

ISOTROPIC MINIMAL SUBMANIFOLDS IN A SPACE FORM

By

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Let $\tilde{M}^m(\tilde{c})$ be an m -dimensional space form of constant curvature \tilde{c} , that is, an m -dimensional Riemannian manifold of constant curvature \tilde{c} . By the Theorem in [5], the author determined n -dimensional minimal submanifolds in $\tilde{M}^m(\tilde{c})$ with the sectional curvature not less than $n\tilde{c}/2(n+1)$. We should pay attention to the value next to $n\tilde{c}/2(n+1)$, so that we could classify minimal submanifolds in $\tilde{M}^m(\tilde{c})$ with the sectional curvature not less than it.

In the present paper, we will classify n -dimensional isotropic minimal submanifolds in $\tilde{M}^m(\tilde{c})$ with the sectional curvature not less than some value. Indeed, we will prove the following

THEOREM A. *Let M be a connected compact n -dimensional ($n \geq 3$) orientable submanifold isotropically and minimally immersed in an $(n+\nu)$ -dimensional space form \tilde{M} of constant curvature \tilde{c} . If the sectional curvature of M is not less than $n\tilde{c}/3(n+2)$, then M is of constant curvature \tilde{c} or $n\tilde{c}/3(n+2)$, or the second fundamental form is parallel.*

We may assume that $0 < \tilde{c}$ by (2.17) and Remark in §2, that is, \tilde{M} is a sphere $S^m(\tilde{c})$ of constant curvature \tilde{c} . When M is of constant curvature, by the results in [2], according as the sectional curvature is \tilde{c} or $n\tilde{c}/3(n+2)$, M is a great sphere of $S^m(\tilde{c})$ or the immersion is the standard minimal one of degree 3 from a sphere into a sphere as stated in [2], which we will call the generalized Veronese submanifold in the present paper. When the second fundamental form is parallel, the above immersion is the planar geodesic one, which is determined in [8]. As a Corollary to Theorem A, using the results in [2], [4] and [8], we have the following

THEOREM B. *Let M be a connected compact n -dimensional ($n \geq 3$) orientable submanifold minimally and isotropically immersed in a sphere $S(\tilde{c})$ of constant curvature \tilde{c} . If the immersion is full and the sectional curvature K_σ satisfies the inequality: $n\tilde{c}/3(n+2) \leq K_\sigma \leq \tilde{c}$, then M is a great sphere of $S(\tilde{c})$, a Veronese submanifold, or a generalized Veronese submanifold.*

§ 1. Preliminaries.

Let M be an n -dimensional submanifold immersed in an $(n+p)$ -dimensional space form $\tilde{M} = \tilde{M}^{n+p}(\tilde{c})$ of constant curvature \tilde{c} , (i.e., Riemannian submanifold with induced Riemannian metric). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation on M (resp. \tilde{M}). Then the second fundamental form (the shape operator) σ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y, \quad \text{where } X \text{ and } Y \text{ are tangent vectors,}$$

and it satisfies $\sigma(X, Y) = \sigma(Y, X)$. We choose a local field of orthonormal frames $e_1, e_2, \dots, e_n, e_{n+1}, \dots, e_{n+p}$ in \tilde{M} in such a way that restricted to M , e_1, e_2, \dots, e_n are tangent to M (and, consequently the remaining vectors are normal to M). Let B be the set of all such frames in \tilde{M} . With respect to the frame field of \tilde{M} chosen above, let $\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_{n+p}$ be the field of dual frames. Then the structure equations of \tilde{M} are given by (*)

$$(1.1) \quad \begin{aligned} d\tilde{\omega}_A &= \sum \tilde{\omega}_{AB} \wedge \tilde{\omega}_B, & \tilde{\omega}_{AB} + \tilde{\omega}_{BA} &= 0, \\ d\tilde{\omega}_{AB} &= \sum \tilde{\omega}_{AC} \wedge \tilde{\omega}_{CB} - \tilde{c} \tilde{\omega}_A \wedge \tilde{\omega}_B. \end{aligned}$$

