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YANG-MILLS CONNECTIONS AND THE INDEX BUNDLES

By

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Dedicated to Prof. Shingo Murakami on his sixtieth birthday

1. Introduction.

Let P be a C^∞ G-principal bundle over a compact connected, oriented Riemannian 4-manifold M and G be a compact semi-simple Lie group. The moduli space of all anti-self-dual Yang-Mills connections on P carries a finite dimensional space structure. As is well known, this moduli space is an effective tool in studying low dimensional topology and complex manifold theory ([4], [5]).

We shall investigate in this paper certain finite dimensional vector bundles, so called *'index bundles'*, over the moduli space, and then develop their geometry from the viewpoint of metrics, connections and also of curvature.

The motivation of this paper is to make a *'gauge theoretical'* study of the following conjecture which may be verifiable by algebro-geometrical methods: "if the base manifold M is a Kähler surface with an ample line bundle, then the moduli space admits a holomorphic vector bundle of positive first Chern class".

This conjecture is also mentioned in Proposition 12, [6] not over the moduli space, but over the set of holomorphic connections.

A Yang-Mills connection is defined as a connection which is stationary under the variation of the Yang-Mills functional. However, any connection is by definition a first order differential operator of special sort, namely a covariant differentiation.

In fact, each connection A on P gives rise to a covariant derivative ∇_A on any associated C^∞ vector bundle $\boldsymbol{E}$.

Here, we suppose that there exists another real (or complex) C^∞ vector bundle V over M with an elliptic operator $\mathcal{D}$; $\Gamma^1(V)\to\Gamma^2(V)$, where $\Gamma^i(V)$ are spaces of sections of certain vector bundles associated with V.

The bundle V is for example a holomorphic vector bundle, and $\mathcal{D}$ is the operator associated with the twisted Dolbeault complex

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$$\xrightarrow{\bar\partial} \Omega^{0,p}(V) \xrightarrow{\bar\partial} \Omega^{0,p+1}(V) \xrightarrow{\bar\partial},$$

namely,

$$\mathscr{D}=(\bar\partial^*, \bar\partial)\,;\ \oplus_{p,odd}\Omega^{0,p}(V) \longrightarrow \oplus_{p,even}\Omega^{0,p}(V).$$

Another example is the Dirac operator defined on the spinor bundles S; $\not\partial: \Gamma(S) \to \Gamma(S)$.

On the tensor product bundle $V\otimes \boldsymbol{E}$ we get a family of elliptic operators $\mathscr{D}_A$; $\Gamma^1(V\otimes \boldsymbol{E})\to\Gamma^2(V\otimes \boldsymbol{E})$ by coupling $\mathscr{D}$ with connections A.

Obviously $\operatorname{Ker}\mathscr{D}_A$ and $\operatorname{Coker}\mathscr{D}_A$ are finite dimensional, and from the Atiyah-Singer index theorem the difference of their dimensions, 'the numerical index' n-index$(\mathscr{D}_A)$, is independent of a choice of connection.

We move the connection A in the space $\mathcal{A}$ of connections on P to obtain a family of formal differences of vector spaces $\operatorname{Ind}\mathscr{D}_{\mathcal{A}}=\{\operatorname{Ind}\mathscr{D}_A;\ A\in\mathcal{A}\}$, where $\operatorname{Ind}\mathscr{D}_A=(\operatorname{Ker}\mathscr{D}_A)-(\operatorname{Coker}\mathscr{D}_A)$. Since the group $\mathcal{G}$ of gauge transformations of P acts equivariantly on operators $\mathscr{D}_A$, $\operatorname{Ind}\mathscr{D}_{\mathcal{A}}$ can be regarded as an element of the K-theoretical group $K(\mathscr{B})$, where $\mathscr{B}=\mathcal{A}/\mathcal{G}$ is the space of gauge equivalence classes of connections on P.

Thus, we can discuss such virtual vector bundles over the infinite dimensional Banach manifold $\mathscr{B}$, and also over a specific finite dimensional subspace of $\mathscr{B}$, the moduli space $\mathcal{M}$ of anti-self-dual connections on P. Since $\mathscr{B}$ is not compact, one has to pay attention to apply the index theorem for family to $\mathscr{B}$.

The following is known with respect to the index bundles. The index formula for a family $\operatorname{Ind}\not\partial_A$ of Dirac operators $\not\partial$ coupled with connections A gives the Chern character formula in an integral form as $\operatorname{ch}(\operatorname{Ind}\not\partial_{\mathcal{A}})=\int_M \hat{a}(M)\operatorname{ch}(\mathcal{E})$, where $\hat{a}(M)$ is the characteristic form of the spinor bundle so that the integral $\int_M \hat{a}(M)$ is the $\hat{A}$-genus and $\mathcal{E}$ is the vector bundle associated to the Poincaré bundle $\boldsymbol{P}$ over $M\times\mathscr{B}$ (see [2], for details). Therefore, the Chern forms are computable in principle.

On the other hand, $\det\operatorname{Ind}\mathscr{D}_A=(\wedge^a(\operatorname{Ker}\mathscr{D}_A))\otimes(\wedge^b(\operatorname{Coker}\mathscr{D}_A))^*$, where $a=\dim\operatorname{Ker}\mathscr{D}_A$, $b=\dim\operatorname{Coker}\mathscr{D}_A$, defines a proper line bundle ([3], [15]). Bismut and Freed applied Quillen's superconnection formalism and also the heat equation method to get a Hermitian connection and compute the curvature of the determinant bundle $\det\operatorname{Ind}\mathscr{D}_{\mathcal{A}}$ ([3], see also [6], [15]).

Their method is analytical. Here, we present a method which is simple and accessible in the case of vanishing $\operatorname{Coker}\mathscr{D}_A$. Namely, we keep the viewpoint that the natural projection $\mathcal{A}\to\mathscr{B}$ defines a principal bundle structure with infinite dimensional structure group $\mathcal{G}$, and further inherits a natural connection.

Regarding $\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}}$ over $\mathscr{B}$ as a (virtual) subbundle of a certain Hilbert space vector bundle associated with the bundle $\mathcal{A}\to\mathscr{B}$, we restrict the connection to $\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}}$. This defines not only a connection but also a second fundamental form. Therefore the vector bundle version of the Gauss equation in the curvature is available and is summarized in the following theorem (see Propositions 2.1 and 3.1).

THEOREM 1. (i) *There is a family of gauge equivariant elliptic operators* $\mathscr{D}_A$; $\Gamma^1(V\otimes \boldsymbol{E})\to\Gamma^2(V\otimes \boldsymbol{E})$ *parametrized by* $A\in\mathcal{A}$, (ii) *if* $\mathrm{Coker}\,\mathscr{D}_A=0$ *for all* A, *then* $\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}}=\{\mathrm{Ker}\,\mathscr{D}_A;\ A\in\mathcal{A}\}$ *defines a* C^∞ *vector bundle over* $\mathscr{B}$, *the space of gauge equivalence classes of connections,* (iii) *the natural projection* $\mathcal{A}\to\mathscr{B}$ *is a fibration with structure group* $\mathcal{G}/Z$. *It is equipped with a natural connection* ω *which induces a connection* $\bar{\boldsymbol{D}}$ *on the associated Hilbert vector bundle* $\mathcal{A}\times_{\mathcal{G}/Z}\Gamma^1(V\otimes\boldsymbol{E})$ *and a connection* $\boldsymbol{D}$ *on the subbundle* $\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}}$ *together with the second fundamental form* σ, *and* (iv) *for the curvature forms* $\bar{\Omega}$ *and* Ω *of* $\bar{\boldsymbol{D}}$ *and* $\boldsymbol{D}$ *the Gauss equation holds*

$$\langle\bar{\Omega}(X, Y)\xi, \eta\rangle=\langle\Omega(X, Y)\xi, \eta\rangle+\langle\sigma_X\xi, \sigma_Y\eta\rangle-\langle\sigma_Y\xi, \sigma_X\eta\rangle.$$

The main results which can be derived from the above Gauss equation are stated as follows.

