

A NOTE ON SOME STRONG WHITNEY-REVERSIBLE PROPERTIES

By

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1. All spaces considered in this paper are assumed to be metric. A continuum means a compact connected space and a map means a continuous function. The letter X will always denote a continuum. Let $C(X)$ denote the hyperspace of all non-empty subcontinua of X with the Hausdorff metric (see [7]). Whitney [10] proved that for every continuum X there exists a map $\mu: C(X) \rightarrow [0, +\infty)$ satisfying

- (1) if $A, B \in C(X)$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and
- (2) $\mu(\{x\}) = 0$ for every $x \in X$.

We shall call any map from $C(X)$ to $[0, +\infty)$ satisfying the above conditions (1) and (2) a *Whitney map for $C(X)$* .

Nadler [7] introduced the concept of a strong Whitney-reversible property. Let P be a topological property. We say that P is a *strong Whitney-reversible property* provided whenever X is a continuum such that $\mu^{-1}(t)$ has the property P for some Whitney map μ for $C(X)$ and every $0 < t \leq \mu(X)$, then so does X . Moreover he has shown that some topological properties are strong Whitney-reversible properties. For example hereditary indecomposability and local connectedness are such properties.

We refer readers to see [1] and [7] for the shape theory and the hyperspace theory respectively if necessary.

2. We shall show that some topological properties are strong Whitney-reversible properties.

THEOREM 1. *Let μ be a Whitney map for $C(X)$. If there is a sequence $\{t_n\}$ $n \geq 1$ in $(0, \mu(X)]$ such that $t_n \rightarrow 0$ as $n \rightarrow +\infty$ and $\mu^{-1}(t_n)$ is an FAR for each $n=1, 2, 3, \dots$, then X is also an FAR.*

Hence the property of being an FAR is a strong Whitney-reversible property.

PROOF. Let M be an arbitrary ANR and $f: X \rightarrow M$ be an arbitrary map. Since M is an ANR and we can identify X with $\mu^{-1}(0) = \{\{x\} \mid x \in X\}$, there are

an open neighborhood U of X and a map $\tilde{f}: U \rightarrow M$ such that $\tilde{f}|_X = f$. Then there is an integer $n \geq 1$ such that $\mu^{-1}([0, t_n]) \subset U$. Since $\mu^{-1}(t_n)$ is an FAR, $\tilde{f}|_{\mu^{-1}(t_n)} \simeq 0$, where 0 is a constant map. Hence there exists a map $g: \mu^{-1}([t_n, \mu(X)]) \rightarrow M$ such that $g|_{\mu^{-1}(t_n)} = \tilde{f}|_{\mu^{-1}(t_n)}$. Now we can define a map $h: C(X) \rightarrow M$ as the following formula;

$$\begin{aligned} h|_{\mu^{-1}([0, t_n])} &= \tilde{f}|_{\mu^{-1}([0, t_n])} \quad \text{and} \\ h|_{\mu^{-1}([t_n, \mu(X)])} &= g. \end{aligned}$$

Since $C(X)$ is an FAR (see [3]), $h \simeq 0$. Hence $f = h|_X \simeq 0$. Therefore X is an FAR.

REMARK 1. By the example of Petrus [8] the converse of Theorem 1 is false.

REMARK 2. By the proof of Theorem 1 the property of being acyclic is a strong Whitney-reversible property. But it is not Whitney property (see [5]) by the same example of Petrus [8].

THEOREM 2. Let μ be a Whitney map for $C(X)$. Let \mathfrak{B} be a class of compact connected polyhedra. If there is a sequence $\{t_n\}$ $n \geq 1$ in $(0, \mu(X)]$ such that $t_n \rightarrow 0$ as $n \rightarrow +\infty$ and $\mu^{-1}(t_n)$ is an hereditarily indecomposable \mathfrak{B} -like continuum (see [6]) for each $n=1, 2, 3, \dots$, then X is also an hereditarily indecomposable \mathfrak{B} -like continuum.

Hence the property of being an hereditarily indecomposable \mathfrak{B} -like continuum is a strong Whitney-reversible property.

