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ON MINIMAL SUBMANIFOLDS IN PRODUCT MANIFOLDS WITH A CERTAIN RIEMANNIAN METRIC

By

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Abstract. We generalize Ejiri's theorem about minimal submanifolds in warped product manifolds and see that there exist minimal immersions of the plane and the catenoid into other Riemannian manifolds.

§1. Introduction

Let (B,g_B) and (F,g_F) be Riemannian manifolds, and f a positive smooth function on B. The *warped product* manifold of (B,g_B) and (F,g_F) by the *warped function* f is defined to be a product manifold $B\times F$ provided with a Riemannian metric $g_B+f^2g_F$, and is denoted by $B\times_f F$. N. Ejiri proved the following theorem:

THEOREM A ([E]). *Let $(B,g_B),(F,g_F)$ and f be as above. Let M be an m-dimensional submanifold in B and N an n-dimensional submanifold in F. Then the product submanifold $M\times N$ in $B\times_f F$ is minimal if and only if both $M\hookrightarrow(B,f^{2n/m}g_B)$ and $N\hookrightarrow(F,g_F)$ are minimal submanifolds.*

For example, the catenoid, which is a minimal surface of revolution in $\boldsymbol{R}^3$, can be considered as a product submanifold in a warped product manifold of the *flat* upper half-plane $(\{y>0\}\subset\boldsymbol{R}^2,dx^2+dy^2)$ and the circle $(S^1,d\theta^2)$ of radius 1, whose warped function is $f(x,y)=y$. So Theorem A implies that the generating curve of the catenoid is a geodesic in the upper half-plane provided with a Riemannian metric $y^2(dx^2+dy^2)$. And it gives a reason why the catenoid is generated by the catenary which is a plane curve formed by a flexible inextensible cable of uniform density hanging from two support.

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In this paper, we deal with product immersions whose ambient manifold possesses a Riemannian structure belonging to $\mathscr{M}$. $\mathscr{M}$ is a set consisting of a certain kind of Riemannian structures of the ambient manifold, which contains warped product structures, and is defined in the following section. We define an equivalent relation in $\mathscr{M}$ and show that the minimality for such immersions is invariant in equivalent classes in $\mathscr{M}$. Especially when the metric in $\mathscr{M}$ is of more distinctive form, we give a necessary and sufficient condition for the minimality. The second result is a generalization of Theorem A. We also give some applications of the results.

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§2. Statement and Proof of Main Theorem

Throughout this paper, manifolds are assumed to be smooth and connected. At first we prepare some notations.

Let $(N_1,g_1),\cdots,(N_l,g_l)$ be Riemannian manifolds and $\dim N_\alpha = n_\alpha$. We put $N := N_1 \times \cdots \times N_l$, and denote by $\mathscr{M}'$ the set of all Riemannian metrics on N. A subset $\mathscr{M}$ of $\mathscr{M}'$ is defined to be

$$\mathscr{M} = \{g \in \mathscr{M}'; g = f_1^2 g_1 + \cdots + f_l^2 g_l\},$$

where $f_1,\cdots,f_l$ are positive smooth functions on N and $g_1,\cdots,g_l$ are considered as tensor fields on N.

$\mathscr{M}$ is bijectively corresponded to $C_+^\infty(N) \times \cdots \times C_+^\infty(N)$ (the set of l-tuples of positive smooth functions on N). Hence we often denote an element $g = \sum f_\alpha^2 g_\alpha$ by $(f_1,\cdots,f_l)$.

Let $d_1,\cdots,d_l$ be positive integers. We say that elements $g=(f_1,\cdots,f_l)$ and $\tilde{g}=(\tilde{f}_1,\cdots,\tilde{f}_l)$ in $\mathscr{M}$ are *$(d_1,\cdots,d_l)$-equivalent* if $f_1^{d_1}\cdots f_l^{d_l} = \tilde{f}_1^{d_1}\cdots \tilde{f}_l^{d_l}$ holds, and denote it by $g \sim_{(d_1,\cdots,d_l)} \tilde{g}$. The relation $\sim_{(d_1,\cdots,d_l)}$ is an equivalent relation in $\mathscr{M}$. We denote by $\mathscr{M}_{(d_1,\cdots,d_l)}$ the quotient set $\mathscr{M} / \sim_{(d_1,\cdots,d_l)}$.