THEOREM 2. *Let* P *be a principal bundle with compact semi-simple structre group* G *over a compact Kähler surface* M. *Let* V *be a Hermitian vector bundle with an Einstein-Hermitian connection* ∇ (*see* (7.1) *in* §7 *for the definition*). *For the elliptic operator* $\mathscr{D}$; $\Omega^1(V)\to(\Omega^0\oplus\Omega^2_+)(V)$ *associated with the connection* ∇ *we assume that* $\mathrm{Coker}\,\mathscr{D}_A=0$ *for all anti-self-dual connections* A *on* P. *Then* (i) *the complexification of the index bundle* $\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}}$ *decomposes as*

$$(\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}})^C=(\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}})^{1,0}\oplus(\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}})^{0,1}$$

with respect to the almost complex structure defined on M, *and* (ii) *when* $(\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}})^C$ *is restricted to the moduli space* $\mathcal{M}$ *of generic anti-self-dual connections on* P, *the curvature from* Ω *of the induced connection* $\boldsymbol{D}$ *on* $(\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}})^{1,0}$ (*or* $(\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}})^{0,1}$) *has type* (1, 1).

The vanishing of the (0, 2)-part of the curvature form gives the integrability of the holomorphic structure, and the complex vector bundles $(\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}})^{1,0}$ and $(\mathrm{Ind}\,\mathscr{D}_{\mathcal{A}})^{0,1}$ inherit a holomorphic vector bundle structure such that a section s is holomorphic if and only if $\boldsymbol{D}^{0,1}s=0$.

Remark that all holomorphic line bundles over a compact Kähler surface admit a Hermitian fibre metric with an Einstein-Hermitian connection.

For these holomorphic vector bundles $(\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}})^{1,0}$ and $(\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}})^{0,1}$ one can ask then how their first Chern class can be calculated.

The first Chern class is represented by the Ricci form up to a universal constant. The Ricci form is the trace of the curvature endomorphism. So, we establish the Ricci form formula by applying the Gauss equation as stated in Theorem 6.1. Namely, the Ricci form Φ of the index bundle $(\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}})^{1,0}$ restricted to the moduli space $\mathcal{M}$ can be expressed in terms of the second fundamental form and the ambient space curvature term $\bar{\Phi}$.

On the other hand, to a principal bundle P over a 4-manifold with a compact, semi-simple group G we associate the so-called Poincaré bundle $\boldsymbol{P}$ over $M\times\mathcal{B}$ with group G which is universal in the sense of deformations of the bundle P ([2]). As is seen in §2, by pulling back the natural connection ω on the bundle $\mathcal{A}\to\mathcal{B}$ to $\boldsymbol{P}$, we get a connection $\boldsymbol{A}$ on it which is the one defined by Atiyah and Singer ([2]).

THEOREM 3. *Let P be a principal bundle over a compact Kähler surface with structure group G, and $\boldsymbol{P}$ the Poincaré bundle defined over $M\times\mathcal{B}$. We restrict $\boldsymbol{P}$ to $M\times\mathcal{M}$, where $\mathcal{M}$ is the moduli space of generic anti-self-dual connections on P. Then the curvature form of the connection $\boldsymbol{A}$ is of type* (1,1) *so that $\boldsymbol{A}$ defines a holomorphic structure on any complex vector bundle associated with $\boldsymbol{P}\to M\times\mathcal{M}$.*

For the discussion of this theorem, see §2 and Proposition 5.2.

Another application of the Gauss equation on the curvature is to derive the curvature formula for the Riemannian structure defined on the moduli space $\mathcal{M}$ of generic anti-self-dual connections. Although the curvature formula is obtained in [10] by tedious calculation, we obtain it here in a transparent manner by making use of the Gauss equation (Theorem 4.3).

In the final section the vanishing criterion on $\mathrm{Coker}\,\mathcal{D}_A$ is discussed in the case where M is a Kähler surface and V is an Einstein-Hermitian vector bundle. We prove the following vanishing theorem

THEOREM 4. *Let P be a principal bundle over a compact Kähler surface (M, h) as before and $\boldsymbol{E}$ an associated complex vector bundle. Let V be a Hermitian vector bundle with an Einstein-Hermitian connection ∇ for the Einstein constant $\lambda=\frac{4\pi}{\mathrm{vol}(M)}\int_M c_1(V)/\mathrm{rank}(V)\wedge\omega_h$. Then* $\mathrm{Coker}\,\mathcal{D}_A$ *vanishes for each anti-self-dual connection A on P provided the following two conditions are satisfied*

(i) $\lambda<\min\left(0, \frac{1}{2}s\right)$, *where s is the scalar curvature of the metric h and*

(ii) *for the Laplace operator* $\Delta_A''=-\sum h^{ij}\nabla_i\nabla_j$ *acting on* $\Omega^0(V\otimes E)$

$$\mathrm{Ker}\,(\Delta_A''+\lambda\,\mathrm{id})\cap\mathrm{Ker}\,\partial_A=0$$

holds (here, ∂_A *means the* (1, 0)*-partial covariant derivative).*

2. A principal bundle with group $\mathcal{G}$.

We denote by $\mathcal{A}$ the space of all irreducible connections on P. The action of $\mathcal{G}$ on $\mathcal{A}$, $(g, A)\mapsto g(A)=g^{-1}dg+g^{-1}\cdot A\cdot g$, defines the space $\mathcal{B}$ of orbits of irreducible connections on P so that the projection $\mathcal{A}\to\mathcal{B}$; $A\to[A]$ is equipped with a principal bundle structure. Since the stabilizer of each irreducible connection is the center of the group $\mathcal{G}$, we should take the quotient group $\mathcal{G}/Z$ instead of $\mathcal{G}$ in order to make the action free.

At each connection A the tangent space $T_A\mathcal{A}$ splits as a sum of vertical and horizontal subspaces

$$T_A\mathcal{A}=\mathcal{V}_A\oplus\mathcal{H}_A.$$

In fact $T_A\mathcal{A}$ is identified with $\Omega^1(\mathrm{ad}\,P)$ for the adjoint bundle $\mathrm{ad}\,P$, the subspace $\mathcal{V}_A$ with $\mathrm{Im}\{D_A: \Omega^0(\mathrm{ad}\,P)\to\Omega^1(\mathrm{ad}\,P)\}$ and $\mathcal{H}_A$ with $\mathrm{Ker}\,D_A^*$, where D_A is the covariant derivative induced by A and D_A^* is its formal adjoint.

Since $\mathcal{G}$ acts also on $\Omega^p(\mathrm{ad}\,P)$ as $g(\phi)=g^{-1}\cdot\phi\cdot g$, in such a way that $D_{g(A)}g(\phi)=g(D_A\phi)$ holds, this splitting is $\mathcal{G}$-equivariant. The vertical subspace is isomorphic through D_A to $\Omega^0(\mathrm{ad}\,P)$, the Lie algebra of the group $\mathcal{G}$. So, we have a distribution of horizontal subspaces which defines a connection on the bundle $\mathcal{A}\to\mathcal{B}$. Its connection form ω: $T\mathcal{A}\to\Omega^0(\mathrm{ad}\,P)$ can be explicitly expressed in the following way.

PROPOSITION 2.1. (i) *The principal bundle* $\mathcal{A}\to\mathcal{B}$ *with group* $\mathcal{G}/Z$ *admits a natural connection with connection form* ω *given by*

$$\omega(\alpha)=G_A(D_A^*\alpha),\ \alpha\in T_A\mathcal{A} \tag{2.1}$$

where G_A *is the inverse of the operator* $D_A^*D_A$; $\Omega^0(\mathrm{ad}\,P)\to\Omega^0(\mathrm{ad}\,P)$.

