

## REDUCTION OF THE CODIMENSION OF TOTALLY REAL SUBMANIFOLDS OF A COMPLEX SPACE FORM

By

U-Hang KI and Hisao NAKAGAWA

### Introduction.

A submanifold  $M$  of a Kaehlerian manifold  $\bar{M}$  is said to be *totally real* if each tangent space to  $M$  is mapped to the normal space by the complex structure of  $\bar{M}$ . Many subjects for totally real submanifolds were studied from various different points of view, as ones of which Naitoh [9] and Naitoh and Takeuchi [10] classified an  $n$ -dimensional totally real submanifold with parallel second fundamental form in  $P_n\mathbf{C}$ , and Ohnita [11] and Urbano [16] showed recently that the second fundamental form of such a submanifold of non-negative curvature is parallel, independently. Besides, the study for 3-dimensional totally real submanifolds of  $S^8$  by Mashimo [8] is also interesting.

In this paper the reduction of Allendoerfer type for the codimension of totally real submanifolds of a complex space form is treated with. As for all sorts of studies mentioned above, it is important that the dimension of the submanifolds is half of that of the ambient space. The purpose of this article is to show that the fact is essential, namely, to verify the following

**THEOREM.** *Let  $\bar{M}$  be a complex space form of complex dimension  $m$ , and  $M$  an  $n$ -dimensional totally real submanifold of  $\bar{M}$ . If the induced  $f$ -structure in the normal bundle is parallel, then there exists a totally geodesic complex space form  $M_0$  of complex dimension  $n$  of  $\bar{M}$  in which  $M$  is totally real.*

In the first section, preliminaries about totally real submanifolds of a complex space form are prepared for and the theorem is proved in the case where the ambient space is complex Euclidean. In §2, a  $(2m+1)$ -dimensional anti-de Sitter space  $H_1^{2m+1}$  and the Sasakian structure on such a manifold are recalled. Lorentzian submanifolds of  $H_1^{2m+1}$  are treated with in the next section and the theorem is proved in §4 provided that the ambient space is hyperbolic. In the last section the proof in  $P_m\mathbf{C}$  will be sketched.

**1. Totally real submanifolds of a complex space form.**

A complete and simply connected Kaehlerian manifold of constant holomorphic sectional curvature is called a *complex space form*. By  $M_m(c)$  a complex space form with constant holomorphic sectional curvature  $c$  and of complex dimension  $m$  is denoted. The complex space form consists of a complex projective space  $P_m\mathbf{C}$ , a complex Euclidean one  $C^m$  or a complex hyperbolic one  $H_m\mathbf{C}$ , according as  $c>0$ ,  $c=0$  or  $c<0$ . Let  $J$  and  $g$  be an almost complex structure and a Hermitian metric which are equipped with in  $M_m(c)$ . Let  $M$  be a real  $n$ -dimensional Riemannian submanifold immersed isometrically in  $M_m(c)$ . By the same symbol  $g$  the Riemannian metric induced on  $M$  from that of the ambient space is denoted.

Let  $\mathcal{X}(M)$  and  $\mathcal{X}^\perp(M)$  be the set of all vector fields tangent to  $M$  and the set of all vector fields normal to  $M$ , respectively. Manifolds and submanifolds which are discussed in this paper will be assumed to be connected and the smoothness of any geometric objects is also assumed to be of class  $C^\infty$ . Linear mappings  $P$  and  $F$  of  $\mathcal{X}(M)$  into  $\mathcal{X}(M)$  and  $\mathcal{X}^\perp(M)$  are defined as follows: for any vector field  $X$  in  $\mathcal{X}(M)$ ,  $PX$  is the tangent part of  $JX$  and  $FX$  is the normal one of  $JX$ . Namely,  $JX=PX+FX$ . Similarly, other two operators  $p$  and  $f$  of  $\mathcal{X}^\perp(M)$  into  $\mathcal{X}(M)$  and  $\mathcal{X}^\perp(M)$  are defined as follows: for any normal field  $\xi$ ,  $p\xi$  and  $f\xi$  are given by the tangent part and the normal one of  $J\xi$ , respectively. That is,  $J\xi=p\xi+f\xi$ . Then it is well known [18] that the following relations between these operators hold true: for any tangent vector fields  $X$  and  $Y$ , and any normal fields  $\xi$  and  $\eta$ ,

$$(1.1) \quad \begin{cases} g(PX, Y) + g(X, PY) = 0, \\ g(FX, \eta) + g(X, p\eta) = 0, \\ g(f\xi, \eta) + g(\xi, f\eta) = 0, \end{cases}$$

$$(1.2) \quad \begin{cases} P^2 = -I - pF, \quad FP + fF = 0, \\ Pp + pf = 0, \quad f^2 = -I - Fp, \end{cases}$$

where  $I$  denotes the identity mapping.

An  $n$ -dimensional submanifold  $M$  of  $M_m(c)$  is said to be *totally real* if  $JM_x \subset M_x^\perp$  for any point  $x$  in  $M$ , where  $M_x (=T_xM)$  and  $M_x^\perp$  denote the tangent space of  $M$  at  $x$  and the normal one to  $M$  at  $x$ , respectively. The submanifold  $M$  is assumed to be always totally real at the rest of this paper. Accordingly,  $JX$  is the normal field to  $M$ , provided that  $X$  is tangent to  $M$ . This implies that  $m \geq n$  and the operator  $P$  is zero, and moreover  $F$  is the linear isomorphism. The orthogonal complement of  $JM_x$  in  $M_x^\perp$  is invariant under the operator  $f$  and hence the orthogonal sum decomposition  $M_x^\perp = FM_x + fM_x^\perp$  is obtained. This yields

that the linear transformation  $f$  satisfies  $f^3+f=0$  and its rank is equal to  $2m-n$ . A non-null tensor field  $\phi$  of type (1,1) on a Riemannian manifold  $N$  is called an  $f$ -structure of rank  $r$  on  $N$  if it satisfies  $\phi^3+\phi=0$  and  $\text{rank } \phi=r$ . It means that  $f$  becomes the  $f$ -structure of rank  $2m-n$  in the normal bundle of  $M$ .