PROOF. By [7] X is hereditarily indecomposable. Hence it is sufficient to show that X is \mathfrak{B} -like. Without loss of generality we may assume that the sequence $\{t_n\}$ $n \geq 1$ is decreasing. Now for each $n=1, 2, 3, \dots$ we define a function $\eta_n: X \rightarrow \mu^{-1}(t_n)$ such that $x \in \eta_n(x) \in \mu^{-1}(t_n)$ for every $x \in X$. Since X is hereditarily indecomposable, for each $n=1, 2, 3, \dots$, η is well-defined and continuous (see [2]). Similarly for each $n=1, 2, 3, \dots$ we can define a map $p_n: \mu^{-1}(t_{n+1}) \rightarrow \mu^{-1}(t_n)$ such that $A \subset p_n(A)$ for each $A \in \mu^{-1}(t_{n+1})$. Then $\{\mu^{-1}(t_n), p_n\}$ is an inverse sequence of \mathfrak{B} -like continua and onto bonding maps. Moreover we hold that $p_n \eta_{n+1} = \eta_n$ for each $n=1, 2, 3, \dots$. Then it is clear that X is homeomorphic to the inverse limit $\varprojlim \{\mu^{-1}(t_n), p_n\}$. Therefore X is \mathfrak{B} -like.

In particular the converse of the result of Krasinkiewicz (4.2. [4]) is hold.

COROLLARY 1. Let μ be a Whitney map for $C(X)$. If there exists a sequence

$\{t_n\}$ $n \geq 1$ in $(0, \mu(X)]$ such that $t_n \rightarrow 0$ as $n \rightarrow +\infty$ and $\mu^{-1}(t_n)$ is an hereditarily indecomposable tree-like continuum for each $n=1, 2, 3, \dots$, then X is also an hereditarily indecomposable tree-like continuum.

The next lemma is useful for our results.

LEMMA (Krasinkiewicz and Nadler [5]). *Let μ be a Whitney map for $C(X)$. If X contains an n -odd ($n \geq 3$), there exists $t_0 > 0$ such that $\mu^{-1}(t_0)$ contains an $(n-1)$ -disk.*

THEOREM 3. *Let μ be a Whitney map for $C(X)$. If $\dim \mu^{-1}(t) \leq n < +\infty$ for every $t \in (0, \mu(X)]$ and one of the following conditions is satisfied, then $\dim X \leq n$:*

- (1) $\dim X < +\infty$,
- (2) $\mu^{-1}(t)$ is locally connected for every $t \in (0, \mu(X)]$,
- (3) $\mu^{-1}(t)$ is hereditarily indecomposable for every $t \in (0, \mu(X)]$.

PROOF. First we shall show the case (1). The following inequality is clearly hold.

$$\dim C(X) \leq 1 + \max \{ \dim \mu^{-1}(t) \mid t \in [0, \mu(X)] \} < +\infty.$$

Then by the result of Rogers [9] $\dim X \leq \dim \mu^{-1}(t)$ for some $t \in (0, \mu(X)]$. Hence $\dim X \leq n$.

Next we shall the case (2). Then X is locally connected by [7]. If $\dim X \geq 2$, for every $m \geq 3$ X contains an $(m+1)$ -odd. But by Lemma this fact contradicts the assumption. Hence $\dim X = 1$.

In the case (3) by the same way of the proof of Theorem 2 we can show that $\dim X \leq n$.

COROLLARY 2. *Let μ be a Whitney map for $C(X)$. If $\mu^{-1}(t)$ is locally connected and $\dim \mu^{-1}(t) \leq n < +\infty$ for every $t \in (0, \mu(X)]$, then X is a finite graph. In particular if $\dim \mu^{-1}(t) = 1$ for every $t \in (0, \mu(X)]$, X is an arc or a circle.*

PROOF. By the proof of Theorem 3 X is one-dimensional and locally connected. If X has infinitely many ramification points or a point with an infinite order, for every $m > 1$ X contains $(m+1)$ -odd. Then by Lemma $\dim \mu^{-1}(t) \geq n$ for some $t \in (0, \mu(X)]$. This contradicts our assumption. Hence X has at most finitely many ramification points and the order of each point of X is finite. Therefore X is a finite graph.

The following corollary is an easy consequence of Theorem 1 and Corollary 2.

COROLLARY 3. *Let μ be a Whitney map for $C(X)$. If $\mu^{-1}(t)$ is locally connected, $\dim \mu^{-1}(t) \leq n < +\infty$ and an FAR for every $t \in (0, \mu(X)]$, X is a tree. In particular if $\dim \mu^{-1}(t) = 1$ for every $t \in (0, \mu(X)]$, X is an arc.*

REMARK 3. Corollary 1 also can be proved by Theorem 1, Theorem 3 and the fact that hereditary indecomposability is a strong Whitney-reversible property.

REMARK 4. The author does not know whether the conditions of Theorem 3 are essential. But it seems not to be essential.

Related to Theorem 1 the following problem is open.

PROBLEM. *Is the property of being an FANR or a movable continuum a strong Whitney-reversible property?*

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