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RETRACTION OF SPACES WITH A σ-ALMOST LOCALLY FINITE BASE

By

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1. Introduction.

In [3], M. Itō and K. Tamano introduced the notion of almost local finiteness and the class of all spaces with a σ-almost locally finite base. Hereafter, σ-$\mathcal{ALF}$ denotes this class and we call a space of σ-$\mathcal{ALF}$ a σ-ALF-space. σ-$\mathcal{ALF}$ is countably productive, hereditary and the closed image of a σ-ALF-space is M_1 (see [3]). But it is not known whether there exists an M_1-space which is not a σ-ALF-space. If M_1-spaces are σ-ALF-spaces, Ceder's long-standing unsolved question will be affirmatively answered; that is, every stratifiable space is M_1.

In this paper, we shall prove that a σ-ALF-space X can be imbedded in a σ-ALF-space $Z(X)$ as a closed subspace in such a way that X is an $AR(\sigma\text{-}\mathcal{ALF})$ (resp. $ANR(\sigma\text{-}\mathcal{ALF})$) if and only if X is a retract (resp. neighborhood retract) of $Z(X)$. Moreover, by using this theorem we shall prove that a space is an $AR(\sigma\text{-}\mathcal{ALF})$ (resp. $ANR(\sigma\text{-}\mathcal{ALF})$) if and only if it is an $AE(\sigma\text{-}\mathcal{ALF})$ (resp. $ANE(\sigma\text{-}\mathcal{ALF})$). In proofs of these theorems, the construction of $Z(X)$, which is constructed by R. Cauty [1], plays an important role. For the analogous results in the class of stratifiable spaces or M_1-spaces, see [1] or [4], respectively.

Throughout this paper, all spaces are assumed to be Hausdorff topological spaces and all maps to be continuous. N denotes the set of all natural numbers. Let Q be a class of spaces. For the definitions of $AR(Q)$, $ANR(Q)$, $AE(Q)$ and $ANE(Q)$, see [2].

2. Auxiliary lemma.

For the definitions of uniformly approaching anti-cover and D-space, see [6]. The following lemma essentially proved in the proof of [5, Lemma 3.3].

LEMMA 2.1. *Let X be a D-space, F a closed subset of X and f a map from F into a space Y. Let Y also denote the natural imbedding of Y in the adjunction space $X\bigcup_f Y=Z$. If $\mathcal{U}=\{U_\alpha : \alpha\in A\}$ is an almost locally finite open family in Y, then for each $\alpha\in A$ there is a family $\{U'_\beta : \beta\in B_\alpha\}$ of open subsets in Z satis-*

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fying the following three conditions:

(C1) $\mathcal{U}'=\{U'_\beta : \beta\in B_\alpha,\ \alpha\in A\}$ *is almost locally finite at each point of* Z,

(C2) *for each* $\beta\in B_\alpha$, $U'_\beta\cap Y=U_\alpha$, *and for every open subset* V *in* Z *with* $V\cap Y=U_\alpha$ *there is* $\beta\in B_\alpha$ *such that* $U_\alpha\subset U'_\beta\subset V$, *and*

(C3) *for every open subset* W *in* Y, *there is an open subset* W' *of* Z *such that* $W'\cap Y=W$ *and* $W'\cap U'_\beta=\emptyset$ *whenever* $\beta\in B_\alpha$ *and* $W\cap U_\alpha=\emptyset$.

PROOF. Let p be the projection from the free union $X\cup Y$ to Z. Since X is monotonically normal, let G be a monotone normality operator for X satisfying the property $G(H, K)\cap G(K, H)=\emptyset$ for a disjoint closed pair (H, K). Since X is a D-space, F has a uniformly approaching anti-cover $\mathcal{V}=\{V_\lambda : \lambda\in \Lambda\}$ in X such that $\mathcal{V}$ is locally finite in $X-F$. For each $U_\alpha\in\mathcal{U}$, let $U'_\alpha=\cup\{G(x, F-p^{-1}(U_\alpha)) : x\in p^{-1}(U_\alpha)\}$. Then U'_α is obviously open in X. For each $\alpha\in A$, let $B_\alpha=\{\gamma(\alpha)\subset\Lambda : p^{-1}(U'_{\gamma(\alpha)})$ is open in $U'_\alpha\}$, where $U'_{\gamma(\alpha)}=U_\alpha\cup p(\cup\{V_\lambda : \lambda\in\gamma(\alpha)\})$. Let $B=\cup\{B_\alpha : \alpha\in A\}$, and $\mathcal{U}'=\{U'_\beta : \beta\in B\}$. Then it is easy to see that the conditions (C2) and (C3) are satisfied by $\mathcal{U}'$.

