

The proof of Corollary is obvious if we replace the term $\|\alpha(t,x,D_x)KH_{10\delta}w\|^2$ in (2.2) and (2.3) by $\mathrm{Re}\,(\alpha(t,x,D_x)KH_{10\delta}w, KH_{10\delta}w)$.

### 3. Proof of Theorem 2

If $\xi_0\notin\Sigma=\{\xi_1+f(0)|\xi|=0\}$ then the theorem is obvious because $q(t,x,\xi)=a(\xi_1+f(t)|\xi|)$ is semi-definite in a small conic neighborhood of $\rho_0=(0,(0,\xi_0))$ and we can apply the result about the propagation of regularities (Hörmander

[6; Proposition 26.6.1]). In what follows we assume $\xi_0 \in \Sigma$ (though we will not use this condition). We apply Theorem 1 by setting

$$\varphi(x, \xi)=(1-h_{5\delta}(x))+(1-h_{5\delta}(\lambda\xi-\xi_0)).$$

Then we have (1.8) and it follows from (1.14) and (1.15) that (1.9) holds with $\alpha(t, x, \xi)=C\alpha(t)$ for a suitable $C>0$ if $\delta$ is small enough. The proof of Theorem 2 would be completed if we could show (1.10). Set

$$a_\lambda(t, x, \xi)=h_{\delta_1}(x)a(t, x, D_x)h_{2\delta_1}(\lambda D_x-\xi_0).$$

Let $A(t)=a_\lambda^w(t, x, D_x)$ denote a pseudodifferential operator with Weyl symbol $a_\lambda(t, x, \xi)$. Setting $B(t)=(D_{x_1}+f(t)|D_x|)h_{\delta_1}(\lambda D_x-\xi_0)$ moreover, we consider $A(t)$, $B(t)$ as a real operator on Hilbert space $\mathcal{H}=L^2(\boldsymbol{R}_x^n)$. Note that for a fixed $\lambda>0$ $B(t)$ is bounded operator in $\mathcal{H}$. If $\Omega_+(t)=\{\xi; \xi_1+f(t)|\xi|>0\}$ and if

$$S_+(t)v(x)=\frac{1}{(2\pi)^n}\int e^{ix\xi}1_{\Omega_+(t)}(\xi)\hat{v}(\xi)d\xi, \quad v\in\mathcal{H}, \quad S_-(t)=Id-S_+(t)$$

then we can define the sign $M(t)$ of $B(t)$, $M(t)=S_+(t)-S_-(t)$ and it follows from (1.12) that

$$(M(t_1)-M(t_2))(t_1-t_2)\geqq 0 \quad \text{on } \mathcal{H}. \tag{3.1}$$

From this condition we have the following lemma given by Lerner [8; §2]:

LEMMA (Lerner [8, 9]). *There exists a $\delta'>0$ independent of $0<\lambda\leqq 1$ such that for any $u(t)\in C_0^1(\boldsymbol{R}_t; \mathcal{H})$ we have*

$$2\int|P_\lambda u(t)|_{\mathcal{H}}dt\geqq\sup|u(t)|_{\mathcal{H}} \quad \textit{if } \operatorname{supp} u\subset\{|t|\leqq\delta'\}, \tag{3.2}$$

*where* $|\cdot|_{\mathcal{H}}=\|\cdot\|_{L^2(\boldsymbol{R}_x^n)}$.

PROOF. By means of [8; Lemma 2.3.1], it follows from (3.1) that

$$\operatorname{Re}\int(\dot{u}(t), M(t)u(t))_{\mathcal{H}}dt\leqq 0, \quad \dot{u}(t)=\frac{du}{dt}(t). \tag{3.3}$$

