TSUKUBA J. MATH.
Vol. 19 No. 2 (1995), 351-367

ON THE GAUSS MAP OF SURFACES OF REVOLUTION IN A 3-DIMENSIONAL MINKOWSKI SPACE

By

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§ 1. Introduction.

For the Gauss map of a surface of revolution in $\boldsymbol{R}^3$ the following theorem is proved by Dillen, Pas and Verstraelen [3].

THEOREM A. *The only surfaces of revolution in* $\boldsymbol{R}^3$ *whose Gauss map* ξ *satisfies*

$$\Delta\xi=A\xi, \qquad A\in Mat(3, \boldsymbol{R}) \tag{1.1}$$

are locally the plane, the sphere and the circular cylinder.

In the case of a Minkowski space, a Gauss map is defined as follows. Let $\boldsymbol{R}_1^{n+1}$ be an $(n+1)$-dimensional Minkowski space with standard coordinate system $\{x_A\}$ whose line element ds^2 is given by $ds^2=-(dx_0)^2+\sum_{i=1}^{n}(dx_i)^2$. Let $S_1^n(c)$ (resp. $H^n(c)$) be an n-dimensional de Sitter space (resp. a hyperbolic space) of constant curvature c in $\boldsymbol{R}_1^{n+1}$. We denote by $M^n(\varepsilon)$ a de Sitter space $S_1^n(1)$ or a hyperbolic space $H^n(-1)$, according as $\varepsilon=1$ or -1. Let M be a n-dimensional space-like or time-like hypersurface in $\boldsymbol{R}_1^{n+1}$ and ξ a unit vector field normal to M. Then, for any point p in M, we can regard $\xi(p)$ as a point in $H^n(-1)$ or $S_1^n(1)$ by translating parallelly to the origin in the ambient space $\boldsymbol{R}_1^{n+1}$, according as the surface M is space-like or time-like. The map ξ of M into $M^n(\varepsilon)$ is called a *Gauss map* of M into $\boldsymbol{R}_1^{n+1}$.

As a Lorentz version of Baikoussis and Blair's result [1], the author [2] proves the following

THEOREM B. *The only space-like or time-like ruled surfaces in* $\boldsymbol{R}_1^3$ *whose Gauss map* $\xi: M\to M^2(\varepsilon)$ *satisfies* (1.1) *are locally the following spaces*:

i. $\boldsymbol{R}_1^2$, $S_1^1\times\boldsymbol{R}^1$ *and* $\boldsymbol{R}_1^1\times S^1$ *if* $\varepsilon=1$,

ii. $\boldsymbol{R}^2$ *and* $H^1\times\boldsymbol{R}^1$ *if* $\varepsilon=-1$.

Partially supported by TERC-KOSEF.
Received October 4, 1993.

Similarly, it seems to be interesting to investigate the Lorentz version of Theorem A. The purpose of this paper is to prove the following

THEOREM. *The only space-like or time-like surfaces of revolution in* $\boldsymbol{R}_1^3$ *whose Gauss map* $\xi: M \to M^2(\varepsilon)$ *satisfies* (1.1) *are locally the following spaces*:

i. $\boldsymbol{R}_1^2$, S_1^2, $S_1^1 \times \boldsymbol{R}^1$ *and* $\boldsymbol{R}_1^1 \times S^1$ *if* $\varepsilon=1$,

ii. $\boldsymbol{R}^2$, H^2 *and* $H^1 \times \boldsymbol{R}^1$ if $\varepsilon=-1$.

In § 2 we define non-degenerate surfaces of revolution in $\boldsymbol{R}_1^3$. Roughly speaking, non-degenerate surfaces of revolution in $\boldsymbol{R}_1^3$ are divided into four types by the axes and the planes containing the axis. The main theorem is proved for each case in § 3 and § 4.

The author would like to express her gratitude to Professor Hisao Nakagawa for his useful advice.

§ 2. Preliminaries.

In this section we will give a definition of a surface of revolution in a 3-dimensional Minkowski space $\boldsymbol{R}_1^3$ and some examples which satisfy the condition (1.1). Throughout this paper, we assume that all objects are smooth and all surfaces are connected, unless otherwise mentioned.

For an open interval J, let $\alpha: J \to \Pi$ be a curve in a plane Π in $\boldsymbol{R}_1^3$ and let l be a straight line in Π which does not intersect the curve α. A surface of revolution M in $\boldsymbol{R}_1^3$ is defined as a non-degenerate surface revolving a *profile curve* α around the *axis* l. In other words, a surface M of revolution with axis l in $\boldsymbol{R}_1^3$ is invariant under the action of the group of motions in $\boldsymbol{R}_1^3$ which fix each point of the line l.

