TSUKUBA J. MATH.
Vol. 16 No. 1 (1992), 107–111

A CRITERION FOR ISOMORPHIC TILED ORDERS OVER A LOCAL DEDEKIND DOMAIN

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Throughout this note R will denote a local Dedekind domain with a unique maximal ideal πR and the quotient ring K.

Let $n(\geqq 2)$ be an integer. As usual, an R-order Λ in the full $n\times n$ matrix ring $(K)_n$ over K is an R-subalgebra of $(K)_n$ such that Λ is finitely generated as an R-module and $\Lambda K=(K)_n$. An R-order in $(K)_n$ is called *tiled* if it contains n orthogonal idempotents. If Λ is a tiled R-order in $(K)_n$ then up to conjugation we may assume that it contains the idempotents e_{ii} $(1\leqq i\leqq n)$, where $\{e_{ij}|1\leqq i, j\leqq n\}$ is the family of usual matrix units in $(K)_n$, and hence that it is of the form $(\pi^{\lambda_{ij}}R)$ for some integers λ_{ij} with $\lambda_{ij}\leqq\lambda_{ik}+\lambda_{kj}$ and $\lambda_{ii}=0$ for all $1\leqq i, j, k\leqq n$ (cf. [2], [3]). Since each $e_{ii}\Lambda e_{ii}\cong R$ is a local ring, Λ is a semiperfect ring. For a basic tiled R-order Λ, the *quiver* $Q(\Lambda)$ of Λ is defined as follows: The vertices of $Q(\Lambda)$ are $\{1, \cdots, n\}$; there exists an arrow from i to j in $Q(\Lambda)$ if $e_{jj}Je_{ii}\subsetneqq e_{jj}J^2e_{ii}$, where J is the Jacobson radical of Λ (c.f. [5]).

In [1, Theorem] we have shown that two basic tiled R-orders between $(\pi R)_n$ and $(R)_n$ are isomorphic as rings if and only if their quivers are equal except for the numbering of the vertices, and in this case, one order is conjugate to the other one with a regular element in $(R)_n$, which is constructed from their graph theoretic properties. In this note we shall remove the hypothesis "between $(\pi R)_n$ and $(R)_n$" and prove the following

THEOREM. *Let Λ and Γ be tiled R-orders in $(K)_n$. If Λ and Γ are isomorphic as rings then there exist a diagonal matrix v and a permutation matrix w in $(R)_n$ such that $\Gamma=vw\Lambda w^{-1}v^{-1}$.*

REMARK. In the proof of Theorem we shall define the matrix v by using given Λ and Γ and a permutation matrix w. Thus it turns out that by a finite number of procedures of taking conjugations we can determine whether given two tiled R-orders are isomorphic, or not.

Received December 14, 1990.

We shall also show that in the case Λ and Γ are basic tiled R-orders between $(\pi R)_n$ and $(R)_n$, the diagonal matrix v in the above theorem coincides with the diagonal matrix u in the proof of [1, Theorem] (Proposition).

In order to prove Theorem, we need the following two lemmas.

LEMMA 1 (c.f. [4, p. 77, Proposition 3]). *Let S be a semiperfect ring. If $\{e_i\}_{i=1}^m$ and $\{f_j\}_{j=1}^n$ are complete sets of orthogonal primitive idempotents of S then $m=n$, and there exist a unit u of S and a permutation σ of $\{1, \cdots, n\}$ such that $ue_iu^{-1}=f_{\sigma(i)}$ for all $i=1, \cdots, n$.*

LEMMA 2. *Let $\Lambda=(\pi^{\lambda_{ij}}R)$ and $\Gamma=(\pi^{\gamma_{ij}}R)$ be tiled R-orders in $(K)_n$, and assume that φ is a ring isomorphism from Λ to Γ such that $\varphi(e_{ii})=e_{ii}$ for all $i=1, \cdots, n$. Then*

(1) *$\varphi(\pi^{\lambda_{ij}+k}Re_{ij})=\pi^{\gamma_{ij}+k}Re_{ij}$ for all $1\leqq i, j\leqq n$ and $k\geqq 0$.*

(2) *$\lambda_{ij}+\lambda_{ji}=\gamma_{ij}+\gamma_{ji}$ for all $1\leqq i, j\leqq n$.*

PROOF. First observe that for all i and j, we have $\varphi(e_{ii}\Lambda e_{jj})=e_{ii}\Gamma e_{jj}$, that is, $\varphi(\pi^{\lambda_{ij}}Re_{ij})=\pi^{\gamma_{ij}}Re_{ij}$.

For each $i=1, \cdots, n$, we define a mapping $\varphi_i: R\to R$ by $a\mapsto a'$, where $\varphi(ae_{ii})=a'e_{ii}$. Then each φ_i is a ring automorphism of R, so $\varphi_i(\pi^kR)=\pi^kR$. Hence we have $\varphi(\pi^kRe_{ii})=\pi^kRe_{ii}$. This shows that $\varphi(\pi^{\lambda_{ij}+k}Re_{ij})=\varphi(\pi^kRe_{ii})\varphi(\pi^{\lambda_{ij}}Re_{ij})=(\pi^kRe_{ii})(\pi^{\gamma_{ij}}Re_{ij})=\pi^{\gamma_{ij}+k}Re_{ij}$, which completes the proof of (1).