If $H(t)$ denotes Heaviside function then for any $T$ we have

$$\begin{aligned}&-\operatorname{Re}\int(\dot{u}(t), \{H(t-T)S_+(t)-H(T-t)S_-(t)\}u(t))_{\mathcal{H}}dt\\ &=-\operatorname{Re}\int(\dot{u}(t), H(t-T)(M+S_-)u(t)+H(T-t)(M-S_+)u(t))_{\mathcal{H}}dt \qquad (3.4)\\ &\geqq-\operatorname{Re}\int(\dot{u}(t), \{H(t-T)S_-(t)-H(T-t)S_+(t)\}u(t))_{\mathcal{H}}dt,\end{aligned}$$

where we have used (3.3) in the last inequality. Adding the left hand side of

(3.4) to both sides of (3.4), we have in view of $S_+ + S_- = Id$

$$\begin{aligned} &-2\mathrm{Re}\int(\dot{u}(t),\ \{H(t-T)S_+(t)-H(T-t)S_-(t)\}\,u(t))_{\mathcal{H}}\,dt \\ (3.5)\qquad &\geqq -\mathrm{Re}\int(\dot{u}(t),\ \{H(t-T)-H(T-t)\}\,u(t))_{\mathcal{H}}\,dt \\ &=2|u(T)|^2_{\mathcal{H}}. \end{aligned}$$

It follows from [8; Lemma 2.3.2] that

$$(3.6)\qquad \mathrm{Re}\,(\pm S_\pm\,\mathrm{Re}\,(AB))\geqq -\frac{10}{3}\|A\|^{1/4}\|[A,\ B]\|^{1/2}\|[B,\ [B,\ A]]\|^{1/4},$$

where $\|A\|$ denotes the operator norm of $A(t)$ in $\mathcal{H}$. Note that the right hand side of (3.6) has the bound independent of $\lambda$. Since the difference between $P_\lambda$ and $D_t+i\,\mathrm{Re}\,(A(t)B(t))$ is bounded in $\mathcal{H}$ uniformly with respect to $0<\lambda\leqq 1$, in view of (3.5) and (3.6) there exists a $C>0$ independent of $\lambda$ such that

$$\begin{aligned} &\mathrm{Re}\int(P_\lambda u(t),\ i\{H(t-T)S_+(t)-H(T-t)S_-(t)\}\,u(t))_{\mathcal{H}}\,dt \\ &\qquad\geqq |u(T)|^2_{\mathcal{H}}-C\int|u(t)|^2_{\mathcal{H}}\,dt\,. \end{aligned}$$

If $\mathrm{supp}\,u\subset\{|t|\leqq\delta'\}$ then the second term of the right hand side is estimated above from $2C\delta'\sup|u(t)|^2_{\mathcal{H}}$, so that we have (3.2) for a small $\delta'>0$ satisfying $4C\delta'\leqq 1$.

By means of the Schwartz inequality it follows from (3.2) that

$$(3.7)\qquad \|P_\lambda u\|\geqq(2\delta')^{-1}\|u\| \quad \text{if } \mathrm{supp}\,u\subset\{|t|\leqq\delta'\}.$$

It follows from (1.16) and (1.17) that for any $\varepsilon>0$ there exists a $\delta_\varepsilon>0$ such that

$$(3.8)\qquad t\alpha(t)\{|\log f'(t)|+|\log\beta(t)|\}\leqq\varepsilon^2 \quad \text{if } |t|\leqq\delta_\varepsilon\,.$$

For the sake of simplicity we assume $\alpha(t)$ is even function (the general case would be clear once we could prove this case), It follows from the monotoness of $\alpha(t)$ that for a small parameter $\lambda>0$ there exists a unique $t_\lambda>0$ such that $t_\lambda\alpha(t_\lambda)|\log\lambda|=2\varepsilon$. Similarly we choose $s_\lambda>0$ such that $s_\lambda\alpha(s_\lambda)|\log\lambda|=\varepsilon$. For a while we assume $\lambda$ is sufficently small such that $s_\lambda<\delta_\varepsilon$. If we set $\delta'=t_\lambda$ in (3.7) then

$$\begin{aligned} (3.9)\qquad 4\varepsilon\|P_\lambda u\| &\geqq \|\alpha(t_\lambda)(\log\lambda)u\| \\ &\geqq \|\alpha(t)(\log\lambda)u\| \qquad \text{if } \mathrm{supp}\,u\subset\{|t|\leqq t_\lambda\}. \end{aligned}$$

If $s_\lambda\leqq|t|\leqq\delta_\varepsilon$ then it follows from (3.8) that

$$\frac{\varepsilon}{|\log\lambda|}\{|\log f'(t)|+|\log\beta(t)|\}\leqq\varepsilon^2,$$

so that if $0<\lambda\leqq\lambda_\varepsilon$ for a sufficiently small $\lambda_\varepsilon$ then

$$f'(t),\ \beta(t)\geqq\lambda^\varepsilon \quad \text{on } s_\lambda<|t|\leqq\delta_1. \tag{3.10}$$