Let  $\nabla$  and  $\bar{\nabla}$  be the Riemannian connections in  $M_m(c)$  and  $M$  respectively, and  $\nabla^\perp$  the normal connection in the normal bundle of  $M$ . Let  $\beta$  be the second fundamental form on  $M$  and  $B$  the shape operator of  $\mathcal{X}^\perp(M) \times \mathcal{X}(M)$  into  $\mathcal{X}(M)$  defined by  $g(B_\xi X, Y) = g(\beta(X, Y), \xi)$ . Since  $M$  is totally real, it follows that  $FZ = JZ$ , which means that for any vector fields tangent to  $M$

$$\begin{aligned} g(\beta(X, Y), FZ) &= g(\beta(X, Y), JZ) = g(\nabla_x Y, JZ) = -g(Y, \bar{\nabla}_x(JZ)) \\ &= g(JY, \nabla_x Z) = g(FY, \beta(X, Z)). \end{aligned}$$

This implies that  $g(\beta(X, Y), FZ)$  is symmetric with respect to  $X, Y$  and  $Z$ .

The covariant derivatives  $\nabla_x P, \nabla_x F, \nabla_x p$  and  $\nabla_x f$  of  $P, F, p$  and  $f$  are defined by

$$\begin{aligned} \nabla_x p(Y) &= \nabla_x(PY) - P\nabla_x Y, \quad \nabla_x F(Y) = \nabla_x^\perp(FY) - F\nabla_x Y, \\ \nabla_x p(\xi) &= \nabla_x(p\xi) - p\nabla_x^\perp \xi, \quad \nabla_x f(\xi) = \nabla_x^\perp(f\xi) - f\nabla_x^\perp \xi. \end{aligned}$$

Then it follows from the Gauss and Weingarten formulas that the following equations hold true :

$$(1.3) \quad \begin{aligned} p\beta(X, Y) &= -B_{Fx}Y, \quad \nabla_x F(Y) = f\beta(X, Y), \\ \nabla_x p(\xi) &= B_{f\xi}X, \quad \nabla_x f(\xi) = -F(B_\xi X) - \beta(X, d\xi). \end{aligned}$$

The  $f$ -structure  $f$  in the normal bundle is said to be *parallel* if it satisfies  $\nabla f=0$ . The operators  $F$  and  $p$  are also said to be *parallel*, provided that the covariant differentials of  $F$  and  $p$  are 0 respectively. For the parallelism of  $f$  one finds the following property, which was pointed out by the referee.

PROPOSITION 1.1. *Let  $M$  be a submanifold of  $M_m(c)$ . If there exists a totally geodesic submanifold  $M_0$  of  $M_m(c)$ , in which  $M$  is totally real, then the  $f$ -structure in the normal bundle is parallel.*

PROOF. For any point  $x$  of  $M$ , the assumption gives the direct sum decomposition of the normal space  $M_x^\perp$  of  $M$  in the ambient space  $\bar{M}$

$$M_x^\perp = JM_x \oplus T_x(M_0)^\perp.$$

Since  $M$  is the totally geodesic submanifold of  $M_0$ , it follows  $B_\xi=0$  for a normal vector  $\xi$  of  $T_x(M_0)^\perp$ , and moreover since  $T_x(M_0)^\perp$  is  $J$ -invariant, it follows that  $p\xi = (J\xi)^T = 0$ , where  $(J\xi)^T$  denotes the tangent part of  $J\xi$ . It implies that

$$\nabla_x f(\xi) = -F(B_\xi X) - \beta(X, p\xi) = 0 \quad \text{for any } \xi \text{ in } T_x(M_0)^\perp$$

by the last equation of (1.3).

Next, for any vector field  $Y$  it follows

$$\begin{aligned} g(F(B_{JY}X), JZ) &= g(J(B_{JY}X), JZ) = g(B_{JY}X, Z) = g(\beta(X, Z), FY) \\ &= g(\beta(X, Y), FZ), \end{aligned}$$

because  $g(\beta(X, Y), FZ)$  is symmetric with respect to  $X$ ,  $Y$  and  $Z$ , and thus we have  $F(B_{JY}X) = \beta(X, Y)$ . Consequently it follows from (1.2) and (1.3) that

$$\nabla_x f(JY) = -F(B_{JY}X) - \beta(X, pJY) = -\beta(X, Y) + \beta(X, Y) = 0.$$

These imply that  $\nabla f = 0$ .

LEMMA 1.2. *Let  $M$  be a totally real submanifold in  $M_m(c)$ . If the  $f$ -structure  $f$  in the normal bundle is parallel, then so do operators  $F$  and  $p$ .*

PROOF. The first equation of (1.3) implies  $Fp\beta(X, Y) + FB_{FY}X = 0$ . By means of (1.2) the assumption of the parallelism of the structure  $f$  yields  $f^2\beta(X, Y) = 0$  and it follows that  $f\beta(X, Y) = 0$  by the property of  $f^3 + f = 0$ , which means that  $F$  is parallel. This is equivalent to  $\nabla p = 0$  in [1].