Let $\varphi_\alpha : M_\alpha \to N_\alpha$ be an immersion of a d_α-dimensional manifold M_α into $N_\alpha \alpha = (1,\cdots,l)$. We denote by Φ the product immersion of $M = M_1 \times \cdots \times M_l$ into $N = N_1 \times \cdots \times N_l$. It is an easy observation that if $g \sim_{(d_1,\cdots,d_l)} \tilde{g}$ then the volume elements of $(M, \Phi^* g)$ and $(M, \Phi^* \tilde{g})$ are coincide.

That the minimality is equivalent to the stationariness of a variational problem about the volume (cf.[L]) gives an implication of the following.

THEOREM 2.1. *Assume that* $g \sim_{(d_1,\cdots,d_l)} \tilde{g}$. *Then* $\Phi : M \to (N,g)$ *is minimal if* *and only if* $\Phi: M \to (N, \tilde{g})$ *is minimal, i.e., the minimality for* Φ *depends only on elements in* $\mathscr{M}_{(d_1,\cdots,d_l)}$.

REMARK 2.2. Theorem 2.1 implies that it does not depend on representative elements of an equivalent class whether Φ is stationary with respect to the *first* variation of volume or not. However, such assertion does not hold for the *second* variation, that is, the stability property is not preserved in the equivalent class. We can see an example for this in §3.

THEOREM 2.3. *Let* F_α *be a positive smooth function on* $N_\alpha (\alpha = 1, \cdots, l)$. *Assume that* $g \sim_{(d_1,\cdots,d_l)} (F_1, \cdots, F_l)$ *where* F_α *is identified with a function on* N. *Then* $\Phi: M \to (N, g)$ *is minimal if and only if each* $\varphi_\alpha : M_\alpha \to (N_\alpha, F_\alpha^2 g_\alpha)$ *is minimal.*

REMARK 2.4. The assumption of Theorem 2.3 means the *separation of variables* of the function $f_1^{d_1} \cdots f_1^{d_1}$.

PROOF OF THEOREM 2.1 AND 2.3. We shall prove Theorem 2.1 and 2.3 by the moving frame method.

Let the convention on the ranges of indices be the following:

$$1 \le \alpha, \beta \le l, \quad 1 \le i_\alpha, j_\alpha \le n_\alpha .$$

Let $e_{(\alpha)1}, \cdots, e_{(\alpha)d_\alpha}, e_{(\alpha)d_\alpha+1}, \cdots, e_{(\alpha)n_\alpha}$ be a local orthonormal frame field of (N_α, g_α) adapted to φ_α, i.e., the restrictions of $e_{(\alpha)1}, \cdots, e_{(\alpha)d_\alpha}$ to M_α are tangent to M_α, and $\theta_{(\alpha)}{}^1, \cdots, \theta_{(\alpha)}{}^{n_\alpha}$ be the dual coframe field. We denote by $(\theta_{(\alpha)}{}^{i_\alpha}_{j_\alpha})$ the connection form of (N_α, g_α) with respect to $\theta_{(\alpha)}{}^1, \cdots, \theta_{(\alpha)}{}^{n_\alpha}$, i.e., $(n_\alpha \times n_\alpha)$ matrix-valued 1-form uniquely determined by the structure equations

$$d\theta_{(\alpha)}{}^{i_\alpha} = -\sum_{j_\alpha} \theta_{(\alpha)}{}^{i_\alpha}_{j_\alpha} \wedge \theta_{(\alpha)}{}^{j_\alpha},$$

$$\theta_{(\alpha)}{}^{i_\alpha}_{j_\alpha} + \theta_{(\alpha)}{}^{j_\alpha}_{i_\alpha} = 0.$$