(ii) *The curvature form* $\Omega^\omega=d\omega+\frac{1}{2}[\omega\wedge\omega]$, *which is the* $\Omega^0(\mathrm{ad}\,P)$*-valued* 2*-form on* $\mathcal{A}$, *is represented by*

$$\Omega^\omega(X, Y)=-2G_A(\{X, Y\}), \tag{2.2}$$

$X, Y\in T_{[A]}\mathcal{B}$ $([A]\in\mathcal{B})$, *where we restrict* Ω^ω *to the origin of a slice* $\mathcal{S}$ *in* $\mathcal{A}$ *at* A.

REMARKS. (i) The bilinear operation $\{\cdot, \cdot\}$; $\Omega^1(\mathrm{ad}\,P)\times\Omega^1(\mathrm{ad}\,P)\to\Omega^0(\mathrm{ad}\,P)$

is defined as

$$\{X, Y\}=\sum_{i,j} h^{ij}[X_i, Y_j], \tag{2.3}$$

$X=\sum_i X_i dx^i$, $Y=\sum_j Y_j dx^j$. Here, (h^{ij}) is the inverse matrix of the base Riemannian metric components (h_{ij}). Then we have $D^*_{A+X}Y-D^*_A Y=-\{X, Y\}$, $X, Y\in\Omega^1(\mathrm{ad}\,P)$. (ii) For each A in $\mathcal{A}$ a slice $\mathcal{S}$ is a subset of $\mathcal{A}$ transversal to gauge orbits and hence it gives a local coordinate neighborhood centered at $[A]$ in $\mathcal{B}$. Actually, $\mathcal{S}$ is defined as $\{\alpha\in\Omega^1(\mathrm{ad}\,P)\,;\, D^*_A\alpha=0,\ |\alpha|<\varepsilon\}$ and the projection restricted to $\mathcal{S}$; $\mathcal{S}\to\mathcal{B}$ covers a neighborhood at $[A]$ so that these slices define a Banach (or Hilbert) manifold structure on $\mathcal{B}$, and then each tangent vector to $\mathcal{B}$ is identified with a vector in $\mathrm{Ker}\,D^*_A$ at the corresponding connection in $\mathcal{S}$ ([10], section 2).

PROOF. (i) We easily see that $\omega(D_A\phi)=\phi$ for any vertical vector $D_A\phi$. Since the usual gauge action on $\mathcal{A}$ gives the right action of the bundle $\mathcal{A}$ over $\mathcal{B}$, we have $\omega_{g(A)}(R_{g*}\alpha)=g(\omega_A(\alpha))$. Therefore the form ω is the connection form of the connection.

(ii) Extending X and Y in $T_{[A]}\mathcal{B}\cong\mathrm{Ker}\,D^*_A$ to vector fields $\tilde{X}$ and $\tilde{Y}$ over $\mathcal{S}$, we have at $[A]$ $\Omega^\omega(X, Y)=X\omega(\tilde{Y})-Y\omega(\tilde{X})$, because $\omega(X)=\omega(Y)=0$. Since $X\omega(\tilde{Y})$ is the derivative at $t=0$ of $\omega(\tilde{Y})$ along a line $A_t=A+tX$, it equals $-G_A(\{X, Y\})$.

With respect to Proposition 2.1 we should comment on the universal connection on the Poincaré bundle. In [2] Atiyah and Singer define the Poincaré bundle $\boldsymbol{P}$, and introduce a connection $\boldsymbol{A}$ on $\boldsymbol{P}$. The action of the gauge group $\mathcal{G}$ on the product $P\times\mathcal{A}\,;(u, A)\mapsto(g(u), g(A))$ induces a bundle $\boldsymbol{P}=(P\times\mathcal{A})/\mathcal{G}\to M\times\mathcal{B}=G\backslash\boldsymbol{P}$. By making use of our natural connection ω on $\mathcal{A}\to\mathcal{B}$, we can define the connection $\boldsymbol{A}$ on $\boldsymbol{P}$ in the following way.

Since the bundle $\boldsymbol{P}$ has a local trivializing neighborhood $U\times\mathcal{S}$ at each $(x, [A])\in M\times\mathcal{B}$ in terms of a trivializing neighborhood U of P at x and a slice $\mathcal{S}$ through $[A]$, we set $\boldsymbol{A}$ to be equal to $A'=A+\alpha$ when restricted to $A'=A+\alpha\in\mathcal{S}$ and equal to $\mathrm{ev}_x(\omega)$ over $\{x\}\times\mathcal{B}$, here ev_x is the evaluation map at $x\,;\,\Omega^0(\mathrm{ad}\,P)\to(\mathrm{ad}\,P)_x$ and $(\mathrm{ad}\,P)_x$ is identified with the Lie algebra of G through the trivialization over U.

So, we readily calculate the curvature $\boldsymbol{F}$ of $\boldsymbol{A}$ with respect to the product space structure; $\boldsymbol{F}=\boldsymbol{F}^{2,0}+\boldsymbol{F}^{1,1}+\boldsymbol{F}^{0,2}$ as $\boldsymbol{F}^{2,0}$ is $F(A)$, the curvature of A on P, $\boldsymbol{F}^{1,1}$ is represented by $\boldsymbol{F}^{1,1}(u, X)=-X(u)$ for $(u, X)\in T_{(x,[A])}(M\times\mathcal{B})$ $(u\in T_xM,\ X\in\mathrm{Ker}\,D^*_A)$, and $\boldsymbol{F}^{0,2}=\mathrm{ev}_x(\Omega^\omega)=-2\mathrm{ev}_x(G_A\{\cdot, \cdot\})$.

3. The index bundle.

Let V be a C^∞ vector bundle over M with a fibre metric. We suppose that V is equipped with an elliptic operator $\mathcal{D}$; $\Gamma^1(V)\to\Gamma^2(V)$.

For the convenience of the reader we assume that $\mathcal{D}$ is the Atiyah-Hitchin-Singer operator;

$$\mathcal{D}=(\nabla^*,\ d^{\nabla,+});\ \Omega^1(V)\longrightarrow(\Omega^0\oplus\Omega^2_+)(V),$$

where ∇^* is the L_2-adjoint of a metric connection ∇ on V, and $d^{\nabla,+}$ is the self-dual part of the exterior derivative d^∇.

We can of course consider the case of Dirac operators on the spinor bundles, and also the case of twisted Dolbeault operators over a complex Kähler surface.

Let E be a vector space on which G acts through a representation ρ and $\boldsymbol{E}$ the associated C^∞ vector bundle $P\times_\rho E$. Then tensoring V with $\boldsymbol{E}$ we form a new vector bundle $V\otimes\boldsymbol{E}$. It is equipped with a family of connections ∇_A, the metric connection ∇ coupled to connections ∇_A^E on $\boldsymbol{E}$ as A moves on P.

The representation ρ on the vector space E induces in a natural way a representation of $\mathcal{G}$ on sections of $\boldsymbol{E}$ and hence on sections of $V\otimes\boldsymbol{E}$. Moreover it induces an action on coupled connections $\nabla_{g(A)}(\rho g)(\phi)=(\rho g)(\nabla_A\phi)$, for $A\in\mathcal{A}$, $\phi\in\Omega^0(V\otimes\boldsymbol{E})$, $g\in\mathcal{G}$.

Consider the first order elliptic operators

$$\mathcal{D}_A=(\nabla_A^*,\ d_A^+);\ \Omega^1(V\otimes\boldsymbol{E})\longrightarrow(\Omega^0\oplus\Omega^2_+)(V\otimes\boldsymbol{E})$$

parametrized by connections on P. Since each $\mathcal{D}_A$ is $\mathcal{G}$-equivariant, the index bundle $\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}}=\{\mathrm{Ind}\,\mathcal{D}_A\}$, $\mathrm{Ind}\,\mathcal{D}_A=(\mathrm{Ker}\,\mathcal{D}_A)-(\mathrm{Coker}\,\mathcal{D}_A)$ is defined as an element of $K(\mathcal{B})$.

From now on, we assume $\mathrm{Coker}\,\mathcal{D}_A=\{0\}$ for each connection A. We will discuss this assumption in the last section.