From definition, we can derive four types of the surfaces of revolution in $\boldsymbol{R}_1^3$. When the axis l is space-like (resp. time-like), there is a Lorentz transformation by which the axis l is transformed to the x_2-axis (resp. the x_0-axis). So we may suppose that the axis is the x_2-axis (resp. the x_0-axis). First of all, we consider that the axis of revolution is space-like. Since the surface M is non-degenerate, it suffices to consider the case that the plane Π is space-like or time-like. So we may suppose that Π is the x_1x_2-plane or the x_0x_2-plane without loss of generality. Then the profile curve α is parametrized as

$$\alpha(u)=(0,\ f(u),\ g(u)), \quad \text{or} \quad (f(u),\ 0,\ g(u)),$$

where f is a positive function and g is a function on J. In the rest of this paper we shall identify a vector (a, b, c) with a transpose ${}^t(a, b, c)$ of (a, b, c).

On the other hand, a subgroup of the Lorentz group which fixes the vector (0, 0, 1) is given by

$$\begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any $v \in \boldsymbol{R}$. Hence the surface M of revolution can be written as

$$x(u, v)=\begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ f(u) \\ g(u) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f(u) \\ 0 \\ g(u) \end{pmatrix}.$$

That is, M can be parametrized by

$$x(u, v)=(f(u)\sinh v,\ f(u)\cosh v,\ g(u)), \tag{2.2}$$

or

$$x(u, v)=(f(u)\cosh v,\ f(u)\sinh v,\ g(u)), \tag{2.3}$$

which is called a surface of revolution of *type I* or *II*.

Next, if the axis is time-like, then we may suppose that Π is the x_0x_1-plane without loss of generality. Then the profile curve α is parametrized as

$$\alpha(u)=(g(u),\ f(u),\ 0),$$

where f is a positive function and g is a function on J. On the other hand, a subgroup of the Lorentz group which fixes the vector (1, 0, 0) is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix}$$

for any $v \in \boldsymbol{R}$. Hence the surface M of revolution can be written as

$$x(u, v)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix}\begin{pmatrix} g(u) \\ f(u) \\ 0 \end{pmatrix}.$$

That is, M is parametrized by

$$x(u, v)=(g(u),\ f(u)\cos v,\ f(u)\sin v), \tag{2.4}$$

which is called a surface of revolution of *type III*.

Last of all, if the axis l is light-like, then we may suppose that it is the line spanned by the vector (1, 1, 0). Since the surface M is non-degenerate, it suffices to consider the case that the plane Π is time-like. So we may suppose that Π is the x_0x_1-plane without loss of generality. Then the profile curve α is parametrized as

$$\alpha(u)=(f(u),\ g(u),\ 0),$$

where f and g are functions such that $f\neq g$ on J. We notice here that a subgroup of the Lorentz group which fixes the vector (1, 1, 0) is given by

$$\begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix}$$

for any $v\in \boldsymbol{R}$. Hence the surface M of revolution can be written as

$$x(u,\ v)=\begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix}\begin{pmatrix} f(u) \\ g(u) \\ 0 \end{pmatrix}$$

That is, M is parametrized by

$$x(u,\ v)=\left(f+\frac{1}{2}v^2h,\ g+\frac{1}{2}v^2h,\ hv\right), \tag{2.5}$$

where we put $h=f-g$. This surface is called a surface of revolution of *type IV*.

Now, let M be a space-like or time-like hypersurface in $\boldsymbol{R}_1^{n+1}$ with locally coordinate system $\{x_i\}$. For the components g_{ij} of the Riemannian metric g on M we denote (g^{ij}) (resp. $\mathfrak{g}$) the inverse matrix (resp. the determinant) of the matrix (g_{ij}). Then the Laplacian Δ on M is given by

$$\Delta=-\frac{1}{\sqrt{|\mathfrak{g}|}}\sum\frac{\partial}{\partial x_i}\left(\sqrt{|\mathfrak{g}|}\,g^{ij}\frac{\partial}{\partial x_j}\right). \tag{2.6}$$

Next we consider some examples mentioned in the theorem which satisfy the condition (1.1).

EXAMPLE 2.1. A Euclidean plane

$$\boldsymbol{R}^2=\{(x_0,\ x_1,\ x_2)\in\boldsymbol{R}_1^3\,|\,x_0=0\}$$

is the totally geodesic space-like surface and the Gauss map ξ is constant. So, the Laplacian $\Delta\xi$ of the Gauss map ξ vanishes. Hence the Euclidean plane satisfies (1.1) with

$$A=\begin{pmatrix}0 & * & * \\ 0 & * & * \\ 0 & * & *\end{pmatrix}.$$

On the other hand, a Minkowski plane

$$\boldsymbol{R}_1^2=\{(x_0,\ x_1,\ x_2)\in\boldsymbol{R}_1^3 \mid x_2=0\}$$

is the totally geodesic time-like surface and the Gauss map ξ is constant. So, the Laplacian $\Delta\xi$ of the Gauss map ξ vanishes. Hence the Minkowski plane satisfies (1.1) with