In fact, if $\lambda_\varepsilon$ is small enough we have $f'(t)$, $\beta(t)\geqq(\lambda_\varepsilon)^\varepsilon$ for $\delta_\varepsilon<|t|\leqq\delta_1$ in view of (1.12) and (1.13). Note that

$$\begin{aligned}\|P_\lambda u\|^2=&\|D_t u\|^2+\|a_\lambda(t,x,D_x)Bu\|^2\\ &+2\operatorname{Re}((\partial_t a_\lambda)Bu,u)\\ &+2\operatorname{Re}(a_\lambda f'(t)|D_x|h_{\delta_1}(\lambda D_x-\xi_0)u,u)\end{aligned} \tag{3.11}$$

Since it follows from (3.10) and (1.14) that

$$a_\lambda(t,x,\xi)\geqq\lambda^\varepsilon \quad \text{on } \{s_\lambda\leqq|t|\leqq\delta_1\}\times\operatorname{supp}H_{\delta_1}(x,\xi;\lambda)$$

the second term of the right hand side of (3.11) is estimated above from

$$C(\lambda^{-\varepsilon}\|a_\lambda Bu\|\|u\|+\|u\|^2)\leqq\|a_\lambda Bu\|^2+C'\lambda^{-2\varepsilon}\|u\|^2.$$

By means of (3.10) again we have, if $\operatorname{supp}u\subset\{s_\lambda\leqq|t|\leqq\delta_1\}$,

$$2\operatorname{Re}(a_\lambda f'(t)|D_x|h_{\delta_1}(\lambda D_x-\xi_0)u,u)\geqq\lambda^{2\varepsilon-1}\|H_{20\delta}u\|^2-C\|u\|^2.$$

Therefore, if $\operatorname{supp}u\subset\{s_\lambda\leqq|t|\leqq\delta_1\}$ then

$$\|P_\lambda u\|^2\geqq\lambda^{2\varepsilon-1}\|H_{20\delta}u\|^2-C\lambda^{-2\varepsilon}\|u\|^2,$$

provided that $0<\lambda\leqq\lambda_\varepsilon$. If $\varepsilon<1/16$ and if $0<\lambda\leqq\min(\lambda_\varepsilon,\varepsilon^2)=\lambda'_\varepsilon$ we have

$$\begin{aligned}\varepsilon\|P_\lambda u\|^2\geqq\lambda^{2\varepsilon-1/2}\|u\|^2-C\lambda^{-2}\|(1-H_{20\delta})u\|^2\\ \text{if } \operatorname{supp}u\subset\{s_\lambda\leqq|t|\leqq\delta_1\}.\end{aligned} \tag{3.12}$$

Let $\chi_0(t)$ be $C^\infty(\boldsymbol{R})$ such that $\chi_0(t)=1$ for $t\leqq0$ and $\chi_0(t)=0$ for $t\geqq1$. Set $\psi_\pm(t)=\chi_0(\pm(t\pm s_\lambda)/(s_\lambda-t_\lambda))$ and $\psi(t)=\psi_+(t)\psi_-(t)$. The fact that $t_\lambda-s_\lambda\geqq c\varepsilon/|\log\lambda|$ for a suitable $c>0$ shows $|\psi^{(j)}(t)|\leqq C_\varepsilon|\log\lambda|^j$ $(j=1,2,\cdots,)$. It follows from (3.12) that

$$\begin{aligned}&\|[P_\lambda,\psi]u\|^2=\|\psi' u\|^2\\ &\leqq C\lambda^{1/2-2\varepsilon}\{\|[P_\lambda,\psi']u\|^2+|\log\lambda|^2(\|P_\lambda u\|^2+\lambda^{-2}\|(1-H_{20\delta})u\|^2)\}.\end{aligned} \tag{3.13}$$

Since similar estimates hold with $\psi$ replaced by $\psi^{(j)}|\log\lambda|^{-j}$, $j=1,2,\cdots$, in view of $u=\psi(t)u+(1-\psi(t))u$, it follows from (3.9) and (3.12) that

$$16\varepsilon\|P_\lambda u\|^2\geqq\|\alpha(t)(\log\lambda)u\|^2-C\lambda^{-2}\|(1-H_{20\delta})u\|^2. \tag{3.14}$$

if $0<\lambda\leqq\lambda'_\varepsilon$. From (3.14), (3.7) and (3.12) we have the desired estimate (1.10) because it is trivial for $\lambda'_\varepsilon<\lambda\leqq 1$ by taking a sufficiently large $C_\varepsilon$ in the right hand side.