REMARK. (1) Lemma 1.2 holds true in the case where the ambient space is only a Kaehlerian manifold. In general, it is also seen in [1] that  $\Delta F = 0$  is equivalent to  $\nabla p = 0$  in a  $CR$ -submanifold in  $M_m(c)$ .

(2) Pak [14] proved that in a totally real submanifold of  $\mathbf{P}_m\mathbf{C}$ , if the second fundamental form is parallel, then so do the operators  $F$ ,  $p$  and  $f$ .

The distribution  $\mathcal{D}$  in the normal bundle of  $M$  is said to be *parallel* with respect to the normal connection, if for any normal field  $\xi$  in  $\mathcal{D}$  the vector field  $\nabla_x \xi$  is contained in  $\mathcal{D}$  for any vector field  $X$  in  $\mathcal{X}(M)$ .

PROPOSITION 1.3. *Let  $M$  be an  $n$ -dimensional totally real submanifold in  $\mathbf{C}^m$ . If the  $f$ -structure  $f$  in the normal bundle is parallel, then there exists a totally geodesic submanifold  $M_0 = \mathbf{C}^n$  in which  $M$  is totally real.*

PROOF. Let  $\mathcal{D}$  be the distribution consisting of subspaces  $FM_x$  in  $M_x^\perp$  at each point  $x$  in  $M$ . Then it is of dimension  $n$  and  $\mathcal{D}(x) = \text{Ker}_x f$ . From Lemma 1.2 it follows that the operator  $F$  is parallel and hence  $\nabla_x \xi(FY) = F\nabla_x Y$  in  $\mathcal{D}$ , which means that  $\mathcal{D}$  is parallel with respect to the normal connection. The first normal space  $M_x^1$  at  $x$  is defined by the orthogonal complement of the following subspace  $\{\xi \in M_x : B_\xi = 0\}$  in  $M_x^\perp$ . Accordingly the first normal space  $M_x^1$  is identified with the linear subspace in  $M_x^\perp$  spanned by  $\beta(u, v)$  for any vectors  $u$  and  $v$  in  $M_x$ . The parallelism of the operator  $p$  together with the property  $\mathcal{D}(x) = \text{Ker}_x f$  implies that the first normal space  $M_x^1$  is contained in the subspace  $\mathcal{D}(x)$ .

By means of the reduction theorem of Erbacher [ 4 ], there exists a  $2n$ -dimensional totally geodesic submanifold  $M_0$  of  $\mathbf{C}^m$  in which  $M$  is a submanifold. This means that  $M_0$  is a complex Euclidean space equipped with the complex structure  $J|_{\mathbf{C}^n}$ , which is denoted by the same  $J$ . Then  $JM_x = FM_x = \mathcal{D}(x)$ , which implies that  $M$  is totally real in  $M_0 = \mathbf{C}^n$ .

This concludes the proof.

## 2. Odd dimensional anti-de Sitter spaces.

In this section, the Sasakian structure on a  $(2m+1)$ -dimensional anti-de Sitter space is recalled [15]. In an  $(m+1)$ -dimensional complex Euclidean space  $\mathbf{C}^{m+1}$  with standard basis, a Hermitian form  $F$  is defined by

$$F(z, w) = -z_0 \bar{w}_0 + \sum_{k=1}^m z_k \bar{w}_k,$$

where  $z = (z_0, z_1, \dots, z_m)$  and  $w = (w_0, w_1, \dots, w_m)$  are in  $\mathbf{C}^{m+1}$ . The space  $(\mathbf{C}^{m+1}, F)$  is called an  $(m+1)$ -dimensional *complex Minkowski space*, which is denoted by  $\mathbf{C}_1^{m+1}$ . A non-degenerate symmetric bilinear form  $g$  of a real vector space  $V$  is called a *scalar product* and the *index* of the scalar product  $g$  is by definition the largest number that is the dimension of a subspace  $W$  of  $V$  on which  $g|_W$  is negative definite. A metric tensor  $g$  on a smooth manifold  $M$  is by definition a symmetric non-degenerate  $(0,2)$  tensor field on  $M$  of constant index  $r$ , and a smooth manifold  $M$  furnished with a metric tensor field  $g$  is called a *semi-Riemannian manifold with semi-Riemannian metric  $g$  of index  $r$* . The common value  $r$  of the index  $g$  on a semi-Riemannian manifold  $M$  is called the *index of  $M$* . The scalar product given by  $\text{Re } F(z, w)$  is a semi-Riemannian metric of index 2 on  $\mathbf{C}_1^{m+1}$ . Let  $H_1^{2m+1}$  be a real hypersurface in  $\mathbf{C}_1^{m+1}$  denoted by

$$H_1^{2m+1} = \{z \in \mathbf{C}_1^{m+1} : F(z, z) = -1\},$$

and let  $G$  be a semi-Riemannian metric on  $H_1^{2m+1}$  induced from the complex Lorentzian metric  $\text{Re } F$  in  $\mathbf{C}_1^{m+1}$ . Then  $(H_1^{2m+1}, G)$  is the Lorentzian manifold with constant curvature  $-1$ , which is called the *anti-de Sitter space* of constant curvature  $-1$  ([ 3 ] and [17]). For any point  $z$  in  $H_1^{2m+1}$  the tangent space  $T_z H_1^{2m+1}$  can be identified through parallel translation in  $\mathbf{C}_1^{m+1}$  with the subspace  $\{w \in \mathbf{C}_1^{m+1} : \text{Re } F(z, w) = 0\}$  of  $\mathbf{C}_1^{m+1}$ . In particular  $iz$  is a point in  $H_1^{2m+1}$  where  $i$  denotes the imaginary unit, and moreover it is contained in the tangent space  $T_z H_1^{2m+1}$ . For any point  $z$  in  $H_1^{2m+1}$ ,  $\xi_z = z$  can be regarded as a unit normal time-like vector to  $H_1^{2m+1}$  up to translation. For the almost complex structure  $J$  in  $\mathbf{C}_1^{m+1}$   $E_z = -J\xi_z = -iz$  is the unit time-like vector of  $H_1^{2m+1}$ , i.e.  $E \in \mathcal{X}(H_1^{2m+1})$  and  $G(E, E) = -1$ .