$$A=\begin{pmatrix}* & * & 0 \\ * & * & 0 \\ * & * & 0\end{pmatrix}.$$

EXAMPLE 2.2. A hyperbolic space

$$H^2(c)=\left\{x=(x_0,\ x_1,\ x_2)\in\boldsymbol{R}_1^3 \mid -x_0^2+x_1^2+x_2^2=\frac{1}{c}=-r^2,\ r>0\right\}$$

is a totally umbilic space-like surface and the Gauss map ξ is given by x/r. The mean curvature vector field H of $H^2(c)$ is given by ξ/r. Since we have $\Delta x=-2H$, the Laplacian $\Delta\xi$ of the Gauss map ξ satisfies

$$\Delta\xi=-\frac{2}{r^2}\xi.$$

Hence the hyperbolic space satisfies (1.1) with

$$A=\begin{pmatrix}-\frac{2}{r^2} & 0 & 0 \\ 0 & -\frac{2}{r^2} & 0 \\ 0 & 0 & -\frac{2}{r^2}\end{pmatrix}.$$

On the other hand, a de Sitter space

$$S_1^2(c)=\left\{x=(x_0,\ x_1,\ x_2)\in\boldsymbol{R}_1^3 \mid -x_0^2+x_1^2+x_2^2=\frac{1}{c}=r^2,\ r>0\right\}$$

is a totally umbilic time-like surface and the Gauss map ξ is given by x/r. The mean curvature vector field H of $S_1^2(c)$ is given by $-\xi/r$. From $\Delta x=-2H$, the Laplacian $\Delta\xi$ of the Gauss map ξ satisfies

$$\Delta\xi=\frac{2}{r^2}\xi .$$

Hence the de Sitter space satisfies (1.1) with

$$A=\begin{pmatrix} \frac{2}{r^2} & 0 & 0 \\ 0 & \frac{2}{r^2} & 0 \\ 0 & 0 & \frac{2}{r^2} \end{pmatrix}.$$

EXAMPLE 2.3. A hyperbolic cylinder

$$H^1(c)\times \boldsymbol{R}=\left\{(x_0,\ x_1,\ x_2)\in \boldsymbol{R}_1^3 \mid -x_0^2+x_1^2=\frac{1}{c}=-r^2,\ r>0\right\}$$

is a space-like surface and the Gauss map ξ is given by $(\xi_0, 0)$, where ξ_0 denotes a Gauss map of the hyperbolic space $H^1(c)$. Since the Laplacian of ξ_0 is to be $-\xi_0/r^2$ by Example 2.2, the Laplacian $\Delta\xi$ of the Gauss map ξ can be expressed as

$$\Delta\xi=-\frac{1}{r^2}\xi .$$

Hence the hyperbolic cylinder satisfies (1.1) with

$$A=\begin{pmatrix} -\frac{1}{r^2} & 0 & * \\ 0 & -\frac{1}{r^2} & * \\ 0 & 0 & * \end{pmatrix}.$$

Next, a Lorentz hyperbolic cylinder

$$S_1^1(c)\times \boldsymbol{R}=\left\{(x_0,\ x_1,\ x_2)\in \boldsymbol{R}_1^3 \mid -x_0^2+x_1^2=\frac{1}{c}=r^2,\ r>0\right\}$$

is a time-like surface and the Gauss map ξ is given by $(\xi_0, 0)$, where ξ_0 denotes a Gauss map of the de Sitter space $S_1^1(c)$. Since the Laplacian of ξ_0 is to be ξ_0/r^2 by Example 2.2, the Laplacian $\Delta\xi$ of the Gauss map ξ can be expressed as

$$\Delta\xi=\frac{1}{r^2}\xi .$$

Hence the Lorentz hyperbolic cylinder satisfies (1.1) with

$$A=\begin{pmatrix} \frac{1}{r^2} & 0 & * \\ 0 & \frac{1}{r^2} & * \\ 0 & 0 & * \end{pmatrix}.$$

On the other hand, a Lorentz circular cylinder

$$R_1^1\times S^1(c)=\left\{(x_0,\ x_1,\ x_2)\in R_1^3 \,|\, x_1^2+x_2^2=\frac{1}{c}=r^2,\ r>0\right\}$$

is a time-like surface and the Gauss map ξ is given by $(0,\ \xi_0)$, where ξ_0 denotes a Gauss map of the circle $S^1(c)$. Since the Laplacian of ξ_0 is to be ξ_0/r^2, the Laplacian $\Delta\xi$ of Gauss map ξ can be expressed as

$$\Delta\xi=\frac{1}{r^2}\xi .$$

Hence the Lorentz circular cylinder satisfies (1.1) with

$$A=\begin{pmatrix} * & 0 & 0 \\ * & \frac{1}{r^2} & 0 \\ * & 0 & \frac{1}{r^2} \end{pmatrix}.$$

REMARK. Other examples about surfaces of revolution with constant mean curvature in R_1^3 are seen by Hano and Nomizu [5].

§ 3. Surfaces of revolution of type I, II and III.