$f_1\theta_{(1)}{}^1, \cdots, f_1\theta_{(1)}{}^{n_1}, \cdots\cdots, f_l\theta_{(l)}{}^1, \cdots, f_l\theta_{(l)}{}^{n_l}$ form an orthonormal coframe field of (N, g). Making use of the structure equations, we can compute that

$$d(f_\alpha \theta_{(\alpha)}{}^{i_\alpha}) = -\sum_{j_\alpha} \frac{1}{f_\alpha} \{(e_{(\alpha)j_\alpha} f_\alpha)\theta_{(\alpha)}{}^{i_\alpha} + f_\alpha \theta_{(\alpha)}{}^{i_\alpha}_{j_\alpha}\} \wedge f_\alpha \theta_{(\alpha)}{}^{j_\alpha}$$
$$- \sum_{\beta \ne \alpha} \sum_{j_\beta} \frac{1}{f_\beta} (e_{(\beta)j_\beta} f_\alpha) \theta_{(\alpha)}{}^{i_\alpha} \wedge f_\beta \theta_{(\beta)}{}^{j_\beta}.$$

Therefore if we put

$$\Theta^{i_\alpha}_{j_\alpha} = \frac{1}{f_\alpha}\{(e_{(\alpha)j_\alpha} f_\alpha)\theta_{(\alpha)}{}^{i_\alpha} - (e_{(\alpha)i_\alpha} f_\alpha)\theta_{(\alpha)}{}^{j_\alpha} + f_\alpha \theta_{(\alpha)}{}^{i_\alpha}_{j_\alpha}\},$$

$$\Theta^{i_\alpha}_{j_\beta} = \frac{1}{f_\beta}\{(e_{(\beta)j_\beta} f_\alpha)\theta_{(\alpha)}{}^{i_\alpha} - \frac{1}{f_\alpha}\}(e_{(\alpha)j_\alpha} f_\beta)\theta_{(\beta)}{}^{i_\beta}, \quad \text{if } \alpha \neq \beta,$$

then the following equations hold:

$$d(f_\alpha \theta_{(\alpha)}{}^{i_\alpha}) = -\sum_\beta \sum_{j_\beta} \Theta^{i_\alpha}_{j_\beta} \wedge f_\beta \theta_{(\beta)}{}^{j_\beta},$$

$$\Theta^{i_\alpha}_{j_\beta} + \Theta^{j_\beta}_{i_\alpha} = 0.$$

So $(\Theta^{i_\alpha}_{j_\beta})$ is the connection form of (N, g).

From now on we shall use the same notations of tensor fields on an ambient space and the restrictions of them to a submanifold, and use the following convention on ranges of indices:

$$1 \leq \alpha, \beta \leq l,$$

$$1 \leq i_\alpha, j_\alpha \leq d_\alpha, \quad d_{\alpha+1} \leq \tilde{i}_\alpha, \tilde{j}_\alpha \leq n_\alpha.$$

The mean curvature normal is the trace of the second fundamental form h divided by the dimension of the submanifold. The minimality is equivalent to that tr$h$ is identically zero.

The second fundamental form h of Φ can be written locally as

$$h = \sum_\alpha \sum_{\tilde{i}_\alpha} \frac{1}{f_\alpha} e_{(\alpha)\tilde{i}_\alpha} \otimes \sum_\beta \sum_{j_\beta} f_\beta \theta_{(\beta)}{}^{j_\beta} \Theta^{\tilde{i}_\alpha}_{j_\beta}$$

by definition. Hence,

$$\mathrm{tr}h = \sum_\alpha \sum_{\tilde{i}_\alpha} \frac{1}{f_\alpha} e_{(\alpha)\tilde{i}_\alpha} \otimes \sum_\beta \sum_{j_\beta} \frac{1}{f_\beta} \Theta^{\tilde{i}_\alpha}_{j_\beta}(e_{(\beta)j_\beta}).$$

On the other hand, it is computed that

$$\sum_{j_\beta} \Theta^{\tilde{i}_\alpha}_{j_\beta}(e_{(\beta)j_\beta}) = \begin{cases} -d_\alpha \dfrac{(e_{(\alpha)\tilde{i}_\alpha} f_\alpha)}{f_\alpha} + \sum_{j_\alpha} \theta_{(\alpha)}{}^{\tilde{i}_\alpha}_{j_\beta}(e_{(\alpha)j_\alpha}) & \text{if } \beta = \alpha, \\ -d_\beta \dfrac{(e_{(\alpha)\tilde{i}_\alpha} f_\beta)}{f_\alpha}, & \text{if } \beta \neq \alpha. \end{cases}$$