For the orthogonal projection  $\pi: T_z\mathbf{C}_1^{2m+1} \rightarrow T_zH_1^{2m+1}$  the linear transformation  $\phi$  of  $\mathcal{X}(H_1^{2m+1})$  into itself is defined by the composition of  $J$  and  $\pi$ , and let  $\omega$  be a 1-form on  $H_1^{2m+1}$  defined by  $JX = \phi X + \omega(X)\xi$  for any tangent vector field  $X$  on  $H_1^{2m+1}$ . Then it is seen in [15] that

$$(2.1) \quad \begin{cases} \omega(X) = -G(X, E), \quad \omega(E) = 1, \\ \phi E = 0, \quad \omega \circ \phi = 0, \\ G(\phi X, Y) + G(X, \phi Y) = 0, \\ \phi^2 = -I + \omega \otimes E, \\ G(\phi X, \phi Y) = G(X, Y) + \omega(X)\omega(Y). \end{cases}$$

Let  $\alpha$  be the second fundamental form for the hypersurface  $H_1^{2m+1}$  of  $\mathbf{C}_1^{m+1}$  and  $A_\xi$  the shape operator with respect to  $\xi$ . Then  $H_1^{2m+1}$  is the totally umbilical hypersurface and  $A = -I$ . Let  $\bar{D}$  be the semi-Riemannian connection of  $H_1^{2m+1}$ . By making use of the Gauss and Weingarten formulas, the following relations are obtained:

$$(2.2) \quad \begin{cases} \bar{D}_V \phi(V) = -\omega(V)U - G(U, V)E, \\ \bar{D}_V \omega(V) = G(\phi U, V), \quad d\omega(U, V) = G(\phi U, V), \\ \bar{D}_V E = -\phi U \end{cases}$$

for any vector fields  $U$  and  $V$  on  $H_1^{2m+1}$ .

Let  $N$  be an odd dimensional manifold equipped with the set  $(\phi, E, \omega, G)$ , which consists of a tensor field  $\phi$  of type (1,1), a unit time-like vector field  $E$ , a 1-form  $\omega$  and a Lorentzian metric  $G$ . The set is called the *Sasakian structure* if it satisfies properties (2.1) and (2.2) [15]. The anti-de Sitter space  $H_1^{2m+1}$  admits the Sasakian structure.

### 3. Lorentzian submanifolds of $H_1^{2m+1}$ .

Let  $H_1^{2m+1}$  be a  $(2m+1)$ -dimensional anti-de Sitter space of constant curvature  $-1$  and  $(\phi, E, \omega, G)$  its admitting Sasakian structure on  $H_1^{2m+1}$ . Let  $N$  be a semi-Riemannian submanifold of  $H_1^{2m+1}$  tangent to the structure vector field  $E$ . By the same  $G$  the semi-Riemannian metric induced on  $N$  from that of  $H_1^{2m+1}$  is denoted. Each tangent space  $N_p$  is by definition a non-degenerate subspace of  $T_p H_1^{2m+1}$ , and hence a property of the linear space with indefinite scalar product gives the direct sum decomposition  $T_p H_1^{2m+1} = N_p \oplus N_p^\perp$ , and the subspace  $N_p^\perp$  which is called the *normal space* at  $p$  to  $N$  is also non-degenerate. Its dimension is equal to  $2m-n$ , where  $\dim N = n+1$ . The index of  $G$  restricted to  $N_p$  is called the *co-index*. In fact, the co-index of  $N$  is independent of the choice of the point  $p$  and it is easily seen that  $\text{ind } H_1^{2m+1} = \text{ind } N + \text{co-ind } N$ . Accordingly  $\text{ind } N = 1$  and  $\text{co-ind } N = 0$ .

Now, by the similar definition to that of 2-sets  $(P, F)$  and  $(p, f)$  of operators defined for the submanifold  $M$  of the Kaehlerian manifold  $\bar{M}$ , operators  $P', F', p'$  and  $f'$  are defined as follows: for any vector field  $U$  in  $\mathcal{X}(N)$  and any normal field  $\tau$  in  $\mathcal{X}^\perp(N)$ ,  $\phi U = P'U + F'U$  and  $\phi\tau = p'\tau + f'\tau$ , where  $P'U$  and  $p'\tau$  are tangent parts of  $\phi U$  and  $\phi\tau$  respectively, and  $F'U$  and  $f'\tau$  are also normal parts of  $\phi U$  and  $\phi\tau$  respectively. Then it is easily seen that for any vector fields  $U$  and  $V$  on  $N$  and any normal fields  $\tau$  and  $\sigma$  on  $N$ , the following relations hold true:

$$(3.1) \quad \begin{cases} G(P'U, V) + G(U, P'V) = 0, \\ G(F'U, \sigma) + G(U, p'\sigma) = 0, \\ G(f'\tau, \sigma) + G(\tau, f'\sigma) = 0, \end{cases}$$

$$(3.2) \quad \begin{cases} F'^2 + p'F' = -I + \omega \otimes E, F'P' + f'F' = 0, \\ P'p' + p'f' = 0, f'^2 = -I - F'p', \\ P'E = F'E = 0. \end{cases}$$