In this section we are concerned with non-degenerate surfaces of revolution of type I, II and III in the 3-dimensional Minkowski space R_1^3. First of all, let M be a surface of revolution of type I with axis x_2-one. Then the profile curve $\alpha=\alpha(u)$ is given by $\alpha(u)=(0,\ f(u),\ g(u))$, where $f>0$. Suppose that it is parametrized by arc-length, i.e., it satisfies $f'^2+g'^2=1$. The surface of revolution of type I in R_1^3 is parametrized by

$$x=x(u,\ v)=(f(u)\sinh v,\ f(u)\cosh v,\ g(u)) \tag{3.1}$$

Then we have the natural frame $\{x_u,\ x_v\}$ given by

$$\begin{aligned} x_u&=(f'(u)\sinh v,\ f'(u)\cosh v,\ g'(u)), \\ x_v&=(f(u)\cosh v,\ f(u)\sinh v,\ 0). \end{aligned} \tag{3.2}$$

Accordingly we see

$$\langle x_u, x_u\rangle=1, \qquad \langle x_u, x_v\rangle=0, \qquad \langle x_v, x_v\rangle=-f^2,$$

which implies that the surface M is time-like. Let ξ be a unit normal to M. It is defined by $f^{-1}x_u\times x_v$, where $\times$ denotes the Lorentz cross product in $\boldsymbol{R}_1^3$. Then we get

$$\xi=(g'(u)\sinh v,\ g'(u)\cosh v,\ -f'(u)). \tag{3.3}$$

Accordingly ξ is the space-like unit normal to M and hence it can be regarded as a Gauss map of M into the 2-dimensional de Sitter space $S_1^2(1)$.

THEOREM 3.1. *The only surfaces of revolution of type I in $\boldsymbol{R}_1^3$ whose Gauss map satisfies*

$$\Delta\xi=A\xi, \qquad A\in Mat(3, \boldsymbol{R}) \tag{3.4}$$

are locally the Minkowski plane $\boldsymbol{R}_1^2$, the de Sitter space S_1^2 and the Lorentz hyperbolic cylinder $S_1^1\times\boldsymbol{R}$.

PROOF Let M be a surface of revolution of type I parametrized by

$$x=x(u, v)=(f(u)\sinh v,\ f(u)\cosh v,\ g(u)).$$

From the natural frame (3.2) the induced Riemannian metric (g_{ij}) of the surface M is given by $g_{11}=1$, $g_{12}=g_{21}=0$ and $g_{22}=-f^2$. It is easy to show that the Laplacian Δ of M can be expressed as

$$\Delta=-\frac{f'}{f}\frac{\partial}{\partial u}-\frac{\partial^2}{\partial u^2}+\frac{1}{f^2}\frac{\partial^2}{\partial v^2}. \tag{3.5}$$

For the Gauss map $\xi=(g'(u)\sinh v,\ g'(u)\cosh v,\ -f'(u))$, we get

$$\frac{\partial\xi}{\partial u}=(g''(u)\sinh v,\ g''(u)\cosh v,\ -f''(u)),$$

$$\frac{\partial^2\xi}{\partial u^2}=(g'''(u)\sinh v,\ g'''(u)\cosh v,\ -f'''(u)),$$

$$\frac{\partial\xi}{\partial v}=(g'(u)\cosh v,\ g'(u)\sinh v,\ 0),$$

$$\frac{\partial^2\xi}{\partial v^2}=(g'(u)\sinh v,\ g'(u)\cosh v,\ 0).$$

Accordingly we get by (3.5)

$$\Delta\xi=\begin{pmatrix}\left(-\frac{f'}{f}g''-g'''+\frac{1}{f^2}g'\right)\sinh v\\ \left(-\frac{f'}{f}g''-g'''+\frac{1}{f^2}g'\right)\cosh v\\ \frac{f'f''}{f}+f'''\end{pmatrix}.$$

By the assumption (3.4) and the above equation we get the following system of differential equations:

$$\text{(3.6)}\qquad\begin{cases}\left(a_{11}g'+\frac{1}{f}f'g''+g'''-\frac{1}{f^2}g'\right)\sinh v+a_{12}g'\cosh v-a_{13}f'=0\,,\\ a_{21}g'\sinh v+\left(a_{22}g'+\frac{1}{f}f'g''+g'''-\frac{1}{f^2}g'\right)\cosh v-a_{23}f'=0\,,\\ a_{31}g'\sinh v+a_{32}g'\cosh v-a_{33}f'-\frac{1}{f}f'f''-f'''=0\,,\end{cases}$$

where a_{ij} $(i, j=1, 2, 3)$ denote components of the matrix A.