Restricting these forms to M , we have the structure equations of M :

$$(1.2) \quad \begin{aligned} \omega_\alpha &= 0, & \omega_{i\alpha} &= \sum h_{ij}^\alpha \omega_j, & h_{ij}^\alpha &= h_{ji}^\alpha, \\ d\omega_i &= \sum \omega_{ij} \wedge \omega_j, & \omega_{ij} + \omega_{ji} &= 0, \\ d\omega_{ij} &= \sum \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, & \Omega_{ij} &= \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \\ R_{ijkl} &= \tilde{c}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \\ d\omega_{\alpha\beta} &= \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \Omega_{\alpha\beta}, & \Omega_{\alpha\beta} &= \frac{1}{2} \sum R_{\alpha\beta ij} \omega_i \wedge \omega_j, \\ R_{\alpha\beta ij} &= \sum (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta). \end{aligned}$$

Then, the second fundamental form σ can be written as

$$\sigma(X, Y) = \sum h_{ij}^\alpha \omega_i(X) \omega_j(Y) e_\alpha.$$

If we define $h_{i_1 i_2 \dots i_{k+1} m}^\alpha$ ($1 \leq k$) by

$$(1.4) \quad \sum h_{i_1 \dots i_{k+1} m}^\alpha \omega_m := d h_{i_1 \dots i_{k+1}}^\alpha + \sum h_{i_1 \dots i_{j-1} m i_{j+1} \dots i_{k+1}}^\alpha \omega_m i_j + \sum h_{i_1 \dots i_{k+1}}^\beta \omega_\beta \alpha,$$

then we have

(*) We use the following convention on the range of indices unless otherwise stated; $A, B, C, \dots = 1, 2, \dots, n+p$; $i, j, k, \dots = 1, 2, \dots, n$; $\alpha, \beta, \gamma, \dots = n+1, n+2, \dots, n+p$. We agree that repeated indices under a summation sign without indication are summed over the respective ranges.

$$(1.5) \quad h_{i_1 \dots i_k l m}^\alpha - h_{i_1 \dots i_k m l}^\alpha = \sum h_{i_1 \dots i_{j-1} r i_{j+1} \dots i_k}^\alpha R_{r i_j l m} + \sum h_{i_1 \dots i_k}^\beta R_{\beta \alpha l m}.$$

If M is of constant curvature c , then we have

$$(1.6) \quad R_{i j k m} = c(\delta_{i k} \delta_{j m} - \delta_{i m} \delta_{j k}).$$

The vector $\sigma(X, X)$ is called the *normal curvature vector* in the direction of a unit vector X . If the normal curvature vector has the same length λ for any unit tangent vector to M , then the immersion is said to be λ -isotropic. The immersion is called *minimal* if $\sum h_{m m}^\alpha = 0$ for all α . We easily see that the immersion is λ -isotropic if and only if the components of the second fundamental form satisfy the following relations ([7]);

$$(1.7) \quad \sum h_{i j}^\alpha h_{k m}^\alpha + \sum h_{i m}^\alpha h_{j k}^\alpha + \sum h_{i k}^\alpha h_{j m}^\alpha = \lambda^2(\delta_{i j} \delta_{k m} + \delta_{i m} \delta_{j k} + \delta_{i k} \delta_{j m}).$$

Now, we consider the functions on M defined by

$$\begin{aligned} S &:= \|\sigma\|^2 = \sum h_{i j}^\alpha h_{i j}^\alpha, & L_N &:= \sum h_{i j}^\alpha h_{k m}^\alpha h_{i j}^\beta h_{k m}^\beta, \\ K_N &:= \sum R_{\alpha \beta i j} R_{\alpha \beta i j} = \sum (h_{i k}^\alpha h_{k j}^\beta - h_{i k}^\beta h_{k j}^\alpha)(h_{i l}^\alpha h_{l j}^\beta - h_{i l}^\beta h_{l j}^\alpha). \end{aligned}$$

Then we know the following differential equation ([1]);

$$(1.8) \quad \frac{1}{2} \Delta S = \|\sigma_3\|^2 + n \tilde{c} S - L_N - K_N, \quad \text{where } \|\sigma_3\|^2 := \sum h_{i j k}^\alpha h_{i j k}^\alpha.$$

§ 2. The proof of Theorems.