Finally to prove (C1), first we consider the case $x\in Z-Y$. In this case, it is easily verified by the local finiteness of $\mathcal{V}$ in $X-F$. Next, we consider the case $x\in Y$. Then there exist an open neighborhood V of x in Y and open finite subsets $\{H_1, \cdots, H_n\}$ of Y such that

$$\mathcal{U}|V=\{U\cap V : U\in\mathcal{U}\}\subset\{H_i\cap W : i=1, \cdots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Y\}.$$

Without loss of generality, we assume that

$$H_i\supset\cup\{U_\alpha\in\mathcal{U} : U_\alpha\cap V=H_i\cap W \text{ for some neighborhood } W \text{ of } x\}.$$

Let $V'=V\cup p(\cup\{G(y, F-p^{-1}(V)) : y\in p^{-1}(V)\cap X\})$ and

$$H'_i=H_i\cup p(\cup\{G(y, F-p^{-1}(H_i)) : y\in p^{-1}(H_i)\cap X\})$$

for each $i=1, \cdots, n$. Then it is easy to see that V' is a neighborhood of x in Z and

$$\mathcal{U}'|V'\subset\{H'_i\cap W : i=1, \cdots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z\}.$$

Thus $\mathcal{U}'$ is almost locally finite at x. This completes the proof.

3. Main theorems.

CONSTRUCTION 3.1. Let X be a space. $M(X)$ denotes the full simplicial complex which has all points of X as the set of vertices. Then there is a canonical bijection i from the 0-skeleton M^0 of $M(X)$ onto X. Let $Z'=M(X)\bigcup_i X$

be the adjunction space and $p' : M(X) \cup X \to Z'$ the projection. By the aid of p', we identify X with $p'(X) \subset Z'$. Since the restriction of p' to $M(X)$ is a bijection from $M(X)$ onto Z', by the abuse of language, a simplex σ of $M(X)$ is said to be contained in a subset U of Z' if $p'(\sigma)$ is contained in U. $Z(X)$ denotes the space such that Z' is the underlying set of $Z(X)$ and the topology of $Z(X)$ has a base which consists of a collection of sets U, which is open in Z', satisfying the following condition:

(C) If σ is a simplex of $M(X)$ such that all vertices of σ are contained in $U \cap X$, then σ is contained in U.

Let $p : M(X) \cup X \to Z(X)$ be the projection. Then p is obviously continuous. Let M^n be the n-skeleton of $M(X)$ and $Z^n = p(M^n \cup X)$.

LEMMA 3.2. *If X is a σ-ALF-space, then $Z(X)$ is also a σ-ALF-space.*

PROOF. For each $n \in N$, let Y be the free union of all $(n+1)$-simplexes of $M(X)$, F the boundary of Y and $f : F \to Z^n$ the map defined by $f(x) = p(x)$ for $x \in F$. Then the set $Y \bigcup_f Z^n$ is equal to the set Z^{n+1}. Let $\{U_\alpha : \alpha \in A\}$ be an almost locally finite open family in Z^n. Since Y is a metric space, Y is a D-space. Therefore the technique of proof of Lemma 2.1 yields that, for each $\alpha \in A$, there is a family $\{U'_\beta : \beta \in B_\alpha\}$ of open subsets in Z^{n+1} satisfying (C1)-(C3). (Note that this proof is slightly different from that of Lemma 2.1; i.e. if σ is $(n+1)$-simplex and U_α contains all vertices of σ, then σ is contained in U'_β, $\beta \in B_\alpha$.)

Now, let $\mathcal{U}_0 = \{U(\alpha_0) : \alpha_0 \in A\}$ be an almost locally finite open family in Z^0 $(=X)$. Then for fixed point $x \in Z^0$ there exist an open neighborhood V_0 of x in Z^0 and open subsets $\{H_1(0), \cdots, H_n(0)\}$ of Z^0 such that

$$\mathcal{U}_0 | V_0 \subset \{H_i(0) \cap W : i=1, \cdots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z^0\}.$$

From the preceding paragraph, we get that every $U(\alpha_0)$ can be extended to open subsets $\{U(\alpha_0, \alpha_1) : \alpha_1 \in A(\alpha_0)\}$ in Z^1 in such a way that the family

$$\mathcal{U}_1 = \{U(\alpha_0, \alpha_1) : \alpha_0 \in A,\ \alpha_1 \in A(\alpha_0)\}$$

satisfies (C1)-(C3). In particular, we may assume that the method of extensions is the same one of Lemma 2.1. Therefore there exist an open neighborhood V_1 of x in Z^1 and open subsets $\{H_1(1), \cdots, H_n(1)\}$ of Z^1 such that

$$\mathcal{U}_1 | V_1 \subset \{H_i(1) \cap W : i=1, \cdots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z^1\},$$