In order to prove this theorem we may solve the above equation and determine the functions f and g. First we suppose that the function f is constant, say r. Since the profile curve $\alpha=(0, f(u), g(u))$ is parametrized by arc-length, we have $g'=\pm 1$ and hence $x(u, v)=(r\sinh v, r\cosh v, \pm u+b)$, $b, r\in\boldsymbol{R}$. That is, the surface M is contained in the Lorentz hyperbolic cylinder $S_1^1(1/r^2)\times\boldsymbol{R}$. Because the functions $\sinh v$ and $\cosh v$ and the constant function are linearly independent, by (3.6) we get $a_{12}=a_{21}=a_{31}=a_{32}=0$ and $a_{11}=a_{22}=r^{-2}>0$. However we have no informations about a_{13}, a_{23} and a_{33}. Therefore the matrix A satisfies

$$A=\begin{pmatrix}\frac{1}{r^2} & 0 & a_{13}\\ 0 & \frac{1}{r^2} & a_{23}\\ 0 & 0 & a_{33}\end{pmatrix}.$$

On the other hand, we suppose that the function g is constant. Then the surface M is contained in the time-like plane parallel to x_0x_1-plane. In this case, by (3.6) the matrix A satisfies

$$A=\begin{pmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ a_{31} & a_{32} & 0\end{pmatrix}.$$

Next, we suppose that the functions f and g are not constant. Let J_1 be a set

$\{u\in J \mid f'(u)\neq 0\}$ and let J_2 be a set $\{u\in J \mid g'(u)\neq 0\}$. Then we know that $J=J_1\cup J_2$ from $f'^2+g'^2=1$ and hence $J_1\cap J_2\neq\emptyset$ by the connectedness of J. Since the matrix A is constant, we may suppose that $J_1\cap J_2$ is an interval. First of all, we consider on $J_1\cap J_2$. From (3.6) we get $a_{12}=a_{23}=a_{21}=a_{23}=a_{31}=a_{32}=0$. Consequently the matrix A satisfies

$$A=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix},$$

and the functions f and g satisfy

$$\left\{\begin{aligned} &\frac{1}{f}f'g''+g'''-\frac{1}{f^2}g'=-a_{11}g', \\ &\frac{1}{f}f'g''+g'''-\frac{1}{f^2}g'=-a_{22}g', \\ &\frac{1}{f}f'f''+f'''=-a_{33}f'. \end{aligned}\right. \tag{3.7}$$

So we get $a_{11}=a_{22}$. We put $a_{11}=a_{22}=\lambda$ and $a_{33}=\mu$. By (3.7) we see

$$f^2g'''+ff'g''+(\lambda f^2-1)g'=0, \tag{3.8}$$

$$f'f''+ff'''+\mu ff'=0, \tag{3.9}$$

$$f'^2+g'^2=1. \tag{3.10}$$

Differentiating (3.10) twice, we get

$$f'f''+g'g''=0, \qquad f''^2+f'f'''+g''^2+g'g'''=0. \tag{3.11}$$

From these equations we eliminate the function g. Using (3.8), (3.10) and (3.11), we have

$$f^2f''^2+ff'(1-f'^2)(ff'''+f'f'')-(\lambda f^2-1)(f'^2-1)^2=0. \tag{3.12}$$

On the other hand, making use of (3.9), we have

$$(ff'')'=-\mu ff'=-\frac{1}{2}\mu(f^2)',$$

which implies by integration

$$ff''=-\frac{1}{2}\mu f^2+a, \qquad a\in\boldsymbol{R}, \tag{3.13}$$

since μ is constant. Solving this differential equation, we get the solution

$$f'^2=-\frac{1}{2}\mu f^2+2a\log f+b, \qquad b\in\boldsymbol{R}. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12) and using (3.9), we get the following polynomial with variable f:

$$\begin{aligned}&\frac{1}{4}\mu^2(\lambda-\mu)f^6\\&\quad-\{2a\mu(\lambda-\mu)\log f+(b-1)\mu(\lambda-\mu)\}f^4\\&\quad+\{4a^2(\lambda-\mu)(\log f)^2+4a(\lambda-\mu)(b-1)\log f+(b-1)^2(\lambda-\mu)+a\mu\}f^2\\&\quad-\{4a^2(\log f)^2+4a(b-1)\log f+(b-1)^2+a^2\}=0\,.\end{aligned}$$

From the coefficients of each term in the above equation we can get

$$a=0, \qquad b=1, \qquad \mu(\lambda-\mu)=0\,.$$

Here, we have that $\mu\neq 0$. In fact, if $\mu=0$, then by (3.14) we get

$$f'^2=b=1\,,$$

which yields that $f'=\pm 1$ and g is constant, a contradiction. Hence we obtain

$$a=0, \qquad b=1, \qquad \lambda=\mu\,.$$

From (3.14), we have

$$f'^2=-\frac{1}{2}\lambda f^2+1\,.$$

Since $g'^2=1-f'^2=\lambda f^2/2$, we get $\lambda>0$. Integrating this equation, we have

$$f=\pm\sqrt{\frac{2}{\lambda}}\sin h(u)\,, \tag{3.15}$$

where $h(u)=\sqrt{\lambda/2}(u+c)$, $c\in\boldsymbol{R}$. From (3.10) and (3.15), we obtain

$$g=\pm\sqrt{\frac{2}{\lambda}}\cos h(u)+d\,, \qquad d\in\boldsymbol{R}\,. \tag{3.16}$$

In this case, we have

$$\langle x(u, v)-\boldsymbol{d},\, x(u, v)-\boldsymbol{d}\rangle=f(u)^2+(g(u)-d)^2=\frac{2}{\lambda}>0, \qquad \boldsymbol{d}=(0, 0, d),$$

which means that the surface M is contained in the de Sitter space $S_1^2(\lambda/2)$ centered at $\boldsymbol{d}$ with radius $\sqrt{2/\lambda}$ on $J_1\cap J_2$ and $A=\lambda E$, where E denotes the unit matrix.