We suppose that M is a λ -isotropic and *minimal* submanifold in a space form \tilde{M} of constant curvature \tilde{c} . Then, from (1.7), we have

$$(2.1) \quad \sum_m h_{i m}^\alpha h_{m j}^\alpha = \frac{(n+2)}{2} \lambda^2 \delta_{i j},$$

that is, M is an Einstein manifold. If $n \geq 3$, as is well known, λ is constant on M . From (1.2) and (1.7) we have

$$(2.2) \quad 3 \sum h_{i j}^\alpha h_{k l}^\alpha = (\lambda^2 - 2\tilde{c}) \delta_{i j} \delta_{k l} + (\lambda^2 + \tilde{c})(\delta_{i k} \delta_{j l} + \delta_{i l} \delta_{j k}) - R_{i l k j} - R_{i k l j},$$

which implies

$$(2.3) \quad \sum h_{k m}^\alpha R_{i k m j} = (\lambda^2 + \tilde{c}) h_{i j}^\alpha - \frac{3}{2} \sum h_{k m}^\alpha h_{k m}^\beta h_{i j}^\beta.$$

Since λ is constant, from (1.7), for all i, j, k and l , we have

$$\sum h_{i j m}^\alpha h_{k l}^\alpha + \sum h_{i k m}^\alpha h_{j l}^\alpha + \sum h_{j k m}^\alpha h_{i l}^\alpha = -\{\sum h_{i j}^\alpha h_{k l m}^\alpha + \sum h_{i k}^\alpha h_{j l m}^\alpha + \sum h_{j k}^\alpha h_{i l m}^\alpha\},$$

which implies

$$(2.4)_{i j m k l} \quad \sum h_{i j m}^\alpha h_{k l}^\alpha - \sum h_{i j}^\alpha h_{k l m}^\alpha = \sum h_{i j l}^\alpha h_{k m}^\alpha + \sum h_{i j k}^\alpha h_{l m}^\alpha + 2(\sum h_{i m}^\alpha h_{k l}^\alpha + \sum h_{j m}^\alpha h_{i k}^\alpha).$$

Making $(2.4)_{ijmkl} + (2.4)_{klmij}$, we have

$$(2.5) \quad \sum h_{im}^\alpha h_{jkl}^\alpha + \sum h_{jm}^\alpha h_{kli}^\alpha + \sum h_{km}^\alpha h_{lij}^\alpha + \sum h_{lm}^\alpha h_{ijk}^\alpha = 0.$$

It follows from $(2.4)_{ijkml}$ and (2.5) that we have

$$(2.6) \quad \sum h_{ij}^\alpha h_{klm}^\alpha - \sum h_{ijm}^\alpha h_{kli}^\alpha = \sum h_{km}^\alpha h_{ijl}^\alpha + \sum h_{lm}^\alpha h_{ijk}^\alpha.$$