Therefore

$$\mathrm{tr}h=\sum_{\alpha}\sum_{\bar{i}_\alpha}\frac{1}{f_\alpha}e_{(\alpha)\bar{i}_\alpha}\otimes\{\frac{1}{f_\alpha}(-d_\alpha\frac{(e_{(\alpha)\bar{i}_\alpha}f_\alpha)}{f_\alpha}+\sum_{j_\alpha}\theta_{(\alpha)}{}^{\bar{i}_\alpha}_{j_\alpha}(e_{(\alpha)j_\alpha}))+\sum_{\beta\neq\alpha}\frac{1}{f_\beta}(-d_\beta\frac{(e_{(\alpha)\bar{i}_\alpha}f_\beta)}{f_\alpha})\}$$

$$=\sum_{\alpha}\sum_{\bar{i}_\alpha}\frac{1}{f_\alpha}e_{(\alpha)\bar{i}_\alpha}\otimes\frac{1}{f_\alpha}\{-\sum_{\beta}d_\beta\frac{(e_{(\alpha)\bar{i}_\alpha}f_\beta)}{f_\beta}+\sum_{j_\alpha}\theta_{(\alpha)}{}^{\bar{i}_\alpha}_{j_\alpha}(e_{(\alpha)j_\alpha})\}$$

$$=\sum_{\alpha}\sum_{\bar{i}_\alpha}\frac{1}{f_\alpha}e_{(\alpha)\bar{i}_\alpha}\otimes\frac{1}{f_\alpha}\{-e_{(\alpha)\bar{i}_\alpha}(\log f_1^{d_1}\cdots f_l^{d_l})+\sum_{j_\alpha}\theta_{(\alpha)}{}^{\bar{i}_\alpha}_{j_\alpha}(e_{(\alpha)j_\alpha})\}.$$

So Theorem 2.1 is proved.

Next we assume that $f_1^{d_1}\cdots f_l^{d_l}=F_1^{d_1}\cdots F_l^{d_l}$ holds for some positive functions $F_1:M_1\to\boldsymbol{R}^+,\cdots,F_l:M_l\to\boldsymbol{R}^+$. Then

$$\mathrm{tr}h=\sum_{\alpha}\sum_{\bar{i}_\alpha}\frac{1}{f_\alpha}e_{\bar{i}_\alpha}\otimes\frac{1}{f_\alpha}\{-e_{(\alpha)\bar{i}_\alpha}(\log F_\alpha^{d_\alpha})+\sum_{j_\alpha}\theta_{(\alpha)}{}^{\bar{i}_\alpha}_{j_\alpha}(e_{(\alpha)j_\alpha})\}.$$

Thus Φ is minimal if and only if

$$-e_{(\alpha)\bar{i}_\alpha}(\log F_\alpha^{d_\alpha})+\sum_{j_\alpha}\theta_{(\alpha)}{}^{\bar{i}_\alpha}_{j_\alpha}(e_{(\alpha)j_\alpha})=0$$

holds for all $\bar{i}_\alpha=d_\alpha+1,\cdots,n_\alpha$ and for all $\alpha=1,\cdots,l$.

Finally, we have only to prove that the above equation for $\bar{i}_\alpha=d_\alpha+1,\cdots,n_\alpha$ is the minimality condition for the immersion $\varphi_\alpha:M_\alpha\to(N_\alpha,F_\alpha^2g_\alpha)$. However, the proof is of similar computations as above, hence we omit it.

§3. Applications

Theorem 2.1 implies that if there is a product minimal submanifold M in a product manifold N provided with a Riemannian metric $g\in\mathscr{M}$ then it is also a minimal submanifold in M with any metric which is $(d_1,\dots,d_l)$-equivalent to g. So we can obtain examples of minimal submanifolds from known examples.

EXAMPLE 3.1. A totally geodesic plane in $\boldsymbol{R}^3$ is a cylindrical minimal surface, which is also interpreted as a product minimal surface in $(\boldsymbol{R}^2\times\boldsymbol{R},(dx^2+dy^2)+dz^2)$. Therefore it is also a minimal surface in $(U,g_f=(dx^2+dy^2)/f^2+f^2dz^2)$ where U is an open set in $\boldsymbol{R}^2\times\boldsymbol{R}$ and f is an arbitrary positive smooth function on U. In particular, if U and f are the following (1)–(3) then (U,g_f) is a Riemannian homogeneous space in each case.