On the other hand, we know that $J=J_1\cap J_2$. In fact, if $J_1-\bar{J}_2$ is not empty, where $\bar{J}_2$ denotes a closure of J_2, then the surface M is contained in the timelike plane parallel to the x_0x_1-plane on $J_1-\bar{J}_2$ and the de Sitter space $S_1^2(\lambda/2)$ on $J_1\cap J_2$. Since the matrix A is constant, λ is zero, a contradiction. Similarly, if $J_2-\bar{J}_1$ is not empty, the surface M is contained in the Lorentz hyperbolic cylinder $S_1^1\times\boldsymbol{R}$ on $J_2-\bar{J}_1$ and the de Sitter space $S_1^2(\lambda/2)$ on $J_1\cap J_2$. Since the matrix A is constant, we have $\lambda=r^{-2}$. This means that the profile curve α is not smooth, a contradiction.

This completes the proof. □

Next, for the case of surfaces of revolution of type II and III, we can get the following theorems.

THEOREM 3.2. *The only space-like (resp. time-like) surfaces of revolution of type II in $\boldsymbol{R}_1^3$ whose Gauss map satisfies* (3.4) *are locally the hyperbolic space H^2 and the hyperbolic cylinder $H^1\times\boldsymbol{R}$ (resp. the Minkowski plane $\boldsymbol{R}_1^2$ and the de Sitter space S_1^2).*

THEOREM 3.3. *The only space-like (resp. time-like) surfaces of revolution of type III in $\boldsymbol{R}_1^3$ whose Gauss map satisfies* (3.4) *are locally the plane $\boldsymbol{R}^2$ and the hyperbolic space H^2 (resp. the de Sitter space S_1^2 and the Lorentz circular cylinder $\boldsymbol{R}_1^1\times S^1$).*

Above theorems are proved by similar discussion to that of Theorem 3.1.

§4. Surfaces of revolution of type IV.

Finally a surfaces of revolution of type IV in $\boldsymbol{R}_1^3$ are characterized in this section. Let M be a surface of revolution of type IV whose axis l is the light-like straight line spanned by (1, 1, 0). Then the profile curve $\alpha=\alpha(u)$ is given by $\alpha(u)=(f(u), g(u), 0)$ where $f\neq g$. Suppose that it is parametrized by arc-length, i.e., it satisfies $-f'^2+g'^2=-\varepsilon(=\pm1)$. The surface of revolution of type IV in $\boldsymbol{R}_1^3$ is parametrized by

$$x=x(u, v)=\left(f(u)+\frac{1}{2}v^2h(u),\ g(u)+\frac{1}{2}v^2h(u),\ vh(u)\right), \tag{4.1}$$

where $h(u)=f(u)-g(u)$. Since the function h has no zero points, we may assume that the function h is positive without loss of generality. The natural frame $\{x_u, x_v\}$ given by

$$\begin{aligned} x_u&=\left(f'+\frac{1}{2}v^2h',\ g'+\frac{1}{2}v^2h',\ vh'\right),\\ x_v&=-(vh,\ vh,\ h). \end{aligned} \tag{4.2}$$

Let ξ be a unit normal to M. It is defined by $h^{-1}x_u\times x_v$. Then we get

$$\xi=\left(\frac{1}{2}v^2h'-g',\ \frac{1}{2}v^2h'-f',\ vh'\right) \quad\text{and}\quad \langle\xi, \xi\rangle=\varepsilon(=\pm1). \tag{4.3}$$

Accordingly ξ can be regarded as a Gauss map of M into the 2-dimensional space form $M^2(\varepsilon)$.