Let  $\bar{D}$  and  $D$  be the semi-Riemannian connections on  $H_1^{2m+1}$  and  $N$  respectively, and  $D^\perp$  the normal connection in the normal bundle of  $N$ . Let  $\alpha$  be the second fundamental form of  $N$  and  $A$  the shape operator of  $\mathcal{X}^\perp(N) \times \mathcal{X}(N)$  into  $\mathcal{X}(N)$  defined by  $G(A, U, V) = G(\alpha(U, V), \tau)$ . The Gauss and Weingarten formulas are also given by  $\bar{D}_U V = D_U V + \alpha(U, V)$  and  $\bar{D}_U \tau = -A_U U + D_U^\perp \tau$ . For example, see [13]. For the vector field  $E$  the last equation of (2.2) gives

$$(3.3) \quad \begin{cases} D_U E = -P'U, \alpha(U, E) = -F'U, \\ \alpha(E, E) = 0, \\ A_U E = p'\tau. \end{cases}$$

The covariant differentials  $DP', DF', Dp'$  and  $Df'$  of these operators are defined similarly. Then it follows from the Gauss and Weingarten formulas that the following relations are given:

$$(3.4) \quad \begin{cases} D_U P'(V) = A_{F'V} U + p'\alpha(U, V) - G(U, V)E - \omega(V)U, \\ D_U F'(V) = -\alpha(U, P'V) + f'\alpha(U, V), \\ D_U p'(\tau) = A_{f'\tau} U - P'A_U U, \\ D_U f'(\tau) = -F'A_U U - \alpha(U, p'\tau). \end{cases}$$

A submanifold  $N$  tangent to the structure vector field  $E$  in  $H_1^{2m+1}$  is said to be *totally real*, if  $\phi N_p$  is contained in the normal space  $N_p^\perp$  at each point  $p$  in  $N$ . In this case, the operator  $P'$  satisfies  $P' = 0$ , and therefore  $f'$  makes the  $f'$ -structure in the normal bundle, because of (3.2). The structure  $f'$  is said to be *parallel*, provided that  $Df' = 0$ . The others  $F'$  and  $p'$  are said to be *parallel*, if  $DF' = 0$  and  $Dp' = 0$  respectively. Similarly to Lemma 1.2, the following property

holds true :

LEMMA 3.1. *Let  $N$  be a totally real submanifold tangent to  $E$  in  $H_1^{2m+1}$ . If the  $f$ -structure  $f'$  in the normal bundle is parallel, then so do the others  $F'$  and  $p'$ .*

PROOF. First of all, the operator  $F'$  is verified. Since  $N$  is totally real, the first equation of (3.4) gives

$$F'A_{F'V}U + F'p'\alpha(U, V) - \omega(V)F'U = 0$$

for any vector fields  $U$  and  $V$ , which together with (3.2) and the assumption yields  $f'^2\alpha(U, V) = 0$ . This means that  $f'\alpha = 0$ , namely, it is equivalent to the fact that  $F'$  is parallel. The second assertion is easily given by the third equation of (3.4) and the total reality.

PROPOSITION 3.2. *Let  $N$  be an  $(n+1)$ -dimensional complete and totally real submanifold tangent to the structure vector field  $E$  in  $H_1^{2m+1}$ . If the  $f$ -structure  $f'$  in the normal bundle is parallel, then there exists a totally geodesic submanifold  $N_0 = H_1^{2n+1}$  in which  $N$  is totally real.*

PROOF. For the given distribution  $\mathcal{D}$  consisting of each tangent space  $N_p$  at each point  $p$  in  $N$ , the distribution consisting of each subspace  $\phi N_p$  of the normal space is defined by  $\phi\mathcal{D}$ . By (3.2),  $F'E = 0$  and because of  $G(F'U, F'V) = G(U, V)$  for any vector fields  $U$  and  $V$  on  $N$  orthogonal to  $E$ , the dimension of the subspace  $F'N_p$  is equal to  $n$  for each point  $p$  in  $N$ , which implies  $\dim \phi\mathcal{D} = n$ . Since the parallelism of the operator  $F'$  induces  $D_U^\perp(F'V) = F'(D_U V)$  for any vector fields  $U$  and  $V$  in  $\mathcal{X}(N)$ , the distribution  $\phi\mathcal{D}$  is parallel with respect to the normal connection.