Since M is minimal, from (2.6) we get

$$(2.7) \quad \sum h_{im}^\alpha h_{jmk}^\alpha = 0.$$

Furthermore, from (2.6) we get

$$(2.8) \quad \begin{aligned} \sum h_{ij}^\alpha h_{klm}^\alpha h_{ijm}^\beta h_{kl}^\beta &= \sum h_{ijm}^\alpha h_{kli}^\alpha h_{kl}^\beta h_{ijm}^\beta + 2 \sum h_{km}^\alpha h_{ijl}^\alpha h_{kl}^\beta h_{ijm}^\beta, \\ \sum h_{ij}^\alpha h_{klm}^\alpha h_{kl}^\beta h_{ijm}^\beta &= - \sum h_{km}^\alpha h_{ijl}^\alpha h_{lm}^\beta h_{ijk}^\beta = - \sum h_{mk}^\alpha h_{ijl}^\alpha h_{kl}^\beta h_{ijm}^\beta, \end{aligned}$$

whcih imply

$$(2.9) \quad 3 \sum h_{ij}^\alpha h_{klm}^\alpha h_{kl}^\beta h_{ijm}^\beta = \sum h_{ij}^\alpha h_{klm}^\alpha h_{ij}^\beta h_{klm}^\beta.$$

Letting $\Delta h_{ij}^\beta = \sum h_{ijm}^\beta$, from (2.6) we have

$$\sum h_{ij}^\beta \Delta h_{kl}^\beta - \sum (\Delta h_{ij}^\beta) h_{kli}^\beta = \sum h_{km}^\beta h_{ijl}^\beta + \sum h_{lm}^\beta h_{ijk}^\beta,$$

which implies that

$$(2.10) \quad \sum h_{ij}^\alpha h_{kli}^\alpha h_{km}^\beta h_{ijl}^\beta = 0.$$

It follows from (1.5) and (2.7) that we have

$$\begin{aligned} \sum h_{ij}^\alpha h_{kli}^\alpha h_{klm}^\beta h_{ijm}^\beta &= - \sum h_{ij}^\alpha h_{kli}^\alpha h_{km}^\beta h_{ijm}^\beta = - \sum h_{ij}^\alpha h_{kli}^\alpha h_{km}^\beta h_{ijl}^\beta \\ &\quad + \sum h_{ij}^\alpha h_{kli}^\alpha h_{km}^\beta \{ h_{pj}^\beta R_{pil m} + h_{pi}^\beta R_{pjl m} + h_{ij}^\beta R_{j\beta l m} \}, \end{aligned}$$

which, together with (1.2), (1.3), (1.7), (2.3) and (2.9), implies

$$(2.11) \quad \begin{aligned} \sum h_{ij}^\alpha h_{kli}^\alpha h_{klm}^\beta h_{ijm}^\beta &= \left\{ \frac{(n+4)\lambda^2}{2} - \tilde{c} \right\} L_N - \lambda^2 (n\tilde{c} + 2\lambda^2) S \\ &\quad + 2 \text{Trace} (H^\alpha H^\beta H^r H^\alpha H^\beta H^r) \\ &= \left\{ \frac{n+10}{2} \lambda^2 - \tilde{c} \right\} L_N - n\lambda^2 (\tilde{c} - \lambda^2) S - \frac{1}{2} \text{Trace} A^3 \\ &\quad - 2 \sum (\text{Trace} H^\alpha H^\beta H^r) (\text{Trace} H^\alpha H^\beta H^r), \end{aligned}$$

where $A = (\sum h_{ij}^\alpha h_{ij}^\beta)$ and $H^\alpha = (h_{ij}^\alpha)$. By means of (1.2), (1.5), (2.1) and (2.3), we have

$$(2.12) \quad \Delta h_{ij}^\alpha = \sum h_{ijm}^\alpha = n(\tilde{c} - \lambda^2) h_{ij}^\alpha - 2 \sum h_{km}^\alpha h_{km}^\beta h_{ij}^\beta.$$