The index bundle is then a finite dimensional subbundle of an infinite dimensional space bundle $Q=\mathcal{A}\times_{\mathcal{G}/Z}\Omega^1(V\otimes\boldsymbol{E})$. We define an L_2-inner product on $\Omega^1(V\otimes\boldsymbol{E})$ thanks to the metric structures on V and $\boldsymbol{E}$ so that the Hilbert space bundle Q is decomposed as $Q=\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}}\oplus(\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}})^\perp$, where the orthogonal complement $(\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}})^\perp$ is spanned by eigenspaces corresponding to positive eigenvalues of the operator $\mathcal{D}_A^*\mathcal{D}_A$.

Letting $\bar{\boldsymbol{D}}$ be the connection on Q induced by the connection ω on the bundle $\mathcal{A}\to\mathcal{B}$, we have

$$\bar{\boldsymbol{D}}_X\xi=\boldsymbol{D}_X\xi+\sigma_X\xi \tag{3.1}$$

where ξ is a section of the bundle $\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}}$ and $X \in T_{[A]}\mathcal{B}$ so that $\boldsymbol{D}$ and σ are a connection and the second fundamental form.

PROPOSITION 3.1 (*Gauss equation*). *Denote by* $\bar{\Omega}$ *and* Ω *the curvature of* $\bar{\boldsymbol{D}}$ *and* $\boldsymbol{D}$, *respectively, and by* $\langle\cdot,\cdot\rangle$ *the* L_2*-inner product on* $\Omega^1(V \otimes \boldsymbol{E})$. *Then*

$$\langle \bar{\Omega}(X, Y)\xi, \eta\rangle = \langle \Omega(X, Y)\xi, \eta\rangle + \langle \sigma_X \xi, \sigma_Y \eta\rangle - \langle \sigma_Y \xi, \sigma_X \eta\rangle \tag{3.2}$$

where ξ *and* η *are in the fibre of* $\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}}$ *at* $[A]$, *and* X, Y *are tangent vectors to* $\mathcal{B}$ *at* $[A]$.

In the case of submanifolds this proposition can be found in any textbook in differential geometry (see for example [11]).

4. The second fundamental from and the curvature formula.

We restrict the bundle $\mathcal{A} \to \mathcal{B}$ to the moduli space $\mathcal{M}$ of anti-self-dual connections on P.

Before getting a formula for the second fundamental form σ, we recall the definition of an anti-self-dual connection. A connection A on P is called anti-self-dual if the curvature form $F(A)$ is anti-self-dual as an $\mathrm{ad}\,P$-valued 2-form. The set of gauge equivalence classes of anti-self-dual connections on P are parametrized by $\mathcal{M}$, which is a subset of $\mathcal{B}$. To each irreducible anti-self-dual connection A we associate the Atiyah-Hitchin-Singer complex

$$0 \longrightarrow \Omega^0(\mathrm{ad}\,P) \xrightarrow{D_A} \Omega^1(\mathrm{ad}\,P) \xrightarrow{d_A^+} \Omega^2_+(\mathrm{ad}\,P) \longrightarrow 0$$

which gives information on the infinitesimal behavior of $\mathcal{M}$ near $[A]$ (see [10] for details). In fact, when the second cohomology group $H_A^2 \cong \mathrm{Ker}\, d_A^+ d_A^{+*}$ vanishes, by the slice argument, the Kuranishi map $\mathcal{M}$ is locally diffeomorphic to a ball in the first cohomology group H_A^1 and the tangent space of $\mathcal{M}$ at $[A]$ can be identified with H_A^1.

We call an irreducible anti-self-dual connection generic if $H_A^2 = \{0\}$. In what follows $\mathcal{M}$ will mean the moduli space of 'generic' anti-self-dual connections on P, which turns out to be a smooth manifold.

Note that as shown by the Atiyah-Singer index theorem $\dim_{\boldsymbol{R}} H_A^1$ is given by the numerical index n-index(D_A^*, d_A^+), which is independent of the choice of A.

Define now a $C^\infty(M)$-bilinear map

$$\Omega^1(\mathrm{ad}\,P) \times \Omega^1(V \otimes \boldsymbol{E}) \longrightarrow (\Omega^0 \oplus \Omega^2_+)(V \otimes \boldsymbol{E})$$

$$(X, \phi) \longmapsto \rho(X)\circ\phi$$

by

(4.1) $$\rho(X)\circ\phi=\mathcal{D}_A\phi-\mathcal{D}_{A+X}\phi$$

for some connection A.

The representation ρ of G on E canonically induces an infinitesimal action ρ of $\Omega^0(\mathrm{ad}\,P)$ on $\Omega^0(V\otimes \boldsymbol{E})$. We extend this action in a natural way over $\Omega^1(\mathrm{ad}\,P)$ so that we have a homomorphism $\rho: \Omega^1(\mathrm{ad}\,P)\to\Omega^1(\mathrm{End}\,(V\otimes\boldsymbol{E}))$, $X\mapsto\rho(X)$.

Then $\rho(X)\circ\phi$ is written as

$$\rho(X)\circ\phi=(\rho(X)\,\lrcorner\,\phi,\ (\rho(X)\wedge\phi)^+),$$

where

$$\rho(X)\,\lrcorner\,\phi=\sum_{i,j} h^{ij}\rho(X_i)\phi_j,\quad X=\sum X_i dx^i,\quad \phi=\sum\phi_j dx^j$$

and $(\rho(X)\wedge\phi)^+$ is the self-dual part of $(\rho(X)\wedge\phi)=\sum\rho(X_i)\phi_j dx^i\wedge dx^j$.

PROPOSITION 4.1. *The second fundamental form σ has the following form,*

(4.2) $$\sigma_X\xi=\boldsymbol{G}_A\mathcal{D}_A^*(\rho(X)\circ\xi),$$

$X\in T_{[A]}\mathcal{M}$, $\xi\in\mathrm{Ker}\,\mathcal{D}_A$. *Here, $\boldsymbol{G}_A$ denotes the Green operator for $\mathcal{D}_A^*\mathcal{D}_A$.*

PROOF. To show (4.2) we have to know the value of the covariant derivative $\boldsymbol{D}_X\xi$ for a section ξ of $Q\to\mathcal{B}$ locally defined at $[A]$. Choose a slice $\mathcal{S}$ at A in $\mathcal{A}$, and denote by $\mathcal{W}$ the image of $\mathcal{S}$ in $\mathcal{B}$ by the projection. Since the assignment $A'\in\mathcal{S}$ to each $[A']\in\mathcal{W}$ gives a section of the principal bundle $\mathcal{A}\to\mathcal{B}$, and hence ξ can be considered as a map; $\mathcal{S}\to\Omega^1(V\otimes\boldsymbol{E})$, by the definition of ω we have $\bar{\boldsymbol{D}}_X\xi=\left(\frac{d}{dt}\xi_t|_{t=0}+\rho(\omega(X))\xi\right)$, where we put $\xi_t=\xi(A+tX)$. Then $\bar{\boldsymbol{D}}_X\xi$ reduces to $\frac{d}{dt}\xi_t|_{t=0}$, because $X\in\mathrm{Ker}\,\mathcal{D}_A^*$ and hence the value $\omega(X)$ vanishes from Proposition 2.1.

Since $\dim\mathrm{Ker}\,\mathcal{D}_{A_t}$ for $A_t=A+tX$ is independent of t, ξ_t is written as $\xi_t=\sum_{i=1}^k\xi^i(t)\beta_i(t)$ with respect to an orthonormal basis $\{\beta_i(t), 1\leqq i\leqq k\}$ of $\mathrm{Ker}\,\mathcal{D}_{A_t}$ which depends smoothly on t. Then the value of the second fundamental form $\sigma_X\xi=(\bar{\boldsymbol{D}}_X\xi)^\perp$ reduces to $\sum\xi^i(0)(\dot\beta_i(0))^\perp$, where the dot indicates the differentiation with respect to t. Moreover observing that $\boldsymbol{G}_A(\mathcal{D}_A^*\mathcal{D}_A)$ gives just the orthogonal projection to the orthogonal complement of $\mathrm{Ker}\,\mathcal{D}_A$ we have that it equals $\sum\xi^i(0)\boldsymbol{G}_A((\mathcal{D}_A^*\mathcal{D}_A)(\dot\beta_i(0))$.