Since $\lambda_{ij}+\lambda_{ji}\geqq\lambda_{ii}=0$, we see from (1) that $\varphi(\pi^{\lambda_{ij}+\lambda_{ji}}Re_{ii})=\pi^{\lambda_{ji}+\lambda_{ij}}Re_{ii}$. On the other hand, $\varphi(\pi^{\lambda_{ij}+\lambda_{ji}}Re_{ii})=\varphi(\pi^{\lambda_{ij}}Re_{ij})\varphi(\pi^{\lambda_{ji}}Re_{ji})=(\pi^{\gamma_{ij}}Re_{ij})(\pi^{\gamma_{ij}}Re_{ji})==\pi^{\gamma_{ij}+\gamma_{ji}}Re_{ii}$. Hence, $\pi^{\lambda_{ij}+\lambda_{ji}}Re_{ii}=\pi^{\gamma_{ij}+\gamma_{ji}}Re_{ii}$, from which we obtain $\lambda_{ij}+\lambda_{ji}=\gamma_{ij}+\gamma_{ji}$.

PROOF OF THEOREM. Let $\Lambda=(\pi^{\lambda_{ij}}R)$ and $\Gamma=(\pi^{\gamma_{ij}}R)$ be tiled R-orders in $(K)_n$, and assume that ϕ is a ring isomorphism from Λ to Γ. According to Lemma 1, there exist a unit u in Λ and a permutation σ of $\{1, \cdots, n\}$ such that $u\phi^{-1}(e_{ii})u^{-1}=e_{\sigma(i)\sigma(i)}$ for all i. Choose a permutation matrix w in $(R)_n$ such that $we_{\sigma(i)\sigma(i)}w^{-1}=e_{ii}$ for all i, and put $w\Lambda w^{-1}=(\pi^{\lambda'_{ij}}R)$. Put $\gamma=\min\{\gamma'\mid\lambda'_{1j}-\gamma_{1j}+\gamma'\geqq 0$ for all $j=1, \cdots, n\}$. Let $v=(v_{ij})\in(R)_n$ such that $v_{jj}=\pi^{\lambda'_{1j}-\gamma_{1j}+\gamma}$ and $v_{ij}=0$ whenever $i\neq j$. Then we put $vw\Lambda w^{-1}v^{-1}=(\pi^{\lambda''_{ij}}R)$. We shall show that $\lambda''_{ij}=\gamma_{ij}$ for all i, j, which will complete the proof. First observe that $\lambda''_{ij}=\lambda'_{ij}+\lambda'_{1i}-\gamma_{1i}-\lambda'_{1j}+\gamma_{1j}$ for i, j. Hence, in particular, $\lambda''_{1j}=\gamma_{1j}$. Since v is a diagonal matrix, $ve_{ii}v^{-1}=e_{ii}$ for all i. Define a mapping $\varphi: vw\Lambda w^{-1}v^{-1}\to\Gamma$ by $vwxw^{-1}v^{-1}\mapsto\phi(u^{-1}xu)$, where $x\in\Lambda$. Then φ is a ring isomorphism fixing each e_{ii}. Now let $1\leqq i, j\leqq n$ be fixed. By Lemma 2(2), it suffices to show that either $\lambda''_{ij}=\gamma_{ij}$ or $\lambda''_{ji}=\gamma_{ji}$. Without loss of generality, we may assume that $\lambda''_{1i} \leqq \lambda''_{1j}$. Put $k=\lambda''_{1j}-\lambda''_{1i}$. Then $\gamma_{1j}-\gamma_{1i}=k \geqq 0$. Note that $\lambda''_{ij}=(\lambda'_{1i}+\lambda'_{ij})-\lambda'_{1j}+\gamma_{1j}-\gamma_{1i} \geqq \lambda'_{1j}-\lambda'_{1j}+\gamma_{1j}-\gamma_{1i}=k \geqq 0$. It then follows from Lemma 2(1) that $\pi^{\gamma_{1j}+\lambda''_{ij}}Re_{1j}=\varphi(\pi^{\lambda''_{1j}+\lambda''_{ij}}Re_{1j})=\varphi(\pi^{\lambda''_{1i}+k}Re_{1i})\varphi(\pi^{\lambda''_{ij}}Re_{ij})=(\pi^{\gamma_{1i}+k}Re_{1i})(\pi^{\gamma_{ij}}Re_{ij})=\pi^{\gamma_{1j}+\gamma_{ij}}Re_{1j}$, from which we conclude that $\gamma_{1j}+\lambda''_{ij}=\gamma_{1j}+\gamma_{ij}$. Therefore, we obtain $\lambda''_{ij}=\gamma_{ij}$, as desired.