On the other hand, the parallelism of the operator  $p'$  reduces to  $A_{f'\tau} = 0$  for any normal  $\tau$  in  $\mathcal{X}^\perp(N)$ , which means that  $f'N_p$  is the subspace of the subspace  $\{\sigma \in N_p^\perp : A\sigma = 0\}$  of the normal space  $N_p^\perp$ . Because the normal space  $N_p^\perp$  has two kinds of orthogonal sum decompositions

$$N_p^\perp = F'N_p \oplus f'N_p^\perp = N_p^1 \oplus \{\sigma \in N_p^\perp : A_\sigma = 0\} ,$$

the first normal space  $N_p^1$  at  $p$  is contained in the subspace  $F'N_p = \phi\mathcal{D}(p)$ . Thus the reduction theorem due to Dajczer [ 2 ] and Magid [ 7 ] in the anti-de Sitter space can be applied to this situation, and hence there exists a  $(2n-1)$ -dimensional complete and totally geodesic submanifold  $N_0$  of  $H_1^{2m+1}$  such that  $N \subset N_0$  and  $T_p(N_0) = N_p \oplus F'N_p$  for any point  $p$  of  $N$ . Fix a point  $p$  in  $N$  and put  $\mathbf{R}_2^{2n+2} = T_p(N_0) \oplus \mathbf{R}_p$ . Then  $\mathbf{R}_2^{2n+2}$  is a complex linear subspace of  $\mathbf{R}_2^{2m+2} = \mathbf{C}_1^{m+1}$  and  $H_1^{2n+1}$



which is defined by the intersection of  $\mathbf{R}_2^{2n+2}$  and  $H_1^{2m+1}$  is the  $(2n+1)$ -dimensional anti-de Sitter space with the metric induced from that of  $H_1^{2m+1}$ . Since the geodesics of  $H_1^{2m+1}$  are just the intersections of  $P$  and  $H_1^{2m+1}$ , where  $P$  is a plane through the origin  $O$  in  $\mathbf{C}_1^{m+1}$  which meets  $H_1^{2m+1}$ , the anti-de Sitter space  $H_1^{2n+1}$  is a complete and totally geodesic submanifold of  $H_1^{2m+1}$ . Obvious it follows  $T_p H_1^{2n+1} = T_p N_0$ . Apart from this viewpoint, it is seen in [13] that for complete and totally geodesic semi-Riemannian submanifolds  $N_1$  and  $N_2$  of a semi-Riemannian manifold if there is a point  $p$  in  $N_1$  and  $N_2$  at which the tangent spaces coincide, then  $N_1$  coincides with  $N_2$ . This implies that  $N_0 = H_1^{2n+1}$ . Moreover, since  $H_1^{2n+1}$  is invariant under the multiplication by  $e^{i\theta}$ ,  $N$  is totally real in  $H_1^{2n+1}$ .

This concludes the proof.

The argument developed in this section can be applied to the case where the submanifold in the unit sphere  $S^{2m+1}$  is totally real. Accordingly the following property is verified. The proof is omitted.

**PROPOSITION 3.3.** *Let  $N$  be an  $(n+1)$ -dimensional complete and totally real submanifold tangent to the structure vector field  $E$  of  $S^{2m+1}$  with the Sasakian structure  $(\phi, E, \omega, G)$ . If the  $f$ -structure in the normal bundle is parallel, there exists a totally geodesic unit sphere  $S^{2n+1}$  of  $S^{2m+1}$ , in which  $N$  is totally real.*

**REMARK.** When the proof of the above Proposition is checked carefully, it is seen that the condition that the  $f$ -structure in the normal bundle is parallel can be replaced by the apparently weaker one that  $f^\alpha$  vanishes identically.

**REMARK.** Under the additional condition that the mean curvature vector field is not trivial and parallel in the normal bundle, the compact  $N$  is minimally contained in a hypersurface of positive curvature in  $S^{2n+1}$  [ 5 ].

#### 4. Lorentzian circle bundles over a submanifold of $H_m C$ .

Let  $H_1^{2m+1}$  be a  $(2m+1)$ -dimensional anti-de Sitter space of  $\mathbf{C}_1^{m+1}$  equipped with the Hermitian form  $F$  on  $\mathbf{C}_1^{m+1}$ . Let  $U(1, m)$  be the set of matrices  $A$  in  $GL(m+1, \mathbf{C})$  such that  $F(Az, Aw) = F(z, w)$  for each  $z$  and  $w$  in  $\mathbf{C}_1^{m+1}$ . Then the group  $U(1, m)$  acts transitively on  $H_1^{2m+1}$  and the group  $S^1 = \{e^{i\theta}\}$  acts freely on  $H_1^{2m+1}$  by  $z \rightarrow e^{i\theta} z$ . The orbit  $\{e^{i\theta} z\}$  lies in negative definite plane spanned by  $z$  and  $iz$ . Let  $\bar{M}$  be the base manifold of the principal fiber bundle  $H_1^{2m+1}$  with the structure group  $S^1$ . For any point  $z$  in  $H_1^{2m+1}$  let  $T_z'$  be the subspace of  $T_z H_1^{2m+1}$  defined by  $T_z' = \{w \in T_z H_1^{2m+1} : \operatorname{Re} F(iz, w) = 0\}$ . Then the restriction of  $F$  to  $T_z'$  is positive definite and the orthogonal sum decomposition  $T_z H_1^{2m+1} = T_z' \oplus$