On the other hand by differentiating $(\mathcal{D}_{A_t}^*\mathcal{D}_{A_t})\beta_i(t)=0$ at $t=0$, we get

(4.3) $$(\mathcal{D}_A^*\mathcal{D}_A)(\dot\beta_i(0))+\left(\frac{d}{dt}\Big|_{t=0}(\mathcal{D}_{A_t}^*\mathcal{D}_{A_t})\right)(\beta_i(0))=0,$$

$1 \leqq i \leqq k$. Since $\beta=\beta_i(0) \in \operatorname{Ker} \mathcal{D}_A$ lies in the ambient space $\Omega^1(V \otimes \boldsymbol{E})$, the second term in the right hand side of (4.3) becomes $\mathcal{D}_A^*(-\rho(X) \circ \beta)$. Now (4.2) follows.

Since from Proposition 2.1 we have $\bar{\Omega}(X, Y)\xi=-2\rho(G_A\{X, Y\})\xi$, we derive the curvature formula from the Gauss equation.

PROPOSITION 4.2. *Let $\boldsymbol{D}$ be the naturally defined connection on the index bundle* $\operatorname{Ind} \mathcal{D}_{\mathcal{A}}$. *Then its curvature is represented as*

$$
\begin{aligned}
\langle \Omega(X, Y)\xi, \eta\rangle = & -2\langle \rho(G_A\{X, Y\})\xi, \eta\rangle \\
& -\langle \boldsymbol{G}_A \mathcal{D}_A^*(\rho(X)\circ\xi), \boldsymbol{G}_A \mathcal{D}_A^*(\rho(Y)\circ\eta)\rangle \\
& +\langle \boldsymbol{G}_A \mathcal{D}_A^*(\rho(Y)\circ\xi), \boldsymbol{G}_A \mathcal{D}_A^*(\rho(X)\circ\eta)\rangle .
\end{aligned} \tag{4.4}
$$

We should comment on the canonical Riemannian structure defined on the moduli space $\mathcal{M}$ of anti-self-dual connections on P. The Riemannian structure was defined by the crucial aid of the Kuranishi map, a map linearizing $\mathcal{M}$ and also the Hodge theory relating the Atiyah-Hitchin-Singer deformation complex ([10]).

Also there are different definitions of Riemannian structures and also different ways to define the Riemannian structure. Indeed, S. Kobayashi applied in [12] the method of submersion due to O'Neill to discuss it. But we are able to apply in a direct way the curvature formula of the index bundle in the trivial case where the vector bundle V is the trivial bundle $M \times \boldsymbol{R}$ and the operator $\mathcal{D}$ is (d^*, d^+); $\Omega^1 \to \Omega^0 \oplus \Omega_+^2$, and $\boldsymbol{E}$ is the adjoint bundle $\operatorname{ad} P$ with the adjoint representation as ρ.

In fact, since the tangent space to the moduli space $\mathcal{M}$ at $[A]$ is identified with $H_A^1 \cong \operatorname{Ker}(D_A^*, d_A^+)$ and, since the inner product on this space defining the Riemannian structure is the restriction of the L_2-inner product on $\Omega^1(\operatorname{ad} P)$, and moreover the canonically induced connection $\boldsymbol{D}$ preserves the structure, we get, by using formulae $\rho(X) \rfloor Y=\{X, Y\}$, $(\rho(X) \wedge Y)^+=[X \wedge Y]^+$, the curvature formula (Theorem 5.1 in [10]).

THEOREM 4.3. *At $[A]$ the Riemannian curvature tensor is expressed by*

$$
\begin{aligned}
\langle R(X, Y)Z, W\rangle = & -\langle \{X, W\}, G_A\{Z, Y\}\rangle \\
& -2\langle \{Z, W\}, G_A\{X, Y\}\rangle + \langle \{Y, W\}, G_A\{Z, X\}\rangle \\
& +\langle [X \wedge W]^+, G_A[Z \wedge Y]^+\rangle - \langle [Y \wedge W]^+, G_A[Z \wedge X]^+\rangle ,
\end{aligned} \tag{4.5}
$$

where X, Y, Z, W are tangent vectors in $T_{[A]}\mathcal{M}$.

5. The holomorphic structure of the index bundle.

We assume in this section that the base 4-manifold M is a Kähler surface. Then the moduli space $\mathcal{M}$ of anti-self-dual connections on P carries in a natural way a complex manifold structure and a Kähler structure ([8], Theorem 2 and [10], §4).

In fact, in a complex analytic way each anti-self-dual connection A yields a (0, 1)-connection $A^{(0,1)}$ whose curvature form $F(A^{(0,1)})=\bar{\partial}A^{(0,1)}+\frac{1}{2}[A^{(0,1)}\wedge A^{(0,1)}]$ vanishes. This (0, 1)-connection induces a holomorphic structure on the complexified bundle P^C. Thus we have a canonical map from the moduli space $\mathcal{M}$ to the moduli space of holomorphic (0, 1)-connections $\mathcal{M}_{\text{hol}}$ modulo complex gauge transformations which carries a natural complex structure so that $\mathcal{M}$ admits a complex manifold structure through this canonical map (see [8] for details).

On the other hand from a differential geometric viewpoint the base almost complex structure $I\,;\,\Omega^1\to\Omega^1$ canonically extends to the space $\Omega^1(\operatorname{ad}P)$ and this restricts well on each $H_A^1(\cong\operatorname{Ker}(D_A^*,\,d_A^+))\cong T_{[A]}\mathcal{M}$. It is observed that the almost complex structure $I|_{H_A^1}$ coincides with the complex structure on $\mathcal{M}$ given by $\mathcal{M}_{\text{hol}}$, and the canonical Riemannian structure becomes on the complex manifold $\mathcal{M}$ a Kähler structure.

We remark that the base almost complex structure I extends also to $\Omega^1(V)$ for any vector bundle V.

The first observation is then the following

PROPOSITION 5.1. *The curvature form Ω^ω of the natural connection ω on the bundle $\mathcal{A}\to\mathcal{B}$ restricted to $\mathcal{M}$ is of type* (1, 1). *Hence, any finite dimensional complex vector bundle associated with $\mathcal{A}|_{\mathcal{M}}\to\mathcal{M}$ is endowed with a holomorphic structure compatible with the induced connection.*

REMARK. Choose a point x in M and consider the $\boldsymbol{E}$-framed moduli space $\mathcal{M}_x$, i.e., the $\mathcal{G}$-quotient of the set $\{(A,\phi)\}$ for the associated vector bundle $\boldsymbol{E}$, where A are anti-self-dual and $\phi\in\boldsymbol{E}_x$. This is a vector bundle over $\mathcal{M}$ with fibre E. Since $\mathcal{M}_x$ is associated with the bundle $\mathcal{A}|_{\mathcal{M}}\to\mathcal{M}$ and the curvature of the induced connection is seen to be $\rho(\operatorname{ev}_x(\Omega^\omega))$, the complexified framed moduli space becomes from the proposition a holomorphic vector bundle.

PROOF. At $[A]\in\mathcal{M}$, $\Omega^\omega(X,Y)=-2G_A\{X,Y\}$, $X,Y\in T_{[A]}\mathcal{M}$. It suffices to show $\{Z,W\}=0$ for any pair of complex tangent vectors Z,W of type (1, 0) or of type (0, 1). Each vector of type (1, 0) (or of type (0, 1)) is a $\pm\sqrt{-1}$ eigenvector of I, respectively. Furthermore, $\{X, IY\}$ is given by the inner product of an ad$P$-valued 2-form $[X\wedge Y]^{+}$ with the base Kähler form, and hence is symmetric with respect to real vectors X, Y (see [9], Lemma 5.3, since this lemma also holds in any Kähler case). Thus $\{Z, W\}$ vanishes. Since the type condition on curvature is exactly the integrability of holomorphic structure ([1], Theorem 5.1), the proposition follows.