Let $\Lambda=(\pi^{\lambda_{ij}}R)$ be a tiled R-order in $(K)_n$. For each nonnegative integer k, we denote the number of i such that $\lambda_{ij}+\lambda_{ji}=k$ for some $1 \leqq j \leqq i \leqq n$ by $t_k(\Lambda)$, and call the sequence $(t_0(\Lambda), t_1(\Lambda), t_2(\Lambda), \cdots)$ the *depth type* of Λ. Since it is immediate that the depth type of tiled orders are invariant under conjugating with permutation matrices and diagonal matrices, we obtain

COROLLARY. *Isomorphic tiled R-orders in $(K)_n$ have the same depth type.*

We shall now clarify the relationship between the diagonal matrix v in the Theorem and the diagonal matrix u in the proof of [1, Theorem].

We set the following assumption and notation. Assume that $\Lambda=(\pi^{\lambda_{ij}}R)$ and $\Gamma=(\pi^{\gamma_{ij}}R)$ are distinct basic tiled R-orders between $(\pi R)_n$ and $(R)_n$ such that $Q(\Lambda)=Q(\Gamma)$. Let $\mathcal{S}$ be as in the proof of [1, Theorem]. Put $u=(u_{ij})$, where $u_{ii}=1$ if $i \notin \mathcal{S}$, $u_{ii}=\pi$ if $i \in \mathcal{S}$ and $u_{ij}=0$ otherwise (as in the proof of [1, Theorem]), and put $\gamma=\min\{\gamma' \mid \lambda_{1j}-\gamma_{1j}+\gamma' \geqq 0$ for all $j\}$ and $v=(v_{ij})$, where $v_{jj}=\pi^{\lambda_{1j}-\gamma_{1j}+\gamma}$ and $v_{ij}=0$ otherwise (as in the proof of Theorem). Then we can make the following observation.

PROPOSITION. *In the above setting, it holds that $u=v$.*

PROOF. We use [1, Lemmas 3, 4 and 6], which says, in other words, that for $1 \leqq i, j \leqq n$, it holds that;

(1) $i, j \in \mathcal{S}$ or $i, j \notin \mathcal{S} \Leftrightarrow \lambda_{ij}=\gamma_{ij}$,

(2) $i \notin \mathcal{S}, j \in \mathcal{S} \Rightarrow \lambda_{ij}=1, \gamma_{ij}=0$,

(3) $i \in \mathcal{S}, j \notin \mathcal{S} \Rightarrow \lambda_{ij}=0, \gamma_{ij}=1$.

If $1 \in \mathcal{S}$ and $\lambda_{1j}=\gamma_{1j}$ for all j then (1) implies that $\lambda_{ij}=\gamma_{ij}$ for all i, j, which contradicts to $\Lambda \neq \Gamma$. Hence it holds that either $1 \notin \mathcal{S}$ or $\lambda_{1j} \neq \gamma_{1j}$ for some j.

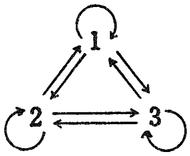
CASE 1. $1 \notin \mathcal{S}$: Then, if $\gamma=1$, that is, $\lambda_{1j}=0$ and $\gamma_{1j}=1$ for some j then this contradicts to (1) and (2). Hence we have $\gamma=0$. Using (1) and (2) again, we have $\lambda_{1j} \neq \gamma_{1j} \Leftrightarrow j \in \mathcal{S} \Leftrightarrow \lambda_{1j}=1, \gamma_{1j}=0$. This, combined with $\gamma=0$, shows that $u=v$.

CASE 2. $1\in\mathcal{S}$ and $\lambda_{1j}\neq\gamma_{1j}$ for some j: Then, a similar argument as in Case 1, using (1) and (3), implies that $\gamma=1$, and $\lambda_{1j}\neq\gamma_{1j}\Leftrightarrow j\notin\mathcal{S}\Leftrightarrow\lambda_{1j}=0$, $\gamma_{1j}=1$. This shows that $u=v$.

Therefore, in both cases, we obtain $u=v$.

REMARK. Let Λ and Γ be basic tiled R-orders between $(\pi R)_n$ and $(R)_n$ such that their quivers are equal except for the numbering of the vertices. Let w be a permutation matrix in $(R)_n$ such that $Q(w\Lambda w^{-1})=Q(\Gamma)$. Then it follows from Proposition and [1, Theorem] that $vw\Lambda w^{-1}v^{-1}=\Gamma$. However, in general, the quivers of tiled R-orders do not determine the permutation matrix w of Theorem.

EXAMPLE. Let $\Lambda=\begin{pmatrix} R & R & R \\ \pi^3 R & R & \pi R \\ \pi^3 R & \pi^2 R & R \end{pmatrix}$ and $\Gamma=\begin{pmatrix} R & R & R \\ \pi^3 R & R & \pi^2 R \\ \pi^3 R & \pi R & R \end{pmatrix}$. Then Λ and Γ have the same quiver



. If the permutation matrix w were determined by the quivers as in [1, Theorem] then we would obtain $w=v=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, but $vw\Lambda w^{-1}v^{-1}\neq\Gamma$. On the other hand, Γ is conjugate to Λ with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Acknowledgement

The work in this paper was done while the second author visited University of Tsukuba from May 1, 1990 to February 28, 1991. He would like to thank the staffs of the Institute of Mathematics for their kind hospitality.

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