Since $S = \sum h_{ij}^\alpha h_{ij}^\alpha = n(n+2)\lambda^2/2$ is constant on M , using (1.7), we rewrite (1.8) as

$$(2.13) \quad \|\sigma_3\|^2 = 2L_N - n(\tilde{c} - \lambda^2)S.$$

Making use of (1.5), (2.7) and (2.12), we have

$$\begin{aligned}\sum h_{ijml}^\alpha - \sum h_{ijml}^\alpha &= \sum h_{pjm}^\alpha R_{pil} + \sum h_{ipm}^\alpha R_{pjl} + \sum h_{ijp}^\alpha R_{pml} + \sum h_{ijm}^\beta R_{\beta\alpha lm} \\ \sum h_{ijlm}^\alpha - \sum h_{ijlm}^\alpha &= \sum h_{pjm}^\alpha R_{pil} + \sum h_{ipm}^\alpha R_{pjl} + \sum h_{ijm}^\beta R_{\beta\alpha lm} \\ &\quad + \sum h_{ij}^\beta h_{klm}^\beta h_{km}^\alpha - \sum h_{ij}^\beta h_{klm}^\beta h_{klm}^\alpha, \\ \sum h_{ijml}^\alpha &= n(\tilde{c} - \lambda^2) h_{ijl}^\alpha - 2 \sum h_{klm}^\alpha h_{km}^\beta h_{ij}^\beta - 2 \sum h_{km}^\alpha h_{klm}^\beta h_{ij}^\beta - 2 \sum h_{km}^\alpha h_{km}^\beta h_{ijl}^\beta.\end{aligned}$$

Summing up these equations and making use of (2.2)~(2.11), we have

$$\begin{aligned}\Delta \|\sigma_3\|^2 &= 2 \sum h_{ijk}^\alpha h_{ijk}^\alpha + 2 \sum h_{ijk}^\alpha \Delta h_{ijk}^\alpha = \{2(2n+3)\tilde{c} - 3(n-2)\lambda^2\} \|\sigma_3\|^2 \\ &\quad + 2 \|\sigma_4\|^2 - 6 \sum h_{ij}^\alpha h_{klm}^\alpha h_{ij}^\beta h_{klm}^\beta - 18 \sum h_{ij}^\alpha h_{kl}^\alpha h_{ijm}^\beta h_{klm}^\beta,\end{aligned}$$

that is,

$$\begin{aligned}(2.14) \quad \Delta \|\sigma_3\|^2 &= 2 \|\sigma_4\|^2 + \{2(2n+3)\tilde{c} - 3(n-2)\lambda^2\} \|\sigma_3\|^2 - 6 \sum h_{ij}^\alpha h_{klm}^\alpha h_{ij}^\beta h_{klm}^\beta \\ &\quad - 18 \left\{ \frac{n+10}{2} \lambda^2 - \tilde{c} \right\} L_N + 18n\lambda^2(\tilde{c} - \lambda^2)S \\ &\quad + 9 \text{Trace } A^3 + 36 \sum (\text{Trace } H^\alpha H^\beta H^r) (\text{Trace } H^\alpha H^\beta H^r),\end{aligned}$$

where $\|\sigma_4\|^2 = \sum h_{ijk}^\alpha h_{ijk}^\alpha$. On the other hand, using (2.9) and (2.12), we have

$$(2.15) \quad \frac{1}{3} \Delta L_N = \frac{3}{4} \sum h_{ij}^\alpha h_{klm}^\alpha h_{ij}^\beta h_{klm}^\beta + \sum h_{ij}^\alpha h_{kl}^\alpha h_{ijm}^\beta h_{klm}^\beta + n(\tilde{c} - \lambda^2) L_N - 2 \text{Trace } A^3.$$

Now, we consider the function $f = (2/9) \|\sigma_3\|^2 + (1/4) L_N$. Making use of (2.14) and (2.15), we have

$$\begin{aligned}(2.16) \quad \Delta f &= \frac{4}{9} \left(\|\sigma_4\|^2 + \left\{ (2n+3)\tilde{c} - \frac{3(n-2)}{2} \lambda^2 \right\} \|\sigma_3\|^2 \right) + 4n\lambda^2(\tilde{c} - \lambda^2)S \\ &\quad + \sum h_{ij}^\alpha h_{klm}^\alpha h_{ijm}^\beta h_{klm}^\beta - \left(4 \left\{ \frac{n+10}{2} \lambda^2 - \tilde{c} \right\} - n(\tilde{c} - \lambda^2) \right) L_N \\ &\quad + 8 \sum (\text{Trace } H^\alpha H^\beta H^r) (\text{Trace } H^\alpha H^\beta H^r).\end{aligned}$$