THEOREM 4.1. *The only space-like (resp. time-like) surface of revolution of type IV in* $\boldsymbol{R}_1^3$ *whose Gauss map satisfies*

$$\Delta\xi = A\xi, \qquad A \in Mat(3, \boldsymbol{R}) \tag{4.4}$$

is locally the hyperbolic space H^2 *(resp. the de Sitter space* S_1^2*).*

PROOF. Let M be a surface of revolution of type IV parametrized by

$$x(u, v) = \left(f(u) + \frac{1}{2}v^2 h(u),\ g(u) + \frac{1}{2}v^2 h(u),\ vh(u)\right).$$

From the natural frame (4.2) the induced Riemannian metric (g_{ij}) of the surface M is given by $g_{11} = -\varepsilon$, $g_{12} = g_{21} = 0$ and $g_{22} = h^2$. It is easy to show that the Laplacian Δ of M can be expressed as

$$\Delta = \varepsilon \frac{h'}{h}\frac{\partial}{\partial u} + \varepsilon \frac{\partial^2}{\partial u^2} - \frac{1}{h^2}\frac{\partial^2}{\partial v^2}, \tag{4.5}$$

For the Gauss map $\xi = ((1/2)v^2 h' - g', (1/2)v^2 h' - f', vh')$ we get

$$\frac{\partial \xi}{\partial u} = \left(\frac{1}{2}v^2 h'' - g'',\ \frac{1}{2}v^2 h'' - f'',\ vh''\right),$$

$$\frac{\partial^2 \xi}{\partial u^2} = \left(\frac{1}{2}v^2 h''' - g''',\ \frac{1}{2}v^2 h''' - f''',\ vh'''\right),$$

$$\frac{\partial \xi}{\partial v} = (vh',\ vh',\ h'),$$

$$\frac{\partial^2 \xi}{\partial v^2} = (h',\ h',\ 0).$$

Accordingly we get by (4.5)

$$\Delta\xi = \begin{pmatrix} \varepsilon \frac{h'}{h}\left(\frac{1}{2}v^2 h'' - g''\right) + \varepsilon\left(\frac{1}{2}v^2 h''' - g'''\right) - \frac{1}{h^2}h' \\ \varepsilon \frac{h'}{h}\left(\frac{1}{2}v^2 h'' v - f''\right) + \varepsilon\left(\frac{1}{2}v^2 h''' - f'''\right) - \frac{1}{h^2}h' \\ \varepsilon\left(\frac{1}{h}h'h''v + h'''v\right) \end{pmatrix}.$$

By the assumption (4.4) and the above equation we get

$$\begin{aligned} &\frac{1}{2}\left\{(a_{11} + a_{12})h' - \varepsilon\left(\frac{1}{h}h'h'' + h'''\right)\right\}v^2 + a_{13}h'v \\ &+ \left\{\varepsilon\left(\frac{1}{h}h'g'' + g'''\right) + \frac{1}{h^2}h' - a_{11}g' - a_{12}f'\right\} = 0, \end{aligned} \tag{4.6}$$

$$\begin{aligned}(4.7)\qquad &\frac{1}{2}\left\{(a_{21}+a_{22})h'-\varepsilon\left(\frac{1}{h}h'h''+h'''\right)\right\}v^2+a_{23}h'v\\ &\quad+\left\{\varepsilon\left(\frac{h'}{h}f''+f'''\right)+\frac{1}{h^2}h'-a_{21}g'-a_{22}f'\right\}=0,\end{aligned}$$

$$(4.8)\qquad \frac{1}{2}(a_{31}+a_{32})h'v^2+\left\{a_{33}h'-\varepsilon\left(\frac{h'}{h}h''+h'''\right)\right\}v-(a_{31}g'+a_{32}f')=0.$$

So we can regard the above equations as polynomials with variable v and from the coefficients we get

$$(4.9)\qquad \begin{cases}(a_{11}+a_{12})h'-\varepsilon\left(\frac{1}{h}h'h''+h'''\right)=0,\\ a_{13}h'=0,\\ a_{11}g'+a_{12}f'-\varepsilon\left(\frac{1}{h}h'g''+g'''\right)-\frac{1}{h^2}h'=0,\end{cases}$$

$$(4.10)\qquad \begin{cases}(a_{21}+a_{22})h'-\varepsilon\left(\frac{1}{h}h'h''+h'''\right)=0,\\ a_{23}h'=0,\\ a_{21}g'+a_{22}f'-\varepsilon\left(\frac{1}{h}h'f''+f'''\right)-\frac{1}{h^2}h'=0,\end{cases}$$

$$(4.11)\qquad \begin{cases}(a_{31}+a_{32})h'=0,\\ a_{33}h'-\varepsilon\left(\frac{1}{h}h'h''+h'''\right)=0,\\ a_{31}g'+a_{32}f'=0.\end{cases}$$

Suppose that the function h' has zero points. Then, at these points, we get $f'=g'$, which implies $f'^2-g'^2=0$, a contradiction. So, h' has no zero points. From (4.9) and (4.10) we get $a_{13}=a_{23}=0$, and by (4.11) $a_{31}+a_{32}=0$ and $a_{31}g'+a_{32}f'=0$. Hence we get $a_{31}=a_{32}=0$. On the other hand, by the first equation of (4.9) and the second equation of (4.11), we have

$$(4.12)\qquad a_{11}+a_{12}=a_{33},\qquad a_{21}+a_{22}=a_{33}.$$

Also, by the third equations of (4.9) and (4.10), we get

$$(a_{12}-a_{22})f'+(a_{11}-a_{21})g'+\varepsilon\left(\frac{1}{h}h'h''+h'''\right)=0,$$

from which together with the second equation of (4.11) and (4.12) it follows that $(a_{11}+a_{22}-2a_{33})h'=0$, i.e.,

$$(4.13)\qquad a_{33}=\frac{1}{2}(a_{11}+a_{22}).$$

We put $a_{11}=\lambda$ and $a_{22}=\mu$. Then, by (4.13) and (4.12), we see

$$a_{33}=\frac{1}{2}(\lambda+\mu) \quad \text{and} \quad a_{12}=-a_{21}=\frac{1}{2}(\mu-\lambda).$$