span  $\{iz\}$  is given. Moreover we have a connection in  $H_1^{2m+1}$  such that  $T_z'$  s,  $z$  in  $H_1^{2m+1}$ , are the horizontal subspaces. The natural projection  $\pi : H_1^{2m+1} \rightarrow M$  induces a linear onto isomorphism  $d\pi : T_z' \rightarrow T_{\pi(z)}M$ . The complex structure  $w \rightarrow iw$  in  $T_z'$  is compatible with the action of  $S^1$  and induces the almost complex structure  $J$  on  $\bar{M}$  such that  $d\pi \circ i = J \circ d\pi$ . A scalar product  $g$  on each tangent space  $\bar{M}_x$  at each point  $x$  in  $\bar{M}$  is defined by  $g(X, Y) = -\text{Re}F(U, V)$ , where  $U$  and  $V$  are elements in  $T_z'$  such that  $\pi(z) = x$ ,  $d\pi(U) = X$  and  $d\pi(V) = Y$ . Then  $(\bar{M}, g, J)$  is the  $m$ -dimensional complex hyperbolic space  $H_m\mathbf{C}$  with constant holomorphic sectional curvature  $-4$  ([3] and [17]). It is well known that  $H_1^{2m+1}$  is the principal  $S^1$ -bundle over  $H_m\mathbf{C}$  with the projection  $\pi : H_1^{2m+1} \rightarrow H_m\mathbf{C}$ , which is the Riemannian submersion in the sence of O'Neill [12] with fundamental tensor  $J$  and totally geodesic time-like fibers. For any point  $z$  in  $H_1^{2m+1}$  we put  $E_z = -Jz$  in  $T_zH_1^{2m+1}$ , and then the orthogonal sum decomposition  $T_zH_1^{2m+1} = T_{z(z)}H_m\mathbf{C} \oplus \text{span}\{E_z\}$  is given up to identification. Let  $(\phi, E, \omega, G)$  be the Sasakian structure equipped in  $H_1^{2m+1}$  and  $(J, g)$  the Kaehlerian structure in  $H_m\mathbf{C}$ . Let  $*$  be the horizontal lift of the Riemannian submersion  $\pi : H_1^{2m+1} \rightarrow H_m\mathbf{C}$ . By the construction.

$$(4.1) \quad (JX)^* = \phi X^*, \quad G(X^*, Y^*) = g(X, Y)$$

for any vector fields  $X$  and  $Y$  on  $\mathcal{X}(H_m\mathbf{C})$ . The above decomposition gives the following relationships between the semi-Riemannian connction  $\bar{D}$  of  $H_1^{2m+1}$  and the Riemannian connection  $\nabla$  of  $H_m\mathbf{C}$ :

$$(4.2) \quad \bar{D}_x^* Y^* = (\nabla_x Y)^* - G(\phi X^*, Y^*)E, \quad \bar{D}_x^* E = -\phi X^*.$$

For an  $n$ -dimensional submanifold  $M$  of  $H_m\mathbf{C}$ , one can construct the Lorentzian submanifold  $N = \pi^{-1}(M)$  which is the principal  $S^1$ -bundle over  $M$  with time-like totally geodesic fibers and the projection  $\pi$  [6]. Moreover,  $\pi$  is compatible with the fibration  $\pi : H_1^{2m+1} \rightarrow H_m\mathbf{C}$ , that is, the diagram

$$\begin{array}{ccc} N & \xrightarrow{i'} & H_1^{2m+1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{i} & H_m\mathbf{C} \end{array}$$

is commutative, where  $i$  and  $i'$  are the respective immersions. This shows that  $N_z = (M_{\pi(z)}) \oplus \text{span}\{E_z\}$ . The Gauss formulas for the immersions  $i'$  and  $i$  and the equation (4.2) yield

$$(4.3) \quad \begin{cases} D_x^* Y^* = (\nabla_x Y)^* - G(\phi X^* Y^*)E, \\ \alpha(X^*, Y^*) = \beta(X, Y)^*, \end{cases}$$

while the Weingarten formulas for the immersions  $i'$  and  $i$  give rise to the following relationships between the shape operators  $A$  and  $B$ :

$$(4.4) \quad \begin{cases} A_\xi^* Y^* = (B_\xi Y)^* + G(A_\xi^* Y^*, E)E, \\ D_x^* \perp \xi^* = (\nabla_x \perp \xi)^*. \end{cases}$$

On the other hand, for orthogonal operators  $(P, F)$ ,  $(p, f)$  and  $(P', F')$ ,  $(p', f')$  of the immersions  $i'$  and  $i$  respectively, (4.1) means that

$$(4.5) \quad \begin{cases} (PX)^* = P'X^*, \quad (FX)^* = F'X^*, \\ (p\xi)^* = p'\xi^*, \quad (f\xi)^* = f'\xi^*, \end{cases}$$

and using (3.3) and (4.4), we have

$$(4.6) \quad (B_\xi X)^* = A_\xi^* X^* + G(F'X^*, \xi^*)E, \quad A_\xi^* E = p'\xi^*.$$

From (4.5) it follows easily that  $N$  is totally real in  $H_1^{2m+1}$  if and only if  $M$  is totally real in  $\mathbf{H}_m\mathbf{C}$  [18].

Now, for the diagram mentioned above, the relation between the parallelism of the operators  $f$  and  $f'$  is investigated.

LEMMA 4.1. *If the  $f$ -structure  $f$  in the normal bundle of  $M$  is parallel, then so does the operator  $f'$ .*

PROOF. Under the assumption that the operator  $f$  is parallel, for any vector field  $X$  in  $\mathcal{X}(M)$  and any normal field  $\xi$  in  $\mathcal{X}^\perp(M)$ , it follows from (4.4), (4.5) and the above equation that  $D_x^* f'(\xi^*) = 0$ . Since  $d\pi: \mathcal{X}^\perp(N) \rightarrow \mathcal{X}^\perp(M)$  is an isometric isomorphism, it means  $D_x^* f' = 0$ .

On the other hand, for any normal field  $\tau$  in  $\mathcal{X}^\perp(N)$ , the last equation of (3.4) is reduced to  $D_E f'(\tau) = -F'(A_\tau E - p'\tau)$ , which together with (3.3) gives  $D_E f' = 0$ . This means that  $f'$  is parallel.