On the other hand, from (1.2) and (2.3) we have

$$(2.17) \quad n \left(\tilde{c} - \frac{\lambda^2}{2} \right) S - \frac{3}{2} L_N = \sum h_{ij}^\alpha h_{klm}^\alpha R_{kijm} + \sum h_{ij}^\alpha h_{kl}^\alpha R_{kmjm}.$$

For each α . let $h_1^\alpha, h_2^\alpha, \dots, h_n^\alpha$ be the eigenvalues of H^α . Then we have

$$\begin{aligned}\sum_{i,j,k,m} (h_{ij}^\alpha h_{klm}^\alpha R_{kijm} + h_{ij}^\alpha h_{kl}^\alpha R_{kmjm}) &= \frac{1}{2} \sum_{i,k} (h_i^\alpha - h_k^\alpha)^2 R_{ikik} \\ &\geq \frac{1}{2} \sum_{i,k} (h_i^\alpha - h_k^\alpha)^2 c = nc \sum_i (h_i^\alpha)^2 = nc \text{Trace } (H^\alpha)^2,\end{aligned}$$

where c is the minimum of the sectional curvature of M and the equality holds only when the sectional curvature is constant. This, together with (2.17), implies

$$n\left(\tilde{c}-\frac{\lambda^2}{2}\right)S-\frac{3}{2}L_N\geq ncS,$$

that is,

$$(2.18) \quad L_N \leq \frac{n}{3} \{2(\tilde{c}-c)-\lambda^2\}S,$$

where the equality holds only when the sectional curvature is constant.

REMARK. It follows from our assumption $n\tilde{c}/3(n+2) \leq c$ that

$$(2.19) \quad 0 \leq \tilde{c}.$$

When \tilde{M} is simply connected and complete, since M is a compact minimal submanifold of \tilde{M} , \tilde{c} must be positive.

It follows from (1.7) and $nL_N \geq K_N$ (see [4]) that we have

$$(2.20) \quad \frac{n\lambda^2}{n-1}S \leq L_N,$$

where the equality holds only when the sectional curvature is constant. From (2.18) and (2.20) we have

$$(2.21) \quad 0 \leq L_N - \frac{n\lambda^2}{n-1}S \leq \frac{2n}{9(n+2)} \left\{ 2(n+3)\tilde{c} - \frac{3(n+2)^2}{2(n-1)}\lambda^2 \right\} S.$$

Since S is constant on M , if $S = \sum h_{ij}^\alpha h_{ij}^\alpha = 0$, then M is totally geodesic. Therefore, from now on we may assume that $S \neq 0$. Let us now find the lower bounds for $\sum h_{ij}^\alpha h_{kl}^\alpha h_{ijm}^\beta h_{klm}^\beta$, $\sum (\text{Trace } H^\alpha H^\beta H^r)(\text{Trace } H^\alpha H^\beta H^r)$ and $\|\sigma_4\|^2$ when $S \neq 0$, that is, we will prove the following.

LEMMA. If $S \neq 0$, then we have the following inequalities;

$$(2.22) \quad \|\sigma_4\|^2 \geq \frac{3(n+2)}{(n+4)} \left\{ \frac{n+1}{n-1} \lambda^2 - \tilde{c} \right\} \|\sigma_3\|^2 + \frac{6(n+2)}{n(n+4)S} \|\sigma_3\|^2 \left\{ L_N - \frac{n\lambda^2}{n-1} S \right\},$$