## 4. Proof of Theorem 3

Since $\rho_0=(0,(0,\xi_0))\in\Gamma$ it follows from (1.20) that $\xi_{01}+b(0,0,\xi'_0)=0$. By taking the canonical transformation such that $\xi_1+b(0,0,\xi')\to\xi_1$ and $\xi'\to\xi'$ we may assume that $\xi_0=(0,\xi'_0)$, $|\xi'_0|=1$. Because $\Gamma$ is of codimension 2 in Char $P$ it follows from (1.20) and (1.6) that $\partial_t b(t,x,\xi')$ has the definite sign. Note that

$$(4.1)\qquad (a^{-1}\partial_t q)|_{\mathrm{Char}\,P}(\pi\gamma_\rho(t))=\partial_t b(t,x,\xi').$$

For each $\rho=(0,x,(0,-b(t,x,\xi'),\xi'))\in\mathrm{Char}\,P\cap V$, let $t(x,\xi')$ denote the extremal point in the condition (1.21). Since it follows from (4.1) that $F_\rho(t)$ in (1.21) equals $(\tilde{e}\partial_t b)(t,x,\xi')$ for some $\tilde{e}(t,x,\xi')\in C^\infty(\boldsymbol{R}_t\times\boldsymbol{R}_{x_1};S^0_{1,0})$, we have in a conic neighborhood of $\rho_0$

$$(4.2)\qquad \begin{aligned}&|(\tilde{e}\partial_t b)(t(x,\xi'),x,\xi')|<|(\tilde{e}\partial_t b)(s,x,\xi')|\leqq|(\tilde{e}\partial_t b)(t,x,\xi')|\\ &\qquad\qquad\text{if } 0<|s-t(x,\xi')|<|t-t(x,\xi')|.\end{aligned}$$

Set $\tilde{b}(t,x,\xi')=\int_{t(x,\xi')}^t\partial_t b(s,x,\xi')ds$ and take the canonical transformation in $T^*(\boldsymbol{R}^n_x)$, keeping $x_1$ variable, such that

$$\xi_1+b(t(x,\xi'),x,\xi')\longrightarrow\xi_1\qquad(\text{and } (0,\xi'_0)\to(0,\xi'_0)).$$

Then $\xi_1+b(t,x,\xi')$ is transformed to $\xi_1+b_0(t,x,\xi')$ of the form:

(4.3) $\quad b_0(t,x,\xi')=\tilde{b}(t,x_1,\Phi(x,\xi'),\Psi(x,\xi'))$ in a small conic neiborhood of $\rho_0$,

where $\Phi(x,\xi')\in S^0_{10}$, $\Psi(x,\xi')\in S^1_{1,0}$. It follows from (4.2) that

$$(4.4)\qquad |\nabla_x b_0(t,x,\xi')|+|\nabla_{\xi'}b_0(t,x,\xi')||\xi|\leqq C|\partial_t b_0(t,x,\xi')|.$$

In fact, for example, the direct calculation gives

$$(4.5)\qquad \begin{aligned}|\partial_{x_2}b_0(t,x,\xi')|&\leqq C_1|\partial_t b(t(x_1,x',\xi'),x',\xi')|_{(x',\xi')=(\Phi(x,\xi'),\Psi(x,\xi'))}|\\ &\quad+C_2\left|\int_{t(x,\xi')}^t|\partial_t\partial_{x_2}b(s,x_1,x',\xi')|_{(x',\xi')=(\Phi(x,\xi'),\Psi(x,\xi'))}|ds\right|.\end{aligned}$$

By means of (4,2), the first term of the right hand side is estimated above from $C|\partial_t b_0(t,x,\xi')|$. Because $\partial_t b$ is semi-definite we have $|\partial_t\partial_{x_2}b|\leqq C|\partial_t b|^{1/2}$ and the second term is estimated above from

$$C\left|\int_{t(x,\xi')}^t|(\tilde{e}\partial_t b)(s,x,\xi')|^{1/2}ds\right|\leqq C'|\partial_t b(t,x,\xi')|^{1/2}$$

with $(x',\xi')=(\Phi(x,\xi'),\Psi(x,\xi'))$. Here we have used (4.2) in the last inequality.