(1) Let $U=\{y>0\}\subset\boldsymbol{R}^2\times\boldsymbol{R}$ and $f(x,y,z)=y$. Then $g_f=(dx^2+dy^2)/y^2+y^2dz^2$. So (U,g_f) is a warped product manifold of the Poincaré upper half-plane

and $\boldsymbol{R}$. Moreover it is isometric to the following Lie group G_1 provided with a left invariant metric:

$$G_1=\left\{\begin{bmatrix} y & 0 & 0 & x \\ 0 & y & 0 & 0 \\ 0 & 0 & 1/y & z \\ 0 & 0 & 0 & 1 \end{bmatrix}; (x,z)\in \boldsymbol{R}^2, y\in \boldsymbol{R}^+\right\},$$

which is a semi-direct product of $(\boldsymbol{R}^2,+)$ and $(\boldsymbol{R}^+,\times)$.

In fact, for an arbitrary $a_1=(x_1,y_1,z_1)\in G_1$, let L_{a_1} denote the left translation by a_1, then

$$L_{a_1}\begin{bmatrix} y & 0 & 0 & x \\ 0 & y & 0 & 0 \\ 0 & 0 & 1/y & z \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} y_1y & 0 & 0 & y_1x+x_1 \\ 0 & y_1y & 0 & 0 \\ 0 & 0 & 1/y_1y & z/y_1+z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} L_{a_1}^* g_f &= \frac{\{d(y_1x+x)\}^2+\{d(y_1y)\}^2}{(y_1y)^2}+(y_1y)^2\{d(z/y_1+z_1)\}^2 \\ &= \frac{{y_1}^2dx^2+{y_1}^2dy^2}{{y_1}^2y^2}+{y_1}^2y^2\frac{1}{{y_1}^2}dz^2 \\ &= g_f. \end{aligned}$$

The obtained minimal surface is totally geodesic if it is defined by the equation $x = \textit{constant}$, otherwise it is not totally geodesic.

(2) Let $U=\{z>0\}\subset \boldsymbol{R}^2\times\boldsymbol{R}$ and $f(x,y,z)=1/z$. In the similar way to (1), we can prove that (U,g_f) is isometric to the following Lie group G_2 provided with a left invariant metric:

$$G_2=\left\{\begin{bmatrix} z & 0 & 0 & 0 \\ 0 & 1/z & 0 & x \\ 0 & 0 & 1/z & y \\ 0 & 0 & 0 & 1 \end{bmatrix}; (x,y)\in \boldsymbol{R}^2, z\in \boldsymbol{R}^+\right\},$$

which is a semi-direct product of $\boldsymbol{R}^2$ and $\boldsymbol{R}^+$. Moreover it is easily checked that (U,g_f) has constant sectional curvature -1, that is, (U,g_f) is isometric to the hyperbolic space, and that the obtained minimal surface is totally geodesic.

(3) Let $U=\boldsymbol{R}^2\times\boldsymbol{R}-\{x=y=0\}\subset \boldsymbol{R}^2\times\boldsymbol{R}$ and $f(x,y,z)=(x^2+y^2)^{1/2}$. Making use of the polar coordinate $x=r\cos\theta, y=r\sin\theta$, (U,g_f) can be written as

$$((\boldsymbol{R}^+\times S^1)\times\boldsymbol{R}, (\frac{1}{r^2}dr^2+d\theta^2)+r^2dz^2),$$

and is isometric to the following Lie group G_3 provided with a left invariant metric, that can be proved similarly to (1):

$$G_3 = \left\{ \begin{bmatrix} r & 0 & 0 & z \\ 0 & 1 & 0 & \theta \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; r \in \boldsymbol{R}^+, z \in \boldsymbol{R}, \theta \in S^1 \right\},$$

which is a semi-direct product of $\boldsymbol{R} \times S^1$ and $\boldsymbol{R}^+$.

The obtained minimal surface is totally geodesic if it is defined by the equation $\theta = constant$, otherwise it is not totally geodesic.