$$(2.23) \quad \sum h_{ij}^\alpha h_{kl}^\alpha h_{ijm}^\beta h_{klm}^\beta \geq \frac{n\lambda^2}{n-1} \|\sigma_3\|^2 + \frac{1}{nS} \|\sigma_3\|^2 \left\{ L_N - \frac{n\lambda^2}{n-1} S \right\}.$$

$$(2.24) \quad 2\sum (\text{Trace } H^\alpha H^\beta H^r)(\text{Trace } H^\alpha H^\beta H^r) \geq \frac{(n-2)(n+4)}{2(n-1)} \lambda^2 L_N \\ - \frac{(n+4)}{2nS} L_N \left\{ L_N - \frac{n\lambda^2}{n-1} S \right\}.$$

PROOF OF LEMMA. Taking the length of the tensors

$$\text{and} \quad K_{ijkm}^\alpha = h_{ijk}^\alpha - h_{ijm}^\alpha + z(\delta_{im} h_{jk}^\alpha + \delta_{jm} h_{ik}^\alpha - \delta_{jk} h_{im}^\alpha - \delta_{ik} h_{jm}^\alpha) \\ L_{ijkm}^\alpha = h_{ijk}^\alpha - x(\delta_{ij} h_{km}^\alpha + \delta_{ik} h_{jm}^\alpha + \delta_{jk} h_{im}^\alpha) + y(\delta_{km} h_{ij}^\alpha + \delta_{jm} h_{ik}^\alpha + \delta_{im} h_{jk}^\alpha),$$

where $z = \|\sigma_3\|^2/nS$, $x = 2y/(n+2)$ and $y = (n+2)\|\sigma_3\|^2/n(n+4)S$, we have (2.22) and (2.23), because we have $\sum(h_{ijk}^\alpha - h_{ijmk}^\alpha)(h_{ijk}^\beta - h_{ijmk}^\beta) = 8\sum h_{ij}^\alpha h_{kl}^\alpha h_{ijm}^\beta h_{klm}^\beta - 4(\lambda + \tilde{c})\|\sigma_3\|^2$ from (1.2), (1.3), (1.5), (2.3) and (2.11). Next, taking the length of the tensor

$$P_{ijkm}^\alpha = \sum h_{ij}^\beta h_{kr}^\beta h_{rm}^\alpha + \sum h_{ir}^\beta h_{km}^\beta h_{rj}^\alpha - a(\delta_{jk} h_{im}^\alpha + 2\delta_{ik} h_{jm}^\alpha + \delta_{im} h_{jk}^\alpha) + b(\delta_{ij} h_{km}^\alpha + \delta_{km} h_{ij}^\alpha).$$

we have

$$(2.25) \quad 2\sum(\text{Trace } H^\alpha H^\beta H^\gamma)(\text{Trace } H^\alpha H^\beta H^\gamma) \geq -2(3n+4)a^2S + 2a\{(3n+8)\lambda^2S - L_N\} + 16abS - 2b^2nS - 4bL_N - (n+2)\lambda^2L_N,$$

because $\sum h_{im}^\alpha h_{mj}^\alpha = \frac{(n+2)\lambda^2}{2}\delta_{ij}$ and $\sum h_{ij}^\alpha h_{kl}^\alpha h_{ijl}^\beta h_{klj}^\beta = \sum(\sum h_{ij}^\alpha h_{kl}^\alpha)\{\sum h_{ij}^\beta h_{kl}^\beta + \tilde{c}(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}) + R_{iklj}\} = \lambda^2S - (1/2)L_N$ from (2.2). For some positive constant x and any positive constant ε such that

$$(2.26) \quad \frac{2x+1}{2(x+1)}\varepsilon + L_N \leq 2aS < nbS \leq L_N + \varepsilon,$$

we set

$$2(3n+4)aS = (3n+8)\lambda^2S - L_N + 4b^2nSx/a + 8(1-x)bS.$$

Then, from (2.25) we have

$$(2.27) \quad 2\sum(\text{Trace } H^\alpha H^\beta H^\gamma)(\text{Trace } H^\alpha H^\beta H^\gamma) \geq (3n+8)\lambda^2Sa - aL_N - 4bL_N - (n+2)\lambda^2L_N + 2\{-(2x+1)nbS + 4(x+1)aS\}b.$$