As stated in the section 1, it follows from (1.19) that Hamilton vector fields $H_1=\partial_t$ and $H_2=H_q$ are transversal to $\Gamma$. In view of (4.4), the fact that $\partial_t b_0(0, 0, \xi_0')=0$ shows that

$$(4.6)\quad \begin{cases} \text{for any small } \mu>0 \text{ there exists a } \delta_\mu>0 \text{ such that} \\ \left\{(t, x, 0, \xi);\ \mu\leqq\max(|t|, |x_1|)\leqq 2\mu,\ \ |x'|+\left|\frac{\xi}{|\xi|}-\xi_0\right|<\delta_\mu\right\}\cap\Gamma=\emptyset. \end{cases}$$

In the new variable we shall apply Corollary of Theorem 1, together with Remark 1. For the brevity we write $b$ instead of $b_0$ in what follows. Set $\varphi(x,\xi)=(1-\chi(x_1/\mu))+(1-h_{2\delta}(x'))+(1-h_{5\delta}(\lambda\xi-\xi_0))$. Choosing $\nu=2\mu/\delta$ in Remark 1 of Corollary we have (1.8). Since $H_q\varphi=a(\partial_{x_1}\varphi+H_b\varphi)+(H_a\varphi)(\xi_1+b)$, in view of $a\neq 0$ it follows from (4.4) and (1.20) that

$$(4.7)\quad \begin{aligned} |H_q\varphi|^2\leqq C((\chi'(x_1/\mu))^2+a\partial_t b/|\xi|+(q/|\xi|)^2) \\ \text{on } \{|t|\leqq\delta_1\}\times\operatorname{supp} H_{100\delta}(x, \xi;\lambda) \end{aligned}$$

because the second term of the right hand side is non-negative by means of (4.1) and (1.6). Putting $\alpha(t, x, \xi)$ equal to the right hand side of (4.7), we shall check (1.10)'. It follows from (4.6) that

$$(4.8)\quad \lambda^{-1}\|\chi'(x_1/\mu)H_{20\delta}u\|^2\leqq C(\|P_\lambda u\|^2+\|u\|^2).$$

Setting $Q_\lambda(t, x, \xi)=Q(t, x, \xi)H_{\delta_1}(x, \xi;\lambda)$ we have

$$\|P_\lambda u\|^2=\|D_t u\|^2+\|Q_\lambda(t, x, D_x)u\|^2+2\operatorname{Re}(Op(\partial_t Q_\lambda(t, x, \xi))u, u),$$

where $Op(r)$ denotes the pseudodifferential operator with symbol $r$. Since the principal symbol of $\partial_t Q_\lambda(t, x, \xi)$ equals $(a\partial_t b+(\partial_t a/a)q)H_{\delta_1}(x, \xi;\lambda)$ it follows from the Schwartz inequality

$$(4.9)\quad \begin{aligned} \|P_\lambda u\|^2 &\geqq \|D_t u\|^2+\|Q_\lambda(t, x, D_x)u\|^2/2 \\ &\quad +2\operatorname{Re}(Op(a\partial_t bH_{\delta_1})u, u)-C\|u\|^2 \\ &\geqq \|D_t u\|^2+\|Q_\lambda(t, x, D_x)u\|^2/2-C'\|u\|^2. \end{aligned}$$

Noting that $(a\partial_t b/|\xi|+(q/|\xi|)^2)H_{20\delta}^2\leqq a\partial_t bH_{\delta_1}/\lambda+Q_\lambda^2/\lambda^2$, by means of the sharp Gårding inequality we have (1.10)' from (4.8) and (4.9), because it follows from the Poincaré inequality that the term $\|u\|^2$ is absorbed by $\|D_t u\|^2$ if $\delta_1$ is small enough.

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