$V_1 \cap Z^0 = V_0$ and $H_i(1) \cap Z^0 = H_i(0)$ for $i=1, \cdots, n$. Repeating this process we get for each $k \in N$ an almost locally finite open family

$$\mathcal{U}_k=\{U(\alpha_0, \alpha_1, \cdots, \alpha_k): \alpha_0\in A, \alpha_1\in A(\alpha_0), \cdots, \alpha_k\in A(\alpha_0, \cdots, \alpha_{k-1})\}$$

in Z^k, an open neighborhood V_k of x in Z^k and open subsets $\{H_1(k), \cdots, H_n(k)\}$ of Z^k such that

$$\mathcal{U}_k|V_k\subset\{H_i(k)\cap W: i=1, \cdots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z^k\},$$

$V_k\cap Z^{k-1}=V_{k-1}$ and $H_i(k)\cap Z^{k-1}=H_i(k-1)$ for $i=1, \cdots, n$. Let

$$\Sigma=\{(\alpha_0, \alpha_1, \alpha_2, \cdots): \alpha_0\in A, \alpha_1\in A(\alpha_0), \alpha_2\in A(\alpha_0, \alpha_1), \cdots\}.$$

For each $(\alpha_0, \alpha_1, \alpha_2, \cdots)\in\Sigma$, let

$$U(\alpha_0, \alpha_1, \alpha_2, \cdots)=\cup\{U(\alpha_0, \alpha_1, \cdots, \alpha_k): k\in N\}.$$

Then $U(\alpha_0, \alpha_1, \alpha_2, \cdots)$ is an open set of $Z(X)$, because for each $k\in N$,

$$U(\alpha_0, \alpha_1, \alpha_2, \cdots)\cap Z^k=U(\alpha_0, \alpha_1, \cdots, \alpha_k)$$

is open in Z^k and $U(\alpha_0, \alpha_1, \alpha_2, \cdots)$ satisfies (C) by the construction of $U(\alpha_0, \alpha_1, \cdots, \alpha_k)$. Next, we claim that

$$\mathcal{U}=\{U(\alpha_0, \alpha_1, \alpha_2, \cdots): (\alpha_0, \alpha_1, \alpha_2, \cdots)\in\Sigma\}$$

is almost locally finite in $Z(X)$. Let $V=\cup\{V_k: k=0, 1, 2, \cdots\}$ and

$$H_i=\cup\{H_i(k): k=0, 1, 2, \cdots\} \qquad \text{for} \quad i=1, \cdots, n.$$

Then it is easily verified that V is an open neighborhood of x in $Z(X)$ and H_i open in $Z(X)$ satisfying

$$\mathcal{U}|V\subset\{H_i\cap W: i=1, \cdots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z(X)\}.$$

Thus $\mathcal{U}$ is almost locally finite at $x\in Z^0$. By the same method, at each point $y\in Z^k$ for any $k\in N$, $\mathcal{U}$ is almost locally finite.

Finally, we shall show that $Z(X)$ has a σ-almost locally finite base. Let $\{\mathcal{U}_n\}$ is a σ-almost locally finite base for Z^0. Then it is easily seen that the extensions $\{\mathcal{U}'_n\}$ of $\{\mathcal{U}_n\}$ to $Z(X)$, by the same method above, is a σ-almost locally finite local base at each point of Z^0. Furthermore, since $M(X)$ has a σ-almost locally finite base by [5, Theorem 4.1] and the open subspace $Z(X)-Z^0$ is homeomorphic to an open subspace of $M(X)$, there is a σ-almost locally finite (in $Z(X)$) local base $\{\mathcal{V}_n\}$ at each point of $Z(X)-Z^0$. Thus $\{\mathcal{U}'_n\}\cup\{\mathcal{V}_n\}$ is a σ-almost locally finite base for $Z(X)$. This completes the proof.

The following lemma was proved in [1, Lemma 1.2].

LEMMA 3.3. *Let X be a space. If Y is a stratifiable space, A a closed subset of Y and $f: A\to X$ a map, then there is a map $F: Y\to Z(X)$ with $F|A=f$.*

The following theorem is an immediate consequence of Lemma 3.2 and 3.3.

THEOREM 3.4. *A σ-ALF-space X is an $AR(\sigma\text{-}\mathcal{ALF})$* (resp. *$ANR(\sigma\text{-}\mathcal{ALF})$*) *if and only if X is a retract* (resp. *neighborhood retract*) *of $Z(X)$*.

The following theorem is a direct consequence of Theorem 3.4 and Lemma 3.3.

THEOREM 3.5. *A space is an $AR(\sigma\text{-}\mathcal{ALF})$* (resp. *$ANR(\sigma\text{-}\mathcal{ALF})$*) *if and only if it is an $AE(\sigma\text{-}\mathcal{ALF})$* (resp. *$ANE(\sigma\text{-}\mathcal{ALF})$*).

REMARK 3.6. The analogous facts of [4, Section 4] can be proved by the same method in [4].

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