Since $L_N = 2(x+1)L_N + (2x+1)\varepsilon - (2x+1)(L_N + \varepsilon) \leq -(2x+1)nbS + 4(x+1)aS$ by (2.26), using (2.26), (2.27) implies

$$(2.28) \quad 2\sum(\text{Trace } H^\alpha H^\beta H^\gamma)(\text{Trace } H^\alpha H^\beta H^\gamma) > \frac{(n-2)(n+4)}{2(n-1)}\lambda^2L_N - \frac{(n+4)}{2nS}L_N\left\{L_N - \frac{n\lambda^2}{n-1}S\right\} + \left(\frac{(3n+8)(2x+1)}{4(x+1)}\lambda^2 - \frac{n+4}{2nS}L_N\right)\varepsilon.$$

Since ε is any positive constant, we have (2.24).

Now, we will prove Theorem A. Making use of (2.22), (2.23) and (2.24), from (2.16) we have

$$(2.29) \quad \Delta f \geq \left(\frac{4(n+1)}{9(n+4)}\left\{2(n+3)\tilde{c} - \frac{3(n+2)^2}{2(n-1)}\lambda^2\right\} - \frac{3n^2+13n+20}{3n(n+4)S}\left\{L_N - \frac{n\lambda^2}{n-1}S\right\}\right)\|\sigma_3\|^2,$$

which, together with (2.21), implies

$$(2.30) \quad \Delta f \geq \frac{2(3n^2+5n-8)}{27(n+4)(n+2)} \left\{ 2(n+3)\tilde{c} - \frac{3(n+2)^2}{2(n-1)}\lambda^2 \right\} \|\sigma_3\|^2 \geq 0,$$

Therefore, if M is compact and orientable, we have $\int_M \Delta f d \text{Vol} M = 0$, and so we get $\Delta f = 0$ on M , that is,

$$f \text{ is constant on } M \text{ and } \left\{ 2(n+3)\tilde{c} - \frac{3(n+2)^2}{2(n-1)}\lambda^2 \right\} \|\sigma_3\| = 0,$$

which implies that if M is not totally geodesic, then

$$\|\sigma_3\| = 0 \quad \text{or} \quad 2(n+3)\tilde{c} = \frac{3(n+2)^2}{2(n-1)}\lambda^2.$$

If $S \neq 0$ and $\|\sigma_3\| \neq 0$, then it follows from (2.20) and (2.21) that M is of constant curvature $\frac{n\tilde{c}}{3(n+2)}$. In this case, as stated in [2] or [6], the immersion may be considered as a standard minimal one of degree 3 from a sphere into a sphere. If $S \neq 0$ and $\|\sigma_3\| = 0$, then the second fundamental form is parallel, and so the immersion is the planar geodesic one which is determined by K. Sakamoto in [8]. Thus we have proved Theorem A.

Next, we will prove Theorem B. If the sectional curvature K_σ of M satisfies the inequality $\frac{n\tilde{c}}{3(n+2)} \leq K_\sigma \leq \tilde{c}$, then, by Theorem A, we see that M is of constant curvature \tilde{c} or $n\tilde{c}/3(n+2)$, or the second fundamental form of M is parallel. Looking over the curvatures of planar geodesic submanifolds in [8], we easily see that M must be of constant curvature \tilde{c} or $n\tilde{c}/2(n+1)$ when $\|\sigma_3\| = 0$. By the results in [5] and [6], according as K_σ is \tilde{c} , $n\tilde{c}/2(n+1)$, or $n\tilde{c}/3(n+2)$, M is a great sphere of $S(\tilde{c})$, a Veronese submanifold, or a generalized Veronese submanifold.

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