EXAMPLE 3.2. A totally geodesic plane minus one point in $\boldsymbol{R}^3$ can be considered as a minimal cone over a great circle in S^2, i.e., a product minimal surface in $\boldsymbol{R}^+ \times_t S^2$, where t is the canonical coordinate for $\boldsymbol{R}^+$. Therefore it is also a minimal surface in $(\boldsymbol{R}^+ \times S^2, g_f = dt^2 / f^2 + t^2 f^2 g_{S^2})$ where f is an arbitrary positive smooth function on $\boldsymbol{R}^+ \times S^2$ and g_{S^2} is the Riemannian metric of S^2 of constant Gaussian curvature $+1$.

In particular, we consider the case of $f = 1/t$. Then $(\boldsymbol{R}^+ \times S^2, g_f)$ is isometric to the *Riemannian product* manifold $\boldsymbol{R}^+ \times S^2$. In this case, it may be considered to be trivial that the surface is minimal in $(\boldsymbol{R}^+ \times S^2, g_f)$, more precisely it is totally geodesic. However this is an easy example for Remark 2.2. In fact, the surface is a stable minimal surface in $\boldsymbol{R}^+ \times_t S^2$ but is an unstable minimal surface in $(\boldsymbol{R}^+ \times S^2, g_f)$.

As another special case, we take the function f to be $\frac{1}{t} \cos \rho(t)$, where $\rho(t)$ is defined by sin $\rho(t) = t^2/2$ in an appropriate interval. Then g_f is a metric of constant sectional curvature $+1$ defined on some open set in $\boldsymbol{R}^+ \times S^2$. It is remarked that the minimal surface obtained in this case is totally geodesic.

EXAMPLE 3.3. Let $\boldsymbol{H}^2 = \{(x,y) \in \boldsymbol{R}^2; y > 0\}$ be the upper half-plane. As mentioned in §1, the catenoid is a product minimal surface in $(\boldsymbol{H}^2 \times \mathrm{S}^1, (dx^2 + dy^2) + y^2 d\theta^2)$, hence is also a minimal surface in $(U, g_f = (dx^2 + dy^2)/f^2 + f^2 y^2 d\theta^2)$ where U is an open set in $\boldsymbol{H}^2 \times S^1$ and f is an arbitrary positive smooth function on U.

In particular, we consider the function $f(x,y,\theta) = y$ and $U = \boldsymbol{H}^2 \times S^1$. Then

$$(U, g_f) = (\boldsymbol{H}^2, \text{the Poincaré metric}) \times_{y^2} S^1.$$

Moreover, in the similar way to Example 3.1, it can be shown that this Riemannian manifold is isometric to a locally homogeneous space $2\pi \boldsymbol{Z} \setminus G$

defined as follows:

G is a Lie group of semi-direct product of $\boldsymbol{R}^2$ and $\boldsymbol{R}^+$, which is realized as a subgroup of $GL(4;\boldsymbol{R})$ as follows:

$$G=\left\{\begin{bmatrix} y & 0 & 0 & x \\ 0 & y & 0 & 0 \\ 0 & 0 & 1/y^2 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix}; x\in\boldsymbol{R}, \theta\in\boldsymbol{R}, y\in\boldsymbol{R}^+\right\}.$$

The metric $(dx^2+dy^2)/y^2+y^4d\theta^2$ on G is left invariant. $2\pi\boldsymbol{Z}$ is a discrete subgroup of G defined by

$$2\pi\boldsymbol{Z}=\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2\pi n \\ 0 & 0 & 0 & 1 \end{bmatrix}; n\in\boldsymbol{Z}\right\}.$$

The obtained minimal surface is not totally geodesic. To see this, we may assume that the surface is defined by $y=c\cosh(x/c)$ for some non-zero constant c, and have only to show that the curve defined by the equation $x=0$ is a geodesic on this surface but is not a geodesic in $2\pi\boldsymbol{Z}\setminus G$.

As an another application we give the following.

EXAMPLE 3.4. In [L-F], L. Lyusternik and A. I. Fet proved that there exists a closed geodesic in any compact Riemannian manifold. So by this theorem together with Theorem 2.1, we immediately have the following:

COROLLARY. *In any product manifold of k numbers of compact manifolds provided with a Riemannian metric which is* $(1,\cdots,1)$*-equivalent to any product metric, there exists a k-dimensional minimal torus.*

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