PROPOSITION 5.2. *Let $\boldsymbol{P}$ be the Poincaré bundle over $M\times\mathcal{M}$ related to a bundle P over a Kähler surface M. Then the connection $\boldsymbol{A}$ has curvature of type* (1, 1), *and hence any complex vector bundle associated with $\boldsymbol{P}$ admits a holomorphic structure compatible with the induced connection.*

The proposition follows from the expression of the curvature $\boldsymbol{F}$ of $\boldsymbol{A}$ given in § 2.

The Poincaré bundle with the connection corresponds to the algebro-geometrical notion of universal bundle which might impose interesting problem to us ([13], [14]). We remark that if we restrict $\boldsymbol{P}$ to $\{x\}\times\mathcal{M}$, then it can be considered as the moduli space of framed anti-self-dual connections on P, that is, the gauge equivalence classes of (A, u) where A is anti-self-dual and $u\in P_x$, the fibre of P over x.

Now, we are in a position to assert the integrability of the holomorphic structure of the index bundle.

THEOREM 5.3. *Let V be a Hermitian vector bundle with an Einstein-Hermitian connection ∇ and $\mathcal{D}_A$ elliptic operators coupled to connections A on P; $\Omega^1(V\otimes\boldsymbol{E})\to(\Omega^0\oplus\Omega^2_+)(V\otimes\boldsymbol{E})$ satisfying* $\mathrm{Coker}\,\mathcal{D}_A=\{0\}$. *Then,* (i) *the complexification of* $\mathrm{Ind}\,\mathcal{D}_{\mathcal{A}}=\{\mathrm{Ker}\,\mathcal{D}_A\}$ *decomposes into subbundles* $\mathrm{Ind}\,\mathcal{D}^{1,0}_{\mathcal{A}}$ *and* $\mathrm{Ind}\,\mathcal{D}^{0,1}_{\mathcal{A}}$ *relative to the almost complex structure I on $\Omega^1(V\otimes\boldsymbol{E})$, and* (ii) *restricted to the subbundle* $\mathrm{Ind}\,\mathcal{D}^{1,0}_{\mathcal{A}}$ *(or* $\mathrm{Ind}\,\mathcal{D}^{0,1}_{\mathcal{A}}$*) the curvature form Ω is a* (1, 1)*-form. Therefore,* $\mathrm{Ind}\,\mathcal{D}^{1,0}_{\mathcal{A}}$ *(or* $\mathrm{Ind}\,\mathcal{D}^{0,1}_{\mathcal{A}}$*) is equipped with a holomorphic structure which is consistent with the induced connection $\boldsymbol{D}$.*

REMARK. Quillen considers a new metric and its curvature of the determinant line bundle of the index bundle over a Riemann surface associated with operators $\boldsymbol{D}_A=\bar{\partial}_A$; $\Omega^0(\boldsymbol{E})\to\Omega^{0,1}(\boldsymbol{E})$ ([3], [6], [15]). This metric which seems to be of interest is defined by the Ray-Singer analytic torsion. However, he defines it only over the affine space $\mathcal{A}$.

PROOF. From Propositions 3.1, 5.1 it suffices to prove that

$$\langle \sigma_Z(\xi-\sqrt{-1}I\xi),\quad \sigma_W(\eta+\sqrt{-1}I\eta)\rangle=0$$

for tangent vectors $Z, W\in T^{1,0}_{[A]}\mathcal{M}$ and ξ, $\eta\in\operatorname{Ker}\mathcal{D}_A$.

We first assert the following

$$\sigma_X\xi=G_A\nabla_A(\rho(X)\rfloor\xi)+I(G_A\nabla_A(\rho(IX)\rfloor\xi))-G_Ad_A^{+*}((\rho(X)\wedge\xi)^2), \tag{5.1}$$

$$\sigma_X(I\xi)=-G_A\nabla_A(\rho(IX)\rfloor\xi)+I(G_A\nabla_A(\rho(X)\rfloor\xi))-G_Ad_A^{+*}((\rho(IX)\wedge\xi)^2) \tag{5.2}$$

and

$$\sigma_{IX}\xi=G_A\nabla_A(\rho(IX)\rfloor\xi)-I(G_A\nabla_A(\rho(X)\rfloor\xi)-G_Ad_A^{+*}((\rho(IX)\wedge\xi)^2) \tag{5.3}$$

where we set $(\rho(X)\wedge\xi)^2=(\rho(X)\wedge\xi)^{2,0}+(\rho(X)\wedge\xi)^{0,2}$ following the splitting $\Omega_+^2\otimes C=\Omega^{2,0}\oplus(\Omega_C^0\otimes\omega_h)\oplus\Omega^{0,2}$ (ω_h is the Kähler form).

We have indeed $(\rho(X)\wedge\xi)^+=(\rho(X)\wedge\xi)^2+(\rho(X)\wedge\xi)^0$ with $(\rho(X)\wedge\xi)^0=\frac{1}{2}(\rho(X)\rfloor(I\xi))\otimes\omega_h$.

So by using Proposition 4.1 we reduce the second fundamental from $\sigma_X\xi$ to (5.1) since $G_Ad_A^{+*}((\rho(X)\wedge\xi)^0)$ is reduced to $I(G_A\nabla_A(\rho(X)\rfloor(I\xi))$ thanks to a simple calculation and also the formula $\rho(IX)\rfloor(I\xi)=\rho(X)\rfloor\xi$ holds.

The formulae (5.2) and (5.3) follow from the formula $(\rho(X)\wedge(I\xi))^2=(\rho(IX)\wedge\xi)$.

By making use of these formulae, we obtain

$$\langle\sigma_X(I\xi),\ \sigma_Y(I\eta)\rangle=\langle\sigma_{IX}\xi,\ \sigma_{IY}\eta\rangle, \tag{5.4}$$

and hence

$$\langle\sigma_Z\xi,\ \sigma_W\eta\rangle+\langle\sigma_Z(I\xi),\ \sigma_W(I\eta)\rangle=0 \tag{5.5}$$

for $Z=X-\sqrt{-1}IX, W=Y-\sqrt{-1}IY\in T^{1,0}_{[A]}\mathcal{M}$ from which one sees that $\langle\sigma_Z(\xi-\sqrt{-1}I\xi),\ \sigma_W(\eta+\sqrt{-1}I\eta)\rangle$ vanishes.

6. The Ricci form.

We considered in § 5 the index bundle of elliptic operators $\mathcal{D}_A$ which is associated to a Hermitian vector bundle V with an Einstein-Hermitian connection, and mainly investigated the integrability of the holomorphic structure of the index bundle provided that the base manifold M is Kähler.

In this section, also under the assumption that M is Kähler, we aim at getting the Ricci form of the index bundle over the moduli space $\mathcal{M}$ of anti-self-dual connections on P.

We recall the definition of the Ricci form of a holomorphic vector bundle. Let F be a holomorphic vector bundle with a Hermitian fibre metric over a complex manifold N. Then the Ricci form Φ of F being a (1, 1)-form over N is defined by the trace of the endomorphism $\Theta(X, Y)_x; F_x \to F_x$, $x \in N$, where Θ is the curvature form of the fibre metric. The real 2-form $\frac{1}{2\pi\sqrt{-1}}\Phi$ represents the first Chern class of the bundle F.

Consider now the index bundle $\mathrm{Ind}\,\mathcal{D}_A$ over $\mathcal{M}$. We already observed that $(\mathrm{Ind}\,\mathcal{D}_A)^{1,0}$ carries a holomorphic structure with the Hermitian L_2-metric (Theorem 5.3). Thus, if we let $\{\xi_i, 1 \leqq i \leqq k\}$ be an orthonormal basis of $\mathrm{Ind}\,\mathcal{D}_A$ at $[A] \in \mathcal{M}$, then the Ricci form Φ of $(\mathrm{Ind}\,\mathcal{D}_A)^{1,0}$ is by definition written as $\Phi(X, Y) = \sum_i \langle \Theta(X, Y)\phi_i, \bar{\phi}_i \rangle$, $\phi_i = \frac{1}{\sqrt{2}}(\xi_i - \sqrt{-1} I\xi_i)$ for tangent vectors X and Y to $\mathcal{M}$ at $[A]$.