Therefore the matrix A satisfies

$$A=\begin{pmatrix} \lambda & \frac{1}{2}(\mu-\lambda) & 0 \\ \frac{1}{2}(\lambda-\mu) & \mu & 0 \\ 0 & 0 & \frac{1}{2}(\lambda+\mu) \end{pmatrix}.$$

Thus, by the first equation of (4.9) and the last equation of (4.10), we get

$$\frac{1}{2}(\lambda+\mu)hh'-\varepsilon(h'h''+hh''')=0\,, \tag{4.14}$$

$$2\varepsilon(hh'f''+h^2f''')+2h'+h^2\{(\lambda-\mu)h'-(\lambda+\mu)f'\}=0\,, \tag{4.15}$$

On the other hand, making use of (4.14), we have

$$(hh'')'=\frac{1}{4}\varepsilon(\lambda+\mu)(h^2)'\,,$$

which implies by integration

$$hh''=\frac{1}{4}\varepsilon(\lambda+\mu)h^2+a\,, \qquad a\in \boldsymbol{R}\,, \tag{4.16}$$

since ε, λ and μ are constant. Solving this differential equation, we get the solution

$$h'^2=\frac{1}{4}\varepsilon(\lambda+\mu)h^2+2a\log h+b\,, \qquad b\in \boldsymbol{R}\,. \tag{4.17}$$

Because of $f'^2-g'^2=\varepsilon$, we have

$$2f'h'=h'^2+\varepsilon\,. \tag{4.18}$$

Differentiating (4.18), we get

$$\begin{aligned} &f''h'+f'h''=h'h''\,, \\ &f'''h'+2f''h''+f'h'''=h'h'''+h''^2\,. \end{aligned} \tag{4.19}$$

Eliminating the functions f, f', f'' and f''' in (4.15), (4.18) and (4.19), and using (4.16) and (4.17), we have the following polynomial with variable h:

$$\begin{aligned} &\frac{1}{8}(\lambda-\mu)(\lambda+\mu)h^6 \\ &\quad+\{2\varepsilon a(\lambda-\mu)(\lambda+\mu)\log h+\varepsilon b(\lambda-\mu)(\lambda+\mu)\}h^4 \\ &\quad+\{8a^2(\lambda-\mu)(\log h)^2+8ab(\lambda-\mu)\log h+2b^2(\lambda-\mu)+2\varepsilon a(\lambda+\mu)\}h^2 \\ &\quad+4(4a^2(\log h)^2+4ab\log h+a^2+b^2)=0\,. \end{aligned}$$

From the coefficient of h^6 and the constant term in the above equation we get $(\lambda-\mu)(\lambda+\mu)=0$ and $a=b=0$. Suppose that $\lambda+\mu=0$. Then by (4.17) the function h must be constant, a contradiction. So we have $\lambda=\mu$. From (4.17) we obtain

$$h'^2=\frac{1}{2}\varepsilon\lambda h^2, \tag{4.20}$$

and hence we get $\varepsilon\lambda>0$. Integrating (4.20), we can calculate

$$h=e^k, \quad \text{where } k(u)=\pm\sqrt{\frac{\varepsilon\lambda}{2}}(u+c), \quad c\in\boldsymbol{R}.$$

From (4.18) and the definition of h, we obtain

$$f=\frac{1}{2}\left(e^k-\frac{2}{\varepsilon}e^{-k}\right)+d$$

and

$$g=-\frac{1}{2}\left(e^k+\frac{2}{\varepsilon}e^{-k}\right)+d, \qquad d\in\boldsymbol{R}.$$

Accordingly, we have

$$\langle x(u, v)-\boldsymbol{d}, x(u, v)-\boldsymbol{d}\rangle=-(f-d)^2+(g-d)^2=\frac{2}{\varepsilon}, \quad \boldsymbol{d}=(d, d, 0).$$

First we consider that the surface M is space-like, i.e., $\varepsilon=-1$. Then we have $\lambda<0$, which means that M is contained in the hyperbolic space $H^2(\lambda/2)$ centered at $\boldsymbol{d}$ with radius $\sqrt{-2/\lambda}$. On the other hand, if the surface M is time-like, i.e., $\varepsilon=1$, then we have $\lambda>0$, which means that M is contained in the de Sitter space $S_1^2(\lambda/2)$ centered at $\boldsymbol{d}$ with radius $\sqrt{2/\lambda}$.

This completes the proof. □

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