THEOREM 4.2. *Let  $\mathbf{H}_m\mathbf{C}$  be a complex  $m$ -dimensional complex hyperbolic space of constant holomorphic sectional curvature  $-4$ , and  $M$  an  $n$ -dimensional complete and totally real submanifold of  $\mathbf{H}_m\mathbf{C}$ . If the  $f$ -structure in the normal bundle of  $M$  is parallel, then there exists a complex  $n$ -dimensional totally geodesic submanifold  $M_0 = \mathbf{H}_n\mathbf{C}$  of  $\mathbf{H}_m\mathbf{C}$  in which  $M$  is totally real.*

PROOF. By means of Proposition 3.2 and Lemma 4.1, there exists a  $(2n+1)$ -dimensional totally geodesic anti-de Sitter space  $H_1^{2n+1} = N_0$  of  $H_1^{2m+1}$ , in which  $N$  is totally real. The restriction of the projection  $\pi: H_1^{2m+1} \rightarrow \mathbf{H}_m\mathbf{C}$  to  $H_1^{2n+1}$  is denoted by the same  $\pi$ , and let  $M_0$  be the image of  $N_0$  under  $\pi$ . Then, since  $N_0$  is invariant under the multiplication by  $e^{i\theta}$ , the image  $M_0$  is a submanifold of  $\mathbf{H}_m\mathbf{C}$ . Induce a metric on  $M_0$  so that the projection  $\pi: N_0 \rightarrow M_0$  is a Riemannian submersion. For the isometric immersion  $i'_0: N \rightarrow N_0$  and the totally geodesic immersion  $i'_1: N_0 \rightarrow H_1^{2m+1}$ , smooth mappings  $i_0$  (resp.  $i_0$ ) of  $M$  into  $M_0$  (resp.  $M_0$  into

$H_m\mathbf{C}$ ) can be chosen in such a way that the diagrams are commutative. Then  $i_0$  and  $i_1$  make both isometric immersions, because  $\pi : N_0 \rightarrow M_0$  is also the Riemannian submersion. On the right diagram, let  $\alpha_1$  and  $\beta_1$  be the second fundamental forms for  $i'_1$  and  $i_1$ . By the similar way to (4.3) we get  $\alpha_1(X^*, Y^*) = \beta_1(X, Y)^*$  for any  $X$  and  $Y$  in  $\mathcal{X}(M_0)$ , which implies that  $\beta_1$  vanishes identically, i.e.,  $M_0$  is also totally geodesic. Since  $M_0$  is a Kaehlerian submanifold of  $H_m\mathbf{C}$ , it is the complex hyperbolic space  $H_n\mathbf{C}$  and  $M$  is totally real in  $H_n\mathbf{C}$ .

This concludes the proof.

**5. Principal circle bundles over a submanifold of  $P_m\mathbf{C}$ .**

Let  $M$  be an  $n$ -dimensional totally real submanifold of a complex  $m$ -dimensional complex projective space  $P_m\mathbf{C}$ . Then one can construct a principal circle bundle over the submanifold  $M$  with the projection  $\pi$  in such a way that  $\pi$  is compatible with the Hopt fibration  $\pi : S^{2m+1} \rightarrow P_m\mathbf{C}$ . Namely, the following diagram is commutative :

$$\begin{array}{ccc} \pi^{-1}(M) & \longrightarrow & S^{2m+1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & P_m\mathbf{C} \end{array}$$

By the similar verification to that stated in the previous section, the following theorem is proved.

**THEOREM 5.1.** *Let  $M$  be an  $n$ -dimensional complete and totally real submanifold of  $P_m\mathbf{C}$ . If the  $f$ -structure in the normal bundle on  $M$  is parallel, then there exists a totally geodesic submanifold  $M_0 = P_n\mathbf{C}$  of  $P_m\mathbf{C}$  in which  $M$  is totally real.*

The proof will be sketched. Let  $(\phi, E, \omega, G)$  be the Sasakian structure admitted in  $S^{2m+1}$  and  $(J, g)$  the Kaehlerian structure in  $P_m\mathbf{C}$ . By these structures, the set  $(P, F)$ ,  $(p, f)$  and  $(P', F')$ ,  $(p', f')$  of orthogonal operators are defined for the isometric immersions  $M \rightarrow P_m\mathbf{C}$  and  $N = \pi^{-1}(M) \rightarrow S^{2m+1}$ . As is well known,  $M$  is totally real in  $P_m\mathbf{C}$  if and only if  $N$  is totally real in  $S^{2m+1}$ , the parallelism of the  $f$ -structure  $f$  derives that  $F$  and  $p$  are both parallel by the same discussion as that developed in the previous sections, and moreover the operators  $F'$ ,  $p'$  and  $f'$  are also so. These properties imply that the distribution  $F'\mathcal{D}$  defined by  $p \rightarrow F'N_p$  is parallel with respect to the normal connection and  $\dim F'\mathcal{D} = n$ , and furthermore the first normal space  $N_p^1$  is the subspace of  $F'N_p$ . By means of the reduction theorem of Erbacher [ 4 ], it yields that there exists a totally geodesic  $S^{2n+1}$  of  $S^{2m+1}$  in which  $N$  is totally real (Proposition 3.3.). The restric-

tion of the Hopf fibration to  $S^{2n+1}$  gives a complete totally geodesic Kaehlerian submanifold  $M_0 = \pi(S^{2n+1})$  of  $P_n\mathbb{C}$ . Hence  $M_0$  is the complex projective space  $P_n\mathbb{C}$  and  $M$  is totally real in  $M_0$

Thus the proof is complete.

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Kyungpook University  
Taegu, 635  
Korea

University of Tsukuba  
Ibaraki, 305  
Japan