Denote by $\bar{\Phi}$ the 2-form on the space $\mathcal{B}$, and hence on the subset $\mathcal{M}$ defined by $\sum_i \langle \bar{\Omega}(X, Y)\phi_i, \bar{\phi}_i \rangle$ for the curvature form $\bar{\Omega}$ of the induced connection $\bar{\boldsymbol{D}}$ on the Hilbert vector bundle Q. This 2-form is the trace of the curvature endomorphism $\bar{\Omega}(X, Y)$ restricted to $\mathrm{Ind}\,\mathcal{D}_A$.

THEOREM 6.1. *The Ricci form satisfies*

$$\Phi(Z, \bar{Z}) - \bar{\Phi}(Z, \bar{Z}) = -4 \sum_{i=1}^{k} \{ |G_A \nabla_A(\rho(X) \lrcorner (I\xi_i)) + I G_A \nabla_A(\rho(X) \lrcorner \xi_i)|^2 \}$$
$$+ 2 \sum_{i=1}^{k} \{ |G_A d_A^{+*}((\rho(X) \wedge \xi_i)^2|^2 + |G_A d_A^{+*}((\rho(IX) \wedge \xi_i)^2)|^2 \}, \tag{6.1}$$

where k is the rank of the index bundle $\mathrm{Ind}\,\mathcal{D}_A$, $Z \in T^{1,0}_{[A]}\mathcal{M}$ *and* $\{\xi_i\}$ *is an orthonormal basis of* $\mathrm{Ind}\,\mathcal{D}_A$ *over* $[A]$.

PROOF. From the Gauss equation (3.2) we have

$$\langle \bar{\Omega}(X, Y)\xi, \xi \rangle = \langle \Omega(X, Y)\xi, \xi \rangle.$$

On the other hand, the pair (X, Y) generates a gauge transformation $g_t = \exp t(G_A\{X, Y\})$ which preserves the L_2-inner product $\langle\ ,\ .\ .\ ,\ \rangle$ on $\Omega^1(V \otimes \boldsymbol{E})$ so that $-\langle \bar{\Omega}(X, Y)\xi, \xi \rangle = \frac{d}{dt}\langle \rho(g_t)\xi, \rho(g_t)\xi \rangle|_{t=0}$ is equal to zero. Hence,

$$\langle \Omega(X, Y)\xi, \xi \rangle = 0, \quad \xi \in (\mathrm{Ind}\,\mathcal{D}_A)_{[A]}. \tag{6.2}$$

Therefore, from this we obtain

$$\langle \Omega(X, Y)\phi_i, \bar{\phi}_i \rangle = \frac{1}{2}\sqrt{-1}\{\langle \Omega(X, Y)\xi_i, I\xi_i \rangle - \langle \Omega(X, Y)I\xi_i, \xi_i \rangle\}$$
$$= \frac{1}{2}\sqrt{-1}\{\langle \bar{\Omega}(X, Y)\xi_i, I\xi_i \rangle - \langle \bar{\Omega}(X, Y)I\xi_i, \xi_i \rangle$$
$$-2\langle \sigma_X I\xi_i, \sigma_Y \xi_i \rangle + 2\langle \sigma_Y I\xi_i, \sigma_X \xi_i \rangle\}.$$

Here, we use the fact $\bar{\Omega}(X, Y)I\xi=I(\bar{\Omega}(X, Y)\xi)$ which stems from the commutability of the almost complex structure I and gauge transformations.

The Ricci form is then represented as

$$\Phi(X, Y)=\sqrt{-1}\sum_i\langle\bar{\Omega}(X, Y)\xi_i, I\xi_i\rangle - \sqrt{-1}\sum_i\{\langle\sigma_X I\xi_i, \sigma_Y\xi_i\rangle-\langle\sigma_Y I\xi_i, \sigma_X\xi_i\rangle\}. \tag{6.3}$$

Therefore, we see for $Z=X-\sqrt{-1}IX\in T^{1,0}_{[A]}\mathcal{M}$ that $\Phi(Z, \bar{Z})=2\sqrt{-1}\Phi(X, IX)$ is given by

$$\Phi(Z, \bar{Z})=-2\sum_i\langle\bar{\Omega}(X, IX)\xi_i, I\xi_i\rangle + 2\sum_i\{\langle\sigma_X I\xi_i, \sigma_{IX}\xi_i\rangle-\langle\sigma_{IX} I\xi_i, \sigma_X\xi_i\rangle\}. \tag{6.4}$$

By making use of formulae (5.1), (5.2), (5.3) we can reduce the terms $\langle\sigma_X I\xi, \sigma_{IX}\xi\rangle-\langle\sigma_{IX}I\xi, \sigma_X\xi\rangle$ for $\xi=\xi_i$ to

$$-|G_A\nabla_A(\rho(IX)\lrcorner\,\xi)-IG_A\nabla_A(\rho(X)\lrcorner\,\xi)|^2+|G_A d_A^{+*}(\rho(IX)\wedge\xi)^2|^2$$
$$-|G_A\nabla_A(\rho(X)\lrcorner\,\xi)+IG_A\nabla_A(\rho(IX)\lrcorner\,\xi)|^2+|G_A d_A^{+*}(\rho(X)\wedge\xi)^2|^2$$

from which (6.1) follows.

7. Discussion of the condition Coker $\mathcal{D}_A=\{0\}$.

We will finally discuss a sufficient condition for the assumption Coker $\mathcal{D}_A=0$ which was made in preceding sections.

Let (V, f) be an Einstein-Hermitian vector bundle over a compact Kähler surface (M, h). Namely we assume that the curvature form $\Theta=\sum_{i,\bar{j}}\Theta_{i,\bar{j}}dz^i\wedge d\bar{z}^{\bar{j}}$ of the fibre metric f satisfies the Einstein-Hermitian condition with a real constant λ

$$\sum_{i,\bar{j}} h^{i\bar{j}}\Theta_{i\bar{j}}=\lambda.\,\mathrm{id}_V, \tag{7.1}$$

$$\lambda=4\pi/\mathrm{vol}(M)\int_M \mathrm{c}_1(V)/\mathrm{r}(V)\wedge\omega_h,$$

($\mathrm{r}(V)$ is the rank of V and ω_h is the Kähler form of h).

Let P be a principal bundle over M, and A be an anti-self-dual connection on P. Then, coupling the Hermitian connection ∇ on V with A yields an elliptic operator

$$\mathcal{D}_A;\ \Omega^1(V\otimes\boldsymbol{E})\longrightarrow(\Omega^0\oplus\Omega^2_+)(V\otimes\boldsymbol{E})$$
$$\xi\longmapsto(\nabla_A^*\xi,\ d_A^+\xi)$$

whose adjoint $\mathcal{D}_A^*$ maps (ϕ, Φ) to $\nabla_A\phi + d_A^{+*}\Phi$ in $\Omega^1(V\otimes \boldsymbol{E})$. So,

$$\mathrm{Ker}\,\mathcal{D}_A^* = \{(\phi, \Phi)\in(\Omega^0\oplus\Omega_+^2)(V\otimes\boldsymbol{E})\,;\,\nabla_A\phi + d_A^{+*}\Phi = 0\}.$$

Now, investigate the equation

$$\nabla_A\phi + d_A^{+*}\Phi = 0. \tag{7.2}$$

Apply ∇_A^* on both sides. Then,

$$\nabla_A^*\nabla_A\phi + (d_A^+\nabla_A)^*\Phi = 0. \tag{7.3}$$

Since $d_A^+\nabla_A$ is the self-dual part of $\Theta+\rho(F(A))$ from the Ricci formula, one can reduce it further using the Einstein-Hermitian condition and the anti-self-dual condition on A to $-\frac{\sqrt{-1}}{2}\lambda\,\mathrm{id}_{V\otimes E}\otimes\omega_h$, $(d_A^+\nabla_A)^*\Phi = \sqrt{-1}\lambda\Phi^0$ holds where Φ^0 is the ω_h-component of Φ. We get then

$$\nabla_A^*\nabla_A\phi + \sqrt{-1}\lambda\Phi^0 = 0. \tag{7.4}$$

We can apply the operator d_A^+ on both sides of (7.2)

$$d_A^+\nabla_A\phi + d_A^+d_A^{+*}\Phi = 0. \tag{7.5}$$

Again, we apply the Einstein-Hermitian condition to the term $d_A^+\nabla_A\phi$ to reduce it to $\frac{-\sqrt{-1}}{2}\lambda\phi\otimes\omega_h$. On the other hand, for $\Phi = \Phi^{2,0} + \Phi^0\otimes\omega_h + \Phi^{0,2}$, $\Psi = d_A^+d_A^{+*}\Phi$ can be written as $\Psi^{2,0} + \Psi^0\otimes\omega_h + \Psi^{0,2}$ where

$$\Psi^{2,0} = (\partial_A\partial_A^* + \partial_A^*\partial_A)\Phi^{2,0},$$

$$\Psi^{0,2} = (\bar\partial_A\bar\partial_A^* + \bar\partial_A^*\bar\partial_A)\Phi^{0,2}, \quad \Psi^0 = \frac{1}{2}(\nabla_A^*\nabla_A\Phi^0)\otimes\omega_h$$

(∂_A and $\bar\partial_A$ denote the partial covariant derivatives). Here, we used the fact that Θ is of type (1, 1) and the following formula (see (2.7), (2.8) in [8])

$$d_A^{+*}\Phi = \partial_A^*\Phi^{2,0} + \bar\partial_A^*\Phi^{0,2} + \sqrt{-1}(\partial_A\Phi^0 - \bar\partial_A\Phi^0). \tag{7.6}$$

So, we get

$$\nabla_A^*\nabla_A\Phi^0 + (-\sqrt{-1})\lambda\phi = 0, \tag{7.7}$$

and also

$$(\partial_A\partial_A^* + \partial_A^*\partial_A)\Phi^{2,0} = 0, \quad (\bar\partial_A\bar\partial_A^* + \bar\partial_A^*\bar\partial_A)\Phi^{0,2} = 0. \tag{7.8}$$

LEMMA (*Bochnor-Weitzenböck formula*). *For* $\Phi^{2,0} = \Phi_{12}dz^1\wedge dz^2 \in \Omega^{2,0}(V\otimes\boldsymbol{E})$, *we have*

$$(7.9) \qquad (\partial_A \partial_A^* + \partial_A^* \partial_A)\Phi^{2,0} = \Psi_{12} dz^1 \wedge dz^2 ,$$

$$\Psi_{12} = -\sum h^{ij} \tilde{\nabla}_j \tilde{\nabla}_i \Phi_{12} + \left(\frac{s}{2} - \lambda\right)\Phi_{12}$$

($\tilde{\nabla}_i$ and $\tilde{\nabla}_j$ are covariant derivatives in the i- and j-directions, respectively, and s denotes the scalar curvature of the metric h).

PROOF. This is proved by a straightforward calculation in the same manner as the proof of [7], Lemma 3.3.

Thus, if the Einstein constant λ satisfies $\lambda < \frac{s}{2}$, then

$$\{\Phi^{2,0} \in \Omega^{2,0}(V \otimes \boldsymbol{E}) ; (\partial_A \partial_A^* + \partial_A^* \partial_A)\Phi^{2,0} = 0\} = \{0\} ,$$

Now, return to the equations (7.4) and (7.7). Set $\psi = \phi - \sqrt{-1}\Phi^0$. Then

$$(7.10) \qquad \nabla_A^* \nabla_A \psi = \lambda \psi .$$

Assume that $\lambda < \min\left(0, \frac{s}{2}\right)$. This is possible only when $\psi = 0$, that is $\phi = \sqrt{-1}\Phi^0$ in $\Omega^0(V \otimes \boldsymbol{E})$. Therefore, the equation $\nabla_A \phi + d_A^{+*}(\Phi^0 \otimes \omega_h) = 0$ becomes $\partial_A \phi = 0$ so that from (7.10) ϕ must further satisfy

$$(7.11) \qquad \Delta_A'' \phi = -\lambda \phi , \quad \Delta_A'' = -\sum h^{ij} \nabla_i \nabla_j .$$

Thus we obtain the following vanishing criterion.

THEOREM 7.2. *Let (M, h) be a compact complex Kähler surface and P be a principal bundle over M with an associated complex vector bundle $\boldsymbol{E}$. Let (V, f) be an Einstein-Hermitian vector bundle over M with Einstein constant λ.*

Assume that 2λ is negative and less than the scalar curvature s of the metric h. If, for any connection ∇_A on $V \otimes \boldsymbol{E}$, the Hermitian connection ∇ on V coupled to an anti-self-dual connection A on P, the Laplace operator $\Delta_A'' = -\sum h^{ij} \nabla_i \nabla_j$ acting on $\Omega^0(V \otimes \boldsymbol{E})$ satisfies

$$(7.12) \qquad \operatorname{Ker}(\Delta_A'' + \lambda \,\mathrm{id}) \cap \operatorname{Ker} \partial_A = \{0\} ,$$

then $\operatorname{Ker} \mathcal{D}_A^*$, which is isomorphic to $\operatorname{Coker} \mathcal{D}_A$, vanishes for each $[A]$ in $\mathcal{M}$.

Since the bundle $V \otimes \boldsymbol{E}$ carries a holomorphic structure induced by the connection ∇_A, by taking its conjugate in the complex vector space $\Omega^0(V \otimes \boldsymbol{E})$, (7.12) is equivalent to that the space of holomorphic sections ψ of $V \otimes \boldsymbol{E}$ satisfying $\Delta_A' \psi + \lambda \psi = 0$ reduces to $\{0\}$.

Suppose now that there is an ample holomorphic line bundle L over M as mentioned in the conjecture made in §1. Then, as is easily shown, L carries an Einstein-Hermitian fibre metric with positive constant λ. The k-fold tensor product of the dual of L has constant $-k\lambda$. So, from this theorem, for a sufficiently large k $\mathcal{D}_A$; $\Omega^1(L^{-k}\otimes E)\to(\Omega^0\oplus\Omega^2_+)(L^{-k}\otimes E)$ has trivial cokernel if and only if $\{\phi\in\Omega^0(L^{-k}\otimes E); \bar{\partial}_A\phi=0,\ \Delta'_A\phi=-\lambda\phi\}$ vanishes.

REMARKS. (i) We can relax the Einstein-Hermitian condition on a vector bundle V in theorem 7.2. In fact, the condition on the value λ can be replaced by the following weaker condition; the curvature Θ of the Hermitian connection ∇ satisfies that $\mathrm{tr}_h\Theta=\sum h^{i\bar{j}}\Theta_{i\bar{j}}$ is negative definite and less than $\frac{s}{2}\mathrm{id}_V$ in the sense of the trace norm of $\mathrm{End}(V)$.

(ii) The condition on $\mathrm{Ker}(\Delta''_A+\lambda\,\mathrm{id})$ in the theorem, which says that $-\lambda$ is not an eigenvalue of the Laplace operator Δ''_A, can be eliminated when we consider the case of elliptic operators $\mathcal{D}_A$ associated with the Dolbeault complex $0\to\Omega^0(V\otimes E)\to\Omega^{0,1}(V\otimes E)\to\Omega^{0,2}(V\otimes E)\to 0$.

Added in proof. After preparing this paper, in the paper 'Poincaré Bundle and Chern Classes' the author establishes the following theorem from which the conjecture stated in §1 is then affirmatively solved: Let (M, h) be a compact Hodge surface and P a principal bundle over M. Then, the moduli space of anti-self-dual connections on P can be endowed with a holomorphic line bundle of positive first